Integrability and Exact results in $\mathcal{N} = 2$ gauge theories

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DESY Theory

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arXiv:1406.3629 with Vladimir Mitev
arXiv:1511.02217 with Vladimir Mitev
work in progress
Motivation: The success story for $\mathcal{N} = 4$ SYM

Possible to compute observables in the strong coupling regime and in some cases to even obtain **Exact results** (for any value of the coupling).

- **AdS/CFT** (gravity/sigma model description)
- **Integrability** (The spectral problem is solved) *at large* $N_c$
- **Localization** (Exact results: e.x. Circular WL) *for any* $N_c$

Which of these properties/techniques are transferable to **more realistic gauge theories** in 4D with less SUSY?
Localization works for $\mathcal{N} = 2$ theories (Pestun)

The path integral localizes to (a Matrix model) an ordinary integral!

$$Z_{S^4} = \int [D\Phi] e^{-S[\Phi]} = \int da |Z(a)|^2$$

An example of exact observable:

$$W(\lambda) = 2 \frac{l_1(\sqrt{\lambda})}{\sqrt{\lambda}} = \begin{cases} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \cdots & \lambda << 1 \\ \sqrt{\frac{2}{\pi}} \lambda^{-\frac{3}{4}} e^{\sqrt{\lambda}} + \cdots & \lambda >> 1 \end{cases}$$

the $\mathcal{N} = 4$ SYM circular Wilson loop in the planar limit.

- Even if the observable cannot be written in a closed form, one can always expand both from the weak and from the strong coupling.
Integrability (only in the planar limit)

- **Perturbation theory**: integrable spin chain (Minahan, Zarembo, ...).
- **Gravity side**: integrable 2D sigma model (Bena, Polchinski, Roiban, ...).
- The spectral problem is solved $\forall \lambda$ (Gromov, Kazakov, Vieira, ...).
- Now other observables: scattering amplitudes, correlation functions...

**Integrability**: 2-body problem $\rightarrow$ $n$-body problem

**Exact result** ($\forall \lambda$) is due to **symmetry**:

- The dispersion relation $\Delta - |r| = \sqrt{1 + h(g) \sin^2 \left( \frac{p}{2} \right)}$ and the 2-body **S-matrix**

are fixed due to the $SU(2|2) \subset PSU(2, 2|4)$ symmetry (Beisert).

- For $\mathcal{N} = 2$ theories we also have it: $SU(2|2) \subset SU(2, 2|2)$!
**AdS/CFT**

**AdS duals** only for a sparse set of 4D theories:

- $D3$ branes in **critical string theory**. (e.g. orbifolds)
- Adjoint and bifundamental matter. (Flavors in the **probe** approx.)

It has been argued that:

- $\mathcal{N} = 1$ SQCD in the Seiberg **conformal** window is dual to 6$d$ non-critical backgrounds of the form $AdS_5 \times S^1$.  
  (Klebanov-Maldacena, Fotopoulos-Niarchos-Prezas, Murthy-Troost,...)

- $\mathcal{N} = 2$ SCQCD is dual to 8$d$ **non-critical** string theory in a background with an $AdS_5 \times S^1$ factor (Gadde-EP-Rastelli)

**Checked at the level of the chiral spectrum.**

For non-protected quantities there is nothing to compare with!
Plan of attack

Discover the string from the "bottom up".

$$AdS_5 \times S^1 \times \mathcal{M}$$

• Probe the $AdS_5 \times S^1$ factor of the geometry: purely gluonic sector
• Probe the compact $S^1 \times \mathcal{M}$ factor: sectors with quarks

$$f_1(g^2) = \frac{R_{AdS}^4}{(2\pi\alpha')^2}, \quad f_2(g^2) = \frac{R_{S^1}^4}{(2\pi\alpha')^2}, \quad f_3(g^2) = \frac{R_{\mathcal{M}}^4}{(2\pi\alpha')^2}$$

using:
• Perturbation theory
• The spin chain description (Symmetry and Integrability)
• Localization
The main statement
Every $\mathcal{N} = 2$ superconformal gauge theory has a purely gluonic $SU(2, 1|2)$ sector integrable in the planar limit

$$H_{\mathcal{N}=2} (g) = H_{\mathcal{N}=4} (g)$$

The **Exact Effective coupling** (relative finite renormalization of $g$)

$$g^2 = f(g^2) = g^2 + g^2 (Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4})$$

we compute using localization

$$W_{\mathcal{N}=2} (g^2) = W_{\mathcal{N}=4} (g^2)$$

**AdS/CFT: effective string tension**

$$f(g^2) = T^2_{\text{eff}} = \left( \frac{R^4}{(2\pi \alpha')^2} \right)_{\text{eff}}$$

Obtain any observable classically in the factor $AdS_5 \times S^1$ of the geometry by replacing $g^2 \rightarrow f(g^2)$. 
\( \mathcal{N} = 2 \) SuperConformal theories

- ADE classification \( \mathcal{N} = 2 \) SCFT: finite/affine Dynkin diagrams

**One parameter family** \( \mathcal{N} = 2 \) SCFT: product gauge group \( SU(N) \times SU(N) \) and **two exactly marginal couplings** \( g \) and \( \tilde{g} \) (Gadde-EP-Rastelli)

\[
\begin{array}{c}
g \\
\rightarrow \\
\tilde{g}
\end{array}
\]

- For \( \tilde{g} \to 0 \) obtain \( \mathcal{N} = 2 \) SCQCD with \( N_f = 2N \)

\[
\begin{array}{c}
\square \\
\rightarrow \\
\bigcirc \\
\rightarrow \\
\square
\end{array}
\]

- For \( \tilde{g} = g \) one finds the well-known \( \mathbb{Z}_2 \) orbifold of \( \mathcal{N} = 4 \) SYM, with an \( AdS_5 \times S^5/\mathbb{Z}_2 \) gravity dual (Kachru-Silverstein, Lawrence-Nekrasov-Vafa, \ldots)
Integrability of the purely gluonic $SU(2,1|2)$ Sector

$\phi, \lambda_+, \mathcal{F}^{++}, D_{+\dot{\alpha}}$
A diagrammatic observation

The only possible way to make diagrams with external fields in the vector mult. different from the $\mathcal{N} = 4$ ones is to make a loop with hyper’s and then in this loop let a checked vector propagate!

(EP-Sieg)

The same with $\mathcal{N} = 4$ SYM

Different from $\mathcal{N} = 4$ SYM but finite!!
A diagrammatic observation

The only possible way to make diagrams with external fields in the vector mult. different from the $\mathcal{N} = 4$ ones is to make a loop with hyper’s and then in this loop let a checked vector propagate! (EP-Sieg)

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Novel Regularization prescription: (Arkani-Hamed-Murayama)

For every individual $\mathcal{N} = 2$ diagram subtract its $\mathcal{N} = 4$ counterpart.
$H_{\mathcal{N}=2}^{(3)}(\lambda) - H_{\mathcal{N}=4}^{(3)}(\lambda) \sim H_{\mathcal{N}=4}^{(1)}(\lambda) \Rightarrow H_{\mathcal{N}=2}^{(3)}(\lambda) = H_{\mathcal{N}=4}^{(3)}(f(\lambda))$

with $f(\lambda) = \lambda + c\lambda^3$
Operator renormalization in the Background Field Gauge

**Background Field Method:** \( \varphi \rightarrow A + Q \)

where \( A \) the classical background and \( Q \) the quantum fluctuation

\[
\begin{align*}
g_{\text{bare}} &= Z_g \, g_{\text{ren}}, \\
A_{\text{bare}} &= \sqrt{Z_A} \, A_{\text{ren}}, \\
Q_{\text{bare}} &= \sqrt{Z_Q} \, Q_{\text{ren}}, \\
\xi_{\text{bare}} &= Z_\xi \, \xi_{\text{ren}}
\end{align*}
\]

In the Background Field Gauge \( Z_g \sqrt{Z_A} = 1 \) and \( Z_Q = Z_\xi \)
Operator renormalization in the Background Field Gauge

**Background Field Method:** \( \varphi \rightarrow A + Q \)

where \( A \) the classical background and \( Q \) the quantum fluctuation

\[
g_{\text{bare}} = Z_g g_{\text{ren}}, \quad A_{\text{bare}} = \sqrt{Z_A} A_{\text{ren}}, \quad Q_{\text{bare}} = \sqrt{Z_Q} Q_{\text{ren}}, \quad \xi_{\text{bare}} = Z_\xi \xi_{\text{ren}}
\]

In the Background Field Gauge \( Z_g \sqrt{Z_A} = 1 \) and \( Z_Q = Z_\xi \)

- Compute \( \langle O(y)A(x_1)\cdots A(x_L) \rangle \) for \( O \sim \text{tr} (\varphi^L) \).

Wick contract \( O_{i}^{\text{ren}} (Q_{\text{ren}}, A_{\text{ren}}) = \sum_j Z_{ij} O_j^{\text{bare}} \left( Z_Q^{1/2} Q, Z_A^{1/2} A \right) \).

+ more diagrams
Background Field Method: No $Q$'s outside, no $A$'s inside!

\[ A(x_1)A(x_m) \quad A(x_{m+1})A(x_L) \]

- $\langle QQAA \rangle$ renormalize as $Z_Q^{2/2} Z_A^{2/2} \langle QQAA \rangle$
- The $Q$ propagators as $Z_Q^{-1}$
- the $\mathcal{O}^{ren}$ has two more $Z_Q^{1/2}$
- all $Z_Q$ will cancel $\forall$ individual diagram (We knew it - gauge invariance!)
- Only $Z = Z_g^2 = Z_A^{-1}$, the combinatorics the same as in $\mathcal{N} = 4$:

\[
H_{\mathcal{N}=2} (g) = H_{\mathcal{N}=4} (g) \quad \text{with} \quad g^2 = f(g^2, \tilde{g}^2) = g^2 + g^2 \left( Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4} \right)
\]
New non-Holomorphic vertices cannot contribute

\[ \Gamma = \Gamma_{\text{ren. tree}} + \Gamma_{\text{new}} = \int d^4 \theta F (\mathcal{W}) + \text{c.c.} + \int d^4 \theta d^4 \bar{\theta} \mathcal{H} (\mathcal{W}, \bar{\mathcal{W}}) \]

- **\( \Gamma_{\text{ren. tree}} \): vertex and self-energy renormalization all encoded in \( Z = Z^2 = Z^{-1}_A \)

- **\( \Gamma_{\text{new}} \): New non-Holomorphic vertices cannot contribute due to the non-renormalization theorem (Fiamberti, Santambrogio, Sieg, Zanon)**
Localization and Exact Effective couplings
Pestun Localization on the sphere

\[ \langle \phi \rangle = \text{diag} (a_1, \ldots, a_N) \]

\[ Z_{S^4} = \int [da] |Z_{Nek}(a, \epsilon_1 = r^{-1}, \epsilon_2 = r^{-1})|^2 \]

\[ \epsilon_{1,2} = r^{-1} \text{ omega deformation parameters serve as an IR regulator} \]

\[ \log (Z_{Nek}(a, \epsilon_1, \epsilon_2)) \sim -\frac{1}{\epsilon_1 \epsilon_2} F(a) \]

- The **UV divergences** on the sphere are the same as those on \( \mathbb{R}^4 \).

The circular wilson loop can be computed

\[ W(g) = Z_{S^4}^{-1} \int [da] \left( \frac{1}{N} \sum_i e^{2\pi a_i} \right) |Z_{Nek}(a, r^{-1})|^2 \]

and is given by a matrix model calculation.
• For $\mathcal{N} = 4$ the matrix model is Gaussian \((\text{Erickson, Semenoff, Zarembo})\)

\[
W_{\mathcal{N}=4}(g) = \frac{l_1(4\pi g)}{2\pi g}
\]

• For $\mathcal{N} = 2$ theories we have a more complicated multi-matrix model

\[
W_{\mathcal{N}=2}(g, \bar{g}) = W_{\mathcal{N}=4}(f(g, \bar{g}))
\]

\[
f(g, \bar{g}) = \left\{ \begin{array}{l}
g^2 + 2(\bar{g}^2 - g^2) \left[ 6\zeta(3)g^4 - 20\zeta(5)g^4 (\bar{g}^2 + 3g^2) \right] + O(g^{10}) \\
\frac{2g\bar{g}}{g + \bar{g}} + O(1)
\end{array} \right.
\]

• Checked with Feynman diagrams calculation (up to 4-loops)

• Agrees with AdS/CFT \((\text{Gadde-EP-Rastelli, Gadde-Liendo-Rastelli-Yan})\)
Cusp anomalous dimension and Bremsstrahlung function

Analytically continue to Minkowski signature $\phi = i\varphi$:

$$W_\varphi \sim e^{-\Gamma_{\text{cusp}}(\varphi) \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}}$$

with $\Lambda_{\text{UV}}$ and $\Lambda_{\text{IR}}$ the UV and IR cutoff.

- For big $\varphi$: light-like cusp anomalous dimension
  $$\Gamma_{\text{cusp}}(\varphi) \sim K\varphi$$
  leading log behavior of the anomalous dims of finite twist operators
  $$\Delta - S \sim K \log S \quad \text{as} \quad S \to \infty$$

- For small $\varphi$:
  $$\Gamma_{\text{cusp}}(\varphi) = B\varphi^2 + O(\varphi^4)$$
  $B \propto$ the energy emitted by an uniformly accelerating probe quark
Bremsstrahlung function from localization on the ellipsoid

Follow (Lewkowycz-Maldacena) and (Fiol-Gerchkovitz-Komargodski)

$$B = \pm \frac{1}{4\pi^2} \frac{d}{db} \log \langle W^\pm (b) \rangle \bigg|_{b=1}$$

- For $\mathcal{N} = 4$:

$$B_{\mathcal{N}=4}(g^2) = \frac{g l_2(4g\pi)}{\pi l_1(4g\pi)}$$

- For $\mathcal{N} = 2$ theories we have a more complicated multi-matrix model

$$B_{\mathcal{N}=2}(g, \tilde{g}) = B_{\mathcal{N}=4}(f(g, \tilde{g}))$$

$$f(g, \tilde{g}) = \begin{cases} g^2 + 2 (\tilde{g}^2 - g^2) \left[ 6\zeta(3)g^4 - 20\zeta(5)g^4 (\tilde{g}^2 + 3g^2) \right] + \mathcal{O}(g^{10}) \\ \frac{2g\tilde{g}}{g+\tilde{g}} + \mathcal{O}(1) \end{cases}$$

The same up to four-loops and the leading term in strong coupling!
Conclusions
Conclusions and outlook

∀ observable in the purely gluonic $SU(2,1|2)$ sector ($AdS_5 \times S^1$) take the $\mathcal{N} = 4$ answer and replace $g^2 \to g^2 = f(g^2) = \frac{R^4}{(2\pi \alpha')^2}$

We need more data! (EP-Mitev), (Leoni-Mauri-Santambrogio) and (Fraser)

$\Gamma_{cusp}(\varphi, g^2) = \Omega(\varphi, K(g^2))$ (Grozin-Henn-Korchemsky-Marquard)

Use the “Exact correlation functions” (Baggio-Niarchos-Papadodimas).

Similar story for:

- asymptotically conformal $\mathcal{N} = 2$ theories (massive quarks) and

- $\mathcal{N} = 1$ SCFTs in 4D (EP-Roček)

- theories in 3D: compare ABJ with ABJM (localization powerful in 3D)
Conclusions and Lessons

- **Lesson**: Think of $\mathcal{N} = 4$ SYM as a regulator! (A.Hamed-Murayama)
  The **integrable** $\mathcal{N} = 4$ model knows all about the **combinatorics**.
  For $\mathcal{N} = 2$: **relative finite renormalization** encoded in $g^2 = f(g^2)$.

- Even explicit calculation the **Feynman diagrams** is not so hard:
  **Only** calculate the **difference**:

  $$g^2 = f(g^2) = g^2 + g^2 (Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4})$$

  **Only** very particular **finite** integrals:

  $$(2n - 1) \frac{\zeta(2n - 1)}{p^2}$$  
  (Broadhurst)
Thank you!