Superintegrability of d-Dimensional Conformal Blocks

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We observe that conformal blocks of scalar four-point functions in a d-dimensional conformal field theory can be mapped to eigenfunctions of a two-particle hyperbolic Calogero-Sutherland Hamiltonian. The latter describes two coupled Pöschl-Teller particles. Their interaction, whose strength depends smoothly on the dimension d, is known to be superintegrable. Our observation enables us to exploit the rich mathematical literature on Calogero-Sutherland models in deriving various results for conformal field theory. These include an explicit construction of conformal blocks in terms of Heckman-Opdam hypergeometric functions. We conclude with a short outlook, in particular, on the consequences of integrability for the theory of conformal blocks.

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Conformal quantum field theories (CFTs) play an important role for modern theoretical physics. In statistical physics, they describe the universal behavior of second order phase transitions. At the same time, CFTs also provide a window into interacting and strongly coupled quantum field theories which are very difficult to access otherwise. In $d = 2$ dimensions, the global conformal algebra is extended to an infinite-dimensional symmetry. This was exploited to construct many such models, paving the way for numerous applications in diverse areas of physics and mathematics.

While the symmetry enhancement of two-dimensional CFT is certainly helpful, it may not be decisive. In fact, CFTs in any dimension d are very strongly constrained by global conformal symmetry. Within the so-called conformal bootstrap program, the solution of CFTs can be reduced to certain integral equations, the crossing symmetry constraints [1–3]. These provide a system of equations for the dynamical coefficients in the operator product expansion involving only the kinematically determined crossing kernel, i.e., group theoretic data. And indeed, recent numerical studies of the crossing symmetry equations, in particular, for the conformal Ising model in $d = 3$ dimensions, have provided ample new precision data on this model; see Refs. [4–8] and references therein.

Analytical progress is lagging behind partly because it is restricted to certain limits in which there exists sufficient control of the kinematical input [9–11]. This is what our work addresses. We will focus on the group theoretic building blocks of scalar four-point functions, the so-called conformal blocks that underly the entire bootstrap program. Partial waves can be characterized through a second order differential equation [12]. So far, a construction of solutions of conformal Casimir equations in terms of hypergeometric functions is only known in even integer dimensions, where they can be obtained from Gauss hypergeometric functions.

Our main observation is that the Casimir equation for conformal blocks in d dimensions may be transformed into the eigenvalue problem for a Calogero-Sutherland (CS) Hamiltonian, whose eigenfunctions are given by Heckman-Opdam (HO) hypergeometric functions [13]. Thereby, we connect the poorly developed theory of conformal blocks to integrability and the modern theory of special functions. The relevant CS Hamiltonian turns out to be superintegrable; i.e., it possesses an additional Runge-Lenz-like integral of motion [14]. The latter is part of the (degenerate) double affine Hecke algebra (DAHA) [15] which provides an extremely powerful algebraic underpinning, introduces a distinguished q deformation, and bridges to the dual Ruijsenaars-Schneider (RS) model [16]. This leads to a wealth of interesting relations for conformal blocks, some of which we will touch upon below.

The plan of this Letter is as follows. In the next section, we will briefly review the characterization of conformal blocks through the conformal Casimir equation. For pedagogical reasons, we will then explore our general theme in $d = 2$ where the relevant CS model decouples into two Pöschl-Teller systems. These are known to be solvable through hypergeometric functions. Then, we turn to the d-dimensional problem and explain how the two Pöschl-Teller systems are coupled in order to describe conformal blocks in d-dimensional conformal field theory. The known eigenfunctions of the resulting CS Hamiltonian are finally used to construct conformal blocks from a q-deformed version of HO hypergeometric functions. We conclude by highlighting a few applications of known mathematical results, many of them quite recent, to the conformal bootstrap program.

Conformal blocks.—In this section, we want to set up the problem by briefly reviewing some material from Ref. [12].

The correlation function of four scalar conformal primary fields of weight $\Delta_i, i = 1, \ldots, 4$ in a d-dimensional conformal field theory can be decomposed as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{1}{x_{12}^{(1/2)(\Delta_1+\Delta_2)} x_{34}^{(1/2)(\Delta_3+\Delta_4)}} \left(\frac{x_{14}}{x_{24}}\right)^a \left(\frac{x_{14}}{x_{13}}\right)^b G(z,\bar{z}), \quad (1)$$

with $x_{ij} = x_i - x_j$ and $2a = \Delta_2 - \Delta_1$, $2b = \Delta_3 - \Delta_4$. The conformal invariants z, $\bar{z}$ were introduced to parametrize the more familiar cross ratios as

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad (2)$$

$$\frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \quad (3)$$

For a Euclidean theory, z, $\bar{z}$ are complex variables. The function G receives contributions from all the primary fields that can appear in the operator product expansion of the field ϕ_1 and ϕ_2

$$G(z,\bar{z}) = \sum_{\Delta,l} \lambda_l^{12}(\Delta)\lambda_l^{34}(\Delta) G_{\Delta,l}(z,\bar{z}). \quad (4)$$

This expansion separates the dynamically determined coefficients λ of the operator product from the kinematic conformal blocks $G_{\Delta,l}$. The latter are eigenfunctions of the conformal Laplacian D_ϵ^2

$$D_\epsilon^2 G(z,\bar{z}) = \frac{1}{2} C_{\Delta,l} G(z,\bar{z}), \quad (5)$$

with eigenvalues

$$C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2) \quad (6)$$

and subject to an additional boundary condition that selects a unique (up to normalization) combination of solutions. The form of the conformal Laplacian can be worked out easily; see, e.g., Ref. [12],

$$D_\epsilon^2 := D^2 + \bar{D}^2 + \epsilon\left[\frac{z\bar{z}}{\bar{z}-z}(\bar{\partial} - \partial) + (z^2\partial - \bar{z}^2\bar{\partial})\right], \quad (7)$$

where $\epsilon = d - 2$ and

$$D^2 = z^2(1-z)\partial^2 - (a+b+1)z^2\partial - abz. \quad (8)$$

$\bar{D}^2$ is defined similarly in terms of $\bar{z}$. In $d = 2$ dimensions, the Hamiltonian splits into a sum of two independent pieces and the corresponding eigenvalue equations are straightforwardly related to hypergeometric differential equations. Our main goal in this work is to solve the eigenvalue problem for the conformal Laplacian in terms of some known special functions.

Pöschl-Teller potential.—In order to get a bit more insight into the structure of the eigenvalue problem for the conformal Laplacian, we will temporarily set $d = 2$. The Laplacian then decomposes into a sum of operators acting on z and $\bar{z}$ only, and we will focus on the eigenvalue problem for D^2. This problem leads to the following second order differential equation

$$D^2 G(z) = h(h-1)G(z).$$

Now, let us now define a new function which is related to G by a "gauge transformation" of the form

$$\psi(x) := \frac{(z-1)^{(a+b)/2+(1/4)}}{\sqrt{z}} G(z), \quad (9)$$

where the coordinates z and x are related by

$$z = -\sinh^{-2}\frac{x}{2}. \quad (10)$$

Note that this relation maps the complex z plane to a semi-infinite strip with $\mathrm{Re}x \geq 0$ and $\mathrm{Im}x \in [0,\pi]$. Inserting these relations, it is easy to see that the function ψ is an eigenfunction of the Pöschl-Teller Hamiltonian with potential

$$V_{\mathrm{PT}}^{(a,b)}(x) = \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 x} - \frac{ab}{\sinh^2(x/2)} \quad (11)$$

for the eigenvalue $\varepsilon := 2mE/\hbar^2 = -(2h-1)^2/4$. The original Schrödinger problem studied by Pöschl and Teller in Ref. [17] was a trigonometric version of Eq. (11). After such rotation to $y = ix$, the associated Schrödinger problem describes a particle that is confined to the interval $y \in [0,\pi]$. The Hamilton operator possesses a discrete spectrum with eigenfunctions given by ordinary Jacobi polynomials.

The hyperbolic version we are dealing with here is also referred to as a Pöschl-Teller Hamiltonian of the second kind. It describes a particle on the half line $x \geq 0$. Since the potential falls off to 0 for large x, the Hamiltonian has a continuous part in its spectrum. The eigenfunctions are given by

$$\psi_h(x) \sim z^{h-(1/2)}(z-1)^{(a+b)/2+(1/4)} {}_2F_1\left({h+a, h+b \atop 2h}; z\right).$$

Before we move on, we stress that the Pöschl-Teller problem is related to some classical theory of special functions. Let us describe this for the trigonometric case in which eigenfunctions are classical Jacobi polynomials. Like all other hypergeometric orthogonal polynomials in a single variable, Jacobi polynomials are obtained from a degeneration of the so-called Askey-Wilson polynomials. The latter may be constructed from the q-deformed version ${}_4\Phi_3$ of the hypergeometric function ${}_4F_3$ by specializing its parameters; see, e.g., Ref. [18]. Of course, all these relations can be lifted to the hyperbolic theory, i.e., from polynomials to functions.

Calogero-Sutherland potential.—Historically, the Schrödinger problem for the Pöschl-Teller potential was solved through the relation with the hypergeometric differential equation. But today, it is much more interesting to look at the relation in the opposite direction. Following work of Calogero, Moser, and Sutherland in the early 1970s (see Refs. [19–21]), the solvable Pöschl-Teller problem has been generalized in several directions. In particular, it was understood that the Pöschl-Teller potential is just the simplest example of a large family of superintegrable Schrödinger problems involving multiple particles. The relevant potentials are associated with reflection groups and give rise to so-called (trigonometric or hyperbolic) Calogero-(Moser)-Sutherland models [22].

In the last section, we recalled that the Casimir equation for blocks in two-dimensional chiral conformal field theory is equivalent to the Pöschl-Teller problem. Our main claim is that this extends to the full Casimir equation for conformal blocks in d dimensions. In complete analogy to the discussion above, it turns out that the Casimir equation is equivalent to the hyperbolic CS model for reflection group BC_2. Its potential is given by

$$V_{\mathrm{CS}}^{(a,b,\epsilon)}(x_1,x_2) = V_{\mathrm{PT}}^{(a,b)}(x_1) + V_{\mathrm{PT}}^{(a,b)}(x_2) + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1-x_2}{2}} + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1+x_2}{2}}. \quad (12)$$

It is built from two Pöschl-Teller systems with an interaction term whose coupling explicitly depends on the dimension d. The six terms of this potential reflect the six positive roots of the BC_2 root system. To relate the associated Schrödinger problem on the BC_2 Weyl chamber with the eigenvalue equation (5) for the conformal Laplacian, we generalize the gauge transformation (9) to become

$$\psi(x_1,x_2) \coloneqq \prod_i \frac{(z_i-1)^{(a+b)/2+(1/4)}}{z_i^{(1/2)+(\epsilon/2)}} |z_1-z_2|^{\epsilon/2} G(z_1,z_2), \quad (13)$$

where $z_1 = z$ and $z_2 = \bar{z}$. It is not difficult to verify that this gauge transformation, along with the relation

$$z_i = -\sinh^{-2}\frac{x_i}{2} \quad (14)$$

between the coordinates z_i and x_i, turns the conformal Laplacian into the CS Hamiltonian for the potential (12), with the eigenvalue $\varepsilon = -d(d-2)/4 - (C_{\Delta,l}+1)/2$. The appearance of the BC_2 root lattice possesses a natural explanation in the corresponding harmonic analysis formulation of the problem [23,24], where it enters as a projection of the root lattice of the conformal algebra $\mathfrak{so}(1,d+1)$, $d \geq 5$, to the two-dimensional plane spanned by the Cartan generators of an embedded $\mathfrak{so}(1,3)$.

Just as the one-dimensional Pöschl-Teller problem is exactly solvable, so are the higher-dimensional CS extensions and hence, by the relation (13), the eigenvalue problem for the conformal Laplacian. Let us note that the coupling constant in front of the interaction term is $\varepsilon = d-2$. In $d=2$ dimensions, we are just dealing with two independent integrable Pöschl-Teller systems. Going away from $d=2$ introduces a new coupling in the potential. It is quite remarkable that this coupling is also integrable. One may notice that in $d=4$ dimensions, the interaction terms vanish once again. This implies that $2d$ and $4d$ conformal blocks are simply related by a gauge transformation. The latter switches between bosonic or fermionic statistics of the wave function. For conformal blocks, the simple relation between $d=2$ and $d=4$ dimensions is indeed consistent with the standard expressions [12].

Some applications.—What makes the observed relation between conformal blocks and the CS Hamiltonian interesting are the connections of the latter with integrability and the modern theory of special functions. The integrability of the CS model can be established using so-called Dunkl operators, i.e., a special set of linear first order operators that involve reflections. From powers of these Dunkl operators, one can construct sufficiently many commuting operators to render the problem integrable, in fact, even superintegrable (in rational or hyperbolic cases). Along with the multiplication by coordinates and Weyl reflections, Dunkl operators generate a (trigonometric) degeneration of the so-called double affine Hecke algebra. The latter involves an additional deformation parameter q that is sent to $q=1$ when dealing with the (undeformed) CS model. In order to understand the origin of the parameter q, one needs to turn to the rational Ruijsenaars-Schneider model, which is related to our hyperbolic CS model by a bispectral duality [25]. Within the dual theory, q controls the deformation from the rational to the hyperbolic version. Many more details on these topics can be found, e.g., in Refs. [15,26].

All this structure is an integral part of the modern theory of special functions. In the context of the (trigonometric) Pöschl-Teller problem, we briefly sketched the relation between classical Jacobi and q-deformed Askey-Wilson polynomials. The latter possess well developed multivariable extensions which are known as q-deformed HO or (Macdonald)-Koornwinder polynomials K. Just as the trigonometric Pöschl-Teller problem can be solved through a degenerate limit of Askey-Wilson polynomials, eigenfunctions of the trigonometric CS Hamiltonian may be obtained from Koornwinder polynomials in the limit

$q \to 1$. This web of interrelations may be lifted from polynomials to functions, i.e., from the trigonometric to the hyperbolic theory. The lift turns Koornwinder or q-deformed HO polynomials into what Rains refers to as *virtual* Koornwinder polynomials $\hat{K}$; see Ref. [27]. We can also think of them as q-deformed HO hypergeometric functions, up to normalization issues.

Before we can spell out a concrete formula, we need to split the data Δ, l that characterize the internal field into a partition (λ_1, λ_2) and a real parameter χ. Upon imposing usual unitarity bounds, this is done as follows:

$$\lambda_2 \coloneqq \lfloor \frac{1}{2}(\Delta - l) \rfloor, \tag{15}$$

$$\lambda_1 \coloneqq \lambda_2 + l, \qquad \chi \coloneqq \frac{1}{2}(\Delta - l) - \lambda_2. \tag{16}$$

The virtual Koornwinder polynomials $\hat{K}^{(2)}_{\lambda_1,\lambda_2}$ for the root system BC_2 are functions of two variables u_i, $i = 1, 2$ that are associated with two-row partitions (λ_1, λ_2). By appropriate choice of their seven parameters, we can obtain conformal blocks as

$$(-4)^{-\Delta}\left(\frac{z\bar{z}}{16}\right)^a G_{\Delta,l}(z_i) = (u_1 u_2)^{\chi+a} \times \lim_{q\to 1^-} \hat{K}^{(2)}_{l_1,l_2}(u_i; q, q^{\epsilon/2}, q^{-\chi-a}; q^{a-b+1}, -q^{a+b+1}, 1, -1), \tag{17}$$

where $\epsilon = d - 2$ and

$$u_i = -\frac{z_i}{(1+\sqrt{1-z_i})^2} \tag{18}$$

for $i = 1, 2$ for $z_1 = z, z_2 = \bar{z}$ are obtained by inverting the relations (14) with $u_i = \exp x_i$. Note that the arguments u_i agree with the radial coordinates of Ref. [28] up to a sign.

Virtual Koornwinder polynomials possess a binomial expansion in terms of Okounkov's BC_n-type interpolation Macdonald polynomials; see Ref. [27], Sec. VII. These should be considered as generalizations of the usual series expansion of a hypergeometric function ${}_2F_1(u)$ in terms of monomials u^k. In the case the BC_2 root system, the interpolation Macdonald reduce to Gegenbauer polynomials upon taking $q \to 1$ and the combinatorial prefactors may be expressed through the hypergeometric functions ${}_4F_3$ [29]. The resulting expansion reproduces a formula for conformal blocks that was found by Dolan and Osborn in Ref. [12].

Conclusion and outlook.—The main observation of this work, that the Casimir equation for conformal blocks is equivalent to the Schrödinger equation for the BC_2 CS model, embeds the central objects in the bootstrap program of d-dimensional conformal field theory into the rich world of superintegrable quantum systems. The deep connections to the modern theory of special functions have powerful implications for conformal blocks of which we have seen just one example in the previous section.

Let us just sketch a few of the more immediate applications. One of them concerns the issue of Euler- and Mellin-Barnes-type integral representations. These are indispensable tools in working with single variable hypergeometric functions such as ${}_2F_1$. To arrive at similar formulas in the multivariate case, one can exploit an important relation between CS models and Knizhnik-Zamolodchikov (KZ) connections [30,31]. This connection takes values in the group algebra of the relevant Weyl group; i.e., it can be represented as an eight-dimensional matrix connection for BC_2. There exist several different approaches to integrate KZ connections. First of all, one may write solutions simply as a path ordered exponentials. These lead to representations of conformal blocks in terms of Chen's iterated integrals [32]. Alternatively, one may exploit Aomoto's theory of hypergeometric functions over Grassmannians to obtain Euler-type (free field) representations [33]. And finally, one can also decide pass to the dual rational RS model, or rather its description in terms of rational dynamical difference equations. In the dual theory, the roles of momenta λ and coordinates y are interchanged, and hence one finds Mellin-Barnes-type representations for conformal blocks which, according to the ideas outlined in Ref. [34], can play a central role in the bootstrap program. All the necessary mathematical background has been developed and applied, although mostly in the context of A_n root systems.

Once Euler- or Mellin-Barnes-type integral formulas are worked out, they can be used to compute the so-called crossing kernel of d-dimensional conformal field theory, the central object of the bootstrap. In the numerical bootstrap program, the crossing symmetry is usually written in terms of conformal blocks, with one side of the equation involving blocks in the so-called s channel while the other side is expressed in terms of t-channel waves. The blocks in the two different channels are related by the crossing kernel, so that crossing symmetry may be expressed in terms of operator product coefficients $\lambda_l(\Delta)$ and the crossing kernel, stripping off the $(z, \bar{z})$-dependent conformal blocks. With the improved analytic control of conformal blocks we have described above, it is possible to obtain new and explicit formulas for the crossing kernels. These nurture hopes for exact analytical solutions of the bootstrap program.

Before we conclude, let us also mention a few more or less obvious extensions of the above. The first one concerns extensions to external fields with spin or superconformal field theory. While quite a few Casimir equations have been worked out, explicit formulas for the corresponding blocks are only known in rare cases; see, e.g., Refs. [35–38] for some interesting recent developments and many references to the earlier literature. All this may be embedded into the theory of matrix CS models, which provide a rather universal framework to construct solutions.

Let us finally stress again that in our entire discussion, the dimension d enters as a continuous parameter which is interpreted as a coupling constant of the CS model. There exist many conformal field theories for $d = 2$ dimensions that can be solved through their higher spin symmetries. It should be possible to combine the results we outlined above with the ideas that were put forward recently in Ref. [11] to study the spectrum of conformal field theories in $2 + \epsilon$ dimensions, at least for small ϵ. We will return to these interesting problems in future work.

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[1] S. Ferrara, A. F. Grillo, and R. Gatto, Ann. Phys. (N.Y.) **76**, 161 (1973).
[2] A. M. Polyakov, Zh. Eksp. Teor. Fiz. **66**, 23 (1974).
[3] G. Mack, Nucl. Phys. **B118**, 445 (1977).
[4] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, J. High Energy Phys. 12 (2008) 031.
[5] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012).
[6] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, J. Stat. Phys. **157**, 869 (2014).
[7] D. Simmons-Duffin, J. High Energy Phys. 06 (2015) 174.
[8] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, arXiv:1603.04436.
[9] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 12 (2013) 004.
[10] Z. Komargodski and A. Zhiboedov, J. High Energy Phys. 11 (2013) 140.
[11] L. F. Alday and A. Zhiboedov, J. High Energy Phys. 06 (2016) 091.
[12] F. A. Dolan and H. Osborn, Nucl. Phys. **B678**, 491 (2004).
[13] G. J. Heckman and E. M. Opdam, Compos. Math. **64**, 329 (1987).
[14] V. B. Kuznetsov, Phys. Lett. A **218**, 212 (1996).
[15] I. Cherednik, *Double Affine Hecke Algebras* (Cambridge University Press, Cambridge, England, 2005).
[16] S. N. M. Ruijsenaars and H. Schneider, Ann. Phys. (N.Y.) **170**, 370 (1986).
[17] G. Pöschl and E. Teller, Z. Phys. **83**, 143 (1933).
[18] T. H. Koornwinder, Scholarpedia **7**, 7761 (2012).
[19] F. Calogero, J. Math. Phys. (N.Y.) **12**, 419 (1971).
[20] J. Moser, Adv. Math. **16**, 197 (1975).
[21] B. Sutherland, Phys. Rev. A **5**, 1372 (1972).
[22] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94**, 313 (1983).
[23] A. Oblomkov, Adv. Math. **186**, 153 (2004).
[24] L. Feher and B. G. Pusztai, Rev. Math. Phys. **22**, 699 (2010).
[25] S. N. M. Ruijsenaars, Commun. Math. Phys. **115**, 127 (1988).
[26] A. A. Kirillov, Jr., arXiv:math/9501219.
[27] M. E. Rains, Trans. Groups **10**, 63 (2005).
[28] M. Hogervorst and S. Rychkov, Phys. Rev. D **87**, 106004 (2013).
[29] T. H. Koornwinder, Sém. Lothar. Combin. **B72a**, '1 (2015).
[30] A. Matsuo, Commun. Math. Phys. **151**, 263 (1993).
[31] I. Cherednik, Commun. Math. Phys. **150**, 109 (1992).
[32] K.-T. Chen, Bull. Am. Math. Soc. **83**, 831 (1977).
[33] K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*, Springer Monographs in Mathematics (Springer, New York, 2011).
[34] G. Mack, arXiv:0907.2407.
[35] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, J. High Energy Phys. 11 (2011) 154.
[36] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, J. High Energy Phys. 04 (2016) 074.
[37] A. C. Echeverri, E. Elkhidir, D. Karateev, and M. Serone, J. High Energy Phys. 02 (2016) 183.
[38] N. Bobev, S. El-Showk, D. Mazac, and M. F. Paulos, J. High Energy Phys. 08 (2015) 142.