

Exact results on $\mathcal{N} = 2$ supersymmetric gauge theories

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The following is meant to give an overview over our special volume. The first three sections 1-3 are intended to give a general overview over the physical motivations behind this direction of research, and some of the developments that initiated this project. These sections are written for a broad audience of readers with interest in quantum field theory, assuming only very basic knowledge of supersymmetric gauge theories and string theory. This will be followed in Section 4 by a brief overview over the different chapters collected in this volume, while Section 5 indicates some related developments that we were unfortunately not able to cover here.

Due to the large number of relevant papers the author felt forced to adopt a very restrictive citation policy. With the exception of very few original papers only review papers will be cited in Sections 1 and 2. More references are given in later sections, but it still seems impossible to list all papers on the subjects mentioned there. The author apologises for any omission that results from this policy. A citation of the form [V: x] refers to article number x in this volume.

1. Background, history and context

1.1 Strong coupling behavior of gauge theories

Gauge theories play a fundamental role in theoretical particle physics. They describe in particular the interactions that bind the quarks into hadrons. It is well understood how these interactions behave at high energies. This becomes possible due to the phenomenon of asymptotic freedom: The effective strength of the interactions depends on the energy scale, and goes to zero for large energies. It is much less well understood how the interactions between quarks behave at low energies: The experimental evidence indicates that the interactions become strong enough to prevent complete separation of the quarks bound in a hadron (confinement). The theoretical understanding of this phenomenon has remained elusive.

When the interactions are weak one may approximate the resulting effects reasonably well using perturbation theory, as can be developed systematically using the existing Lagrangian formula-

tions. However, the calculation of higher order effects in perturbation theory gets cumbersome very quickly. It is furthermore well-known that additional effects exist that can not be seen using perturbation theory. Exponentially suppressed contributions to the effective interactions are caused, for example, by the existence of nontrivial solutions to the Euclidean equations of motion called instantons. The task to understand the strong coupling behavior of gauge theories looks rather hopeless from this point of view: It would require having a complete resummation of all perturbative and non perturbative effects. Understanding the strong coupling behaviour of general gauge theories remains an important challenge for quantum field theory. However, there exist examples in which substantial progress has recently been made on this problem: Certain important physical quantities like expectation values of Wilson loop observables can even be calculated exactly. What makes these examples more tractable is the existence of supersymmetry. It describes relations between bosons and fermions which may imply that most quantum corrections from bosonic degrees of freedom cancel against similar contributions coming from the fermions. Whatever remains may be exactly calculable.

Even if supersymmetry has been crucial for getting exact results up to now, it seems likely that some of the lessons that can be learned by analysing supersymmetric field theories will hold in much larger generality. One may in particular hope to deepen our insights into the origin of quantum field theoretical duality phenomena by analysing supersymmetric field theories, as will be discussed in more detail below. As another example let us mention that it was expected for a long time that instantons play a key role for the behaviour of gauge theories at strong coupling. This can now be illustrated beautifully with the help of the new exact results to be discussed in this volume. We believe that the study of supersymmetric field theories offers a promising path to enter into the mostly unexplored world of non-perturbative phenomena in quantum field theory.

1.2 Electric-magnetic duality conjectures

It is a hope going back to the early studies of gauge theories that there may exist asymptotic strong coupling regions in the gauge theory parameter space in which a conventional (perturbative) description is recovered using a suitable new set of field variables. This phenomenon is called a duality. Whenever this occurs, one may get access to highly nontrivial information about the gauge theory at strong coupling.

For future reference let us formulate a bit more precisely what it means to have a duality. Let us consider a family $\{\mathcal{F}_z; z \in \mathcal{M}\}$ of quantum theories having a moduli space \mathcal{M} of parameters z . The quantum theory \mathcal{F}_z is for each fixed value of z abstractly characterised by an algebra of observables \mathcal{A}_z and a linear functional on \mathcal{A}_z which assigns to each observable $O \in \mathcal{A}_z$ its vacuum expectation value $\langle O \rangle_z$. We say that $\{\mathcal{F}_z; z \in \mathcal{M}\}$ is a quantum field theory with fields

Φ and action $S_\tau[\Phi]$ depending on certain parameters τ (like masses and coupling constants) if there exists a point z_0 in the boundary of the moduli space \mathcal{M} , a coordinate $\tau = \tau(z)$ in the vicinity of z_0 , and a map \mathcal{O} assigning to each $O \in \mathcal{A}_z$ a functional $\mathcal{O}_{O,\tau}[\Phi]$ such that

$$\langle O \rangle_z \simeq \int [\mathcal{D}\Phi] e^{-S_\tau[\Phi]} \mathcal{O}_{O,\tau}[\Phi], \quad (1.1)$$

where \simeq means equality of asymptotic expansions around z_0 and the right hand side is defined in terms of the action $S[\Phi]$ using path integral methods.

We say that a theory with fields Φ , action $S_\tau[\Phi]$ and parameters τ is dual to a theory characterised by similar data $S'_{\tau'}[\Phi']$ if there exists a family of quantum theories $\{\mathcal{F}_z; z \in \mathcal{M}\}$ with moduli space \mathcal{M} having boundary points z_0 and z'_0 such that the vacuum expectation values of \mathcal{F}_z have an asymptotic expansion of the form (1.1) near z_0 , and also an asymptotic expansion

$$\langle O \rangle_z \simeq \int [\mathcal{D}\Phi'] e^{-S'_{\tau'}[\Phi']} \mathcal{O}'_{O,\tau'}[\Phi'], \quad (1.2)$$

near z'_0 , with \mathcal{O}' being a map assigning to each $O \in \mathcal{A}_z$ a functional $\mathcal{O}'_{O,\tau'}[\Phi']$.

A class of long-standing conjectures concerning the strong coupling behavior of gauge theories are referred to as the electric-magnetic duality conjectures. Some of these conjectures concern the infrared (IR) physics as described in terms of low-energy effective actions, others are about the full ultraviolet (UV) descriptions of certain gauge theories. The main content of the first class of such conjectures is most easily described for theories having an effective description at low energies involving in particular an abelian gauge field A and some charged matter q . The effective action $S(A, q; \tau_{\text{IR}})$ will depend on an effective IR coupling constant τ_{IR} . The phenomenon of an electric magnetic duality would imply in particular that the strong coupling behavior of such a gauge theory can be represented using a dual action $S'(A', q'; \tau'_{\text{IR}})$ that depends on the dual abelian gauge field A' related to A simply as

$$F'_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \quad (1.3)$$

The relation between the dual coupling constant τ'_{IR} and τ_{IR} is also conjectured to be very simple,

$$\tau'_{\text{IR}} = -\frac{1}{\tau_{\text{IR}}}. \quad (1.4)$$

The relation expressing q' in terms of q and A may be very complicated, in general. In many cases one expects that q' is the field associated to solitons, localized particle-like excitations associated to classical solutions of the equations of motion of $S(A, q; \tau_{\text{IR}})$. Such solitons are usually very heavy at weak coupling but may become light at strong coupling where they may be identified with fundamental particle excitations of the theory with action $S'(A', q'; \tau'_{\text{IR}})$.

For certain theories there exist even deeper conjectures predicting dualities between different perturbative descriptions of the full *ultraviolet* quantum field theories. Such conjectures, often

referred to as S-duality conjectures originated from the observations of Montonen and Olive [MoOl, GNO], and were subsequently refined in [WO, Os], leading to the conjecture of a duality between the $N = 4$ supersymmetric Yang-Mills theory with gauge group G and coupling τ on the one hand, and the $N = 4$ supersymmetric Yang-Mills theory with gauge group ${}^L G$ and coupling $-1/n_G \tau$ on the other hand. ${}^L G$ is the Langlands dual of a group G having as Cartan matrix the transpose of the Cartan matrix of G , and n_G is the lacing number¹ of the Lie algebra of G .

A given UV action S can be used to define such expectation values perturbatively, as well as certain non-perturbative corrections like the instantons. The question is whether all perturbative and non perturbative corrections can be resummed to get the cross-over to the perturbation theory defined using a different UV action S' .

A non-trivial strong-coupling check for the S-duality conjecture in the $N = 4$ supersymmetric Yang-Mills theory was performed in [VW].² Generalised S-duality conjectures have been formulated in [Ga09] (see [V:1] for a review) for a large class of $\mathcal{N} = 2$ supersymmetric gauge theories which are ultraviolet finite and therefore have well-defined bare UV coupling constants τ_r . It is of course a challenge to establish the validity of such conjectures in any nontrivial example.

1.3 Seiberg-Witten theory

A breakthrough was initiated by the discovery of exact results for the low energy effective action of certain $\mathcal{N} = 2$ supersymmetric gauge theories by Seiberg and Witten [SW1, SW2]. There are several good reviews on the subject, see e.g. [Bi, Le, Pe97, DPh, Tac] containing further references.³

The constraints of $\mathcal{N} = 2$ supersymmetry restrict the low-energy physics considerably. As a typical example let us consider a gauge theory with $SU(M)$ gauge symmetry. The gauge field sits in a multiplet of $\mathcal{N} = 2$ supersymmetry containing a scalar field ϕ in the adjoint representation of $SU(M)$. $\mathcal{N} = 2$ supersymmetry allows parametric families of vacuum states. The vacuum states in the Coulomb branch can be parameterised by the vacuum expectation values of gauge-invariant functions of the scalars like $u^{(k)} := \langle \text{Tr}(\phi^k) \rangle$, $k = 2, \dots, M$. For generic values of these quantities one may describe the low-energy physics in terms of a Wilsonian effective action $S^{\text{eff}}[A]$ which is a functional of $d = M - 1$ vector multiplets A_k , $k = 1, \dots, d$, having scalar components a_k and gauge group $U(1)_k$, respectively. The effective action $S^{\text{eff}}[A]$

¹The lacing number n_G is equal to 1 if the Lie algebra of G is simply-laced, 2 if it is of type B_n , C_n and F_4 , and 3 if it is of type G_2 .

²The result of [Sen] furnishes a nontrivial check of a prediction following from the Montonen-Olive conjecture.

³A fairly extensive list of references to the early literature can be found e.g. in [Le].

turns out to be completely determined by a single holomorphic function $\mathcal{F}(a)$ of d variables $a = (a_1, \dots, a_d)$ called the prepotential. It completely determines the (Wilsonian) low energy effective action as $S^{\text{eff}} = S_{\text{bos}}^{\text{eff}} + S_{\text{fer}}^{\text{eff}}$, where

$$S_{\text{bos}}^{\text{eff}} = \frac{1}{4\pi} \int d^4x \left(\text{Im}(\tau^{kl}) \partial_\mu \bar{a}_k \partial^\mu a_l + \frac{1}{2} \text{Im}(\tau^{kl}) F_{k,\mu\nu} F_l^{\mu\nu} + \frac{1}{2} \text{Re}(\tau^{kl}) F_{k,\mu\nu} \tilde{F}_l^{\mu\nu} \right), \quad (1.5)$$

while $S_{\text{fer}}^{\text{eff}}$ is the sum of all terms containing fermionic fields, uniquely determined by $\mathcal{N} = 2$ supersymmetry. The a -dependent matrix $\tau^{kl}(a)$ in (1.5) is the matrix of second derivatives of the prepotential,

$$\tau^{kl}(a) := \partial_{a_k} \partial_{a_l} \mathcal{F}(a). \quad (1.6)$$

Based on physically motivated assumptions about the strong coupling behavior of the gauge theories under consideration, Seiberg and Witten proposed a precise mathematical definition of the relevant functions $\mathcal{F}(a)$ for $M = 2$. This type of description was subsequently generalised to large classes of $\mathcal{N} = 2$ supersymmetric gauge theories including the cases with $M > 2$.

The mathematics underlying the definition of $\mathcal{F}(a)$ is called special geometry. In many cases including the examples discussed above one may describe $\mathcal{F}(a)$ using an auxilliary Riemann surface Σ called the Seiberg-Witten curve which in suitable local coordinates can be described by a polynomial equation $P(x, y) = 0$. The polynomial $P(x, y)$ has coefficients determined by the mass parameters, the gauge coupling constants, and the values $u^{(k)}$ parameterising the vacua. Associated to Σ is the canonical one form $\lambda_{\text{SW}} = y dx$ on Σ . Picking a canonical basis for the first homology $H_1(\Sigma, \mathbb{Z})$ of Σ , represented by curves $\alpha_1, \dots, \alpha_d$ and β_1, \dots, β_d with intersection index $\alpha_r \circ \beta_s = \delta_{rs}$ one may consider the periods

$$a_r = \int_{\alpha_r} \lambda_{\text{SW}}, \quad a_r^D = \int_{\beta_r} \lambda_{\text{SW}}. \quad (1.7)$$

Both $a_r \equiv a_r(u)$ and $a_r^D \equiv a_r^D(u)$, $r = 1, \dots, d$, represent sets of complex coordinates for the d -dimensional space of vacua, in our example parameterised by $u = (u^{(2)}, \dots, u^{(M)})$. It must therefore be possible to express a^D in terms of a . It turns out that the relation can be expressed using a function $\mathcal{F}(a)$, $a = (a_1, \dots, a_d)$, from which the coordinates a_r can be obtained via $a_r^D = \partial_{a_r} \mathcal{F}(a)$. It follows that $\mathcal{F}(a)$ is up to an additive constant defined by Σ and the choice of a basis for $H_1(\Sigma, \mathbb{Z})$.

The choice of the field coordinates a_k is not unique. Changing the basis $\alpha_1, \dots, \alpha_d$ and β_1, \dots, β_d to $\alpha'_1, \dots, \alpha'_d$ and $\beta'_1, \dots, \beta'_d$ will produce new coordinates a'_r , a'^D_r , $k = 1, \dots, d$ along with a new function $\mathcal{F}'(a')$ which is the prepotential determining a dual action $S'_{\text{eff}}[a']$. The actions $S_{\text{eff}}[a]$ and $S'_{\text{eff}}[a']$ give us equivalent descriptions of the low-energy physics. This gives an example for an IR duality.

1.4 Localization calculations of SUSY observables

Having unbroken SUSY opens the possibility to compute some important quantities exactly using a method called localization [W88]. This method forms the basis for much of the recent progress in this field.

Given a supersymmetry generator Q such that $Q^2 = P$, where P is the generator of a bosonic symmetry. Let $S = S[\Phi]$ be an action such that $QS = 0$. Let us furthermore introduce an auxiliary fermionic functional $V = V[\Phi]$ that satisfies $PV = 0$. We may then consider the path integral defined by deforming the action by the term tQV , with t being a real parameter. In many cases one can argue that expectation values of supersymmetric observables $\mathcal{O} \equiv \mathcal{O}[\Phi]$, $Q\mathcal{O} = 0$, defined by the deformed action are in fact independent of t , as the following formal calculation indicates. Let us consider

$$\begin{aligned} \frac{d}{dt} \int [\mathcal{D}\Phi] e^{-S-tQV} \mathcal{O} &= \int [\mathcal{D}\Phi] e^{-S-tQV} QV \mathcal{O} \\ &= \int [\mathcal{D}\Phi] Q(e^{-S-tQV} V \mathcal{O}) = 0, \end{aligned} \quad (1.8)$$

if the path-integral measure is SUSY-invariant, $\int [\mathcal{D}\Phi] Q(\dots) = 0$. This means that

$$\langle \mathcal{O} \rangle := \int [\mathcal{D}\Phi] e^{-S} \mathcal{O} = \lim_{t \rightarrow \infty} \int [\mathcal{D}\Phi] e^{-S-tQV} \mathcal{O}. \quad (1.9)$$

If V is such that QV has positive semi-definite bosonic part, the only non-vanishing contributions are field configurations satisfying $QV = 0$. There are cases where the space \mathcal{M} of solutions of $QV = 0$ is finite-dimensional.⁴ The arguments above then imply that the expectation values can be expressed as an ordinary integral over the space \mathcal{M} which may be calculable.

The reader should note that this argument bypasses the actual definition of the path integral in an interesting way. For the theories at hand, the definition of $\int [\mathcal{D}\Phi] e^{-S-tQV}$ represents a rather challenging task which is not yet done. What the argument underlying the localisation method shows is the following: If there is ultimately *any* definition of the theory that ensures unbroken supersymmetry in the sense that $\int [\mathcal{D}\Phi] Q(\dots) = 0$, the argument (1.8) will be applicable, and may allow us to calculate *certain* expectation values exactly even if the precise definition of the full theory is unknown.

1.5 Instanton calculus

The work of Seiberg and Witten was based on certain assumptions on the strong coupling behavior of the relevant gauge theories. It was therefore a major progress when it was shown

⁴In other cases \mathcal{M} may be a union of infinitely many finite-dimensional components of increasing dimensions, as happens in the cases discussed in Section 1.5.

in [N, NO03, NY, BE] that the mathematical description for the prepotential conjectured by Seiberg and Witten can be obtained by an honest calculation of the quantum corrections to a certain two-parameter deformation of the prepotential to all orders in the instanton expansion.

To this aim it turned out to be very useful to define a regularisation of certain IR divergences called Omega-deformation by adding terms to the action breaking Lorentz symmetry in such a way that a part of the supersymmetry is preserved [N]⁵,

$$S \rightarrow S_{\epsilon_1 \epsilon_2} = S + R_{\epsilon_1 \epsilon_2} . \quad (1.10)$$

One may then consider the partition function \mathcal{Z} defined by means of the path integral defined by the action $S_{\epsilon_1 \epsilon_2}$. As an example let us again consider a theory with $SU(M)$ gauge group. This partition function $\mathcal{Z} = \mathcal{Z}(a, m, \tau; \epsilon_1, \epsilon_2)$ depends on the eigenvalues $a = (a_1, \dots, a_{M-1})$ of the vector multiplet scalars at the infinity of \mathbb{R}^4 , the collection m of all mass parameters of the theory, and the complexified gauge coupling τ formed out of the gauge coupling constant g and theta-angle θ as

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} . \quad (1.11)$$

The unbroken supersymmetry can be used to apply the localisation method briefly described in Section 1.4, here leading to the conclusion that the path integral defining \mathcal{Z} can be reduced to a sum of ordinary integrals over instanton moduli spaces. The culmination of a long series of works⁶ were explicit formulae for the summands $\mathcal{Z}^{(k)}(a, m; \epsilon_1, \epsilon_2)$ that appear in the resulting infinite series⁷ of instanton corrections

$$\mathcal{Z}(a, m, \tau; \epsilon_1, \epsilon_2) = \mathcal{Z}^{\text{pert}}(a, m, \tau; \epsilon_1, \epsilon_2) \left(1 + \sum_{k=1}^{\infty} q^k \mathcal{Z}^{(k)}(a, m; \epsilon_1, \epsilon_2) \right) , \quad (1.12)$$

with $q = e^{2\pi i \tau}$ in the ultraviolet finite cases, while it is related to the running effective scale Λ otherwise. The explicitly known prefactor $\mathcal{Z}^{\text{pert}}(a, m, \tau; \epsilon_1, \epsilon_2)$ is the product of the simple tree-level contribution with a one-loop determinant. The latter is independent of the coupling constants q_r , and can be expressed in terms of known special functions.

In order to complete the derivation of the prepotentials proposed by Seiberg and Witten it then remained to argue that $\mathcal{F}(a) \equiv \mathcal{F}(a, m, \tau)$ is related to the partition function \mathcal{Z} as

$$\mathcal{F}(a, m; \tau) = - \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \epsilon_2 \mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2) , \quad (1.13)$$

⁵The regularisation introduced in [N] provides a physical interpretation of a regularisation for integrals over instanton moduli spaces previously used in [LNS, MNS1].

⁶The results presented in [N, NO03] were based in particular on the previous work [LNS, MNS1, MNS2]. Similar results were presented in [FPS, Ho1, Ho2, FP, BFMT]; for a review see [V:3].

⁷The infinite series (1.12) are probably convergent. This was verified explicitly for the example of pure $SU(2)$ Super-Yang-Mills theory in [ILTy], and it is expected to follow for UV finite gauge theories from the relations with conformal field theory to be discussed in the next section.

and to derive the mathematical definition of $\mathcal{F}(a)$ proposed by Seiberg and Witten from the exact results on $\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2)$ obtained in [N, NO03, NY, BE].

2. New exact results on $\mathcal{N} = 2$ supersymmetric field theories

2.1 Localisation on curved backgrounds

Another useful way to regularise IR-divergences is to consider the quantum field theory on four-dimensional Euclidean space-times M^4 of finite volume. The finite-size effects encoded in the dependence of physical quantities with respect to the volume or other parameters of M^4 contain profound physical information. It has recently become possible to calculate some of these quantities exactly. One may, for example, consider gauge theories on a four-sphere S^4 [Pe07], or more generally four-dimensional ellipsoids [HH],

$$S_{\epsilon_1, \epsilon_2}^4 := \{ (x_0, \dots, x_4) \mid x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1 \}. \quad (2.14)$$

The spaces $S_{\epsilon_1, \epsilon_2}^4$ have sufficient symmetry for having an unbroken supersymmetry Q such that Q^2 is the sum of a space-time symmetry plus possibly an internal symmetry. Expectation values of supersymmetric observables on $S_{\epsilon_1, \epsilon_2}^4$ therefore represent candidates for quantities that may be calculated by the localisation method. Interesting physical quantities are the partition function on S^4 , and the values of Wilson- and 't Hooft loop observables. Wilson loop observables can be defined as path-ordered exponentials of the general form $W_{r,i} := \text{Tr } \mathcal{P} \exp \left[\oint_{\mathcal{C}} ds (i\dot{x}^\mu A_\mu^r + |\dot{x}| \phi^r) \right]$. The 't Hooft loop observables $T_{r,i}$, $i = 1, 2$, can be defined semiclassically by performing a path integral over field configurations with a specific singular behavior near a curve \mathcal{C} describing the effect of parallel transport of a magnetically charged probe particle along \mathcal{C} . Choosing the support of the loop observables to be the circles \mathcal{C}_1 or \mathcal{C}_2 defined by $x_0 = x_3 = x_4 = 0$ or $x_0 = x_1 = x_2 = 0$, respectively, one gets operators commuting with part of the supersymmetries of the theory.

However, applying the localisation method to the field theories with $\mathcal{N} = 2$ supersymmetry is technically challenging [Pe07, HH, GOP]. A review of the necessary technology and of some subsequent developments in this direction can be found in the articles [V:5, V:6]. Appropriately modifying the action defining the theory under consideration on \mathbb{R}^4 gives a Q -invariant action S for the theory on $S_{\epsilon_1, \epsilon_2}^4$. A functional V is found in [Pe07] such that QV is positive definite. The field configurations solving $QV = 0$ have constant values of the scalar fields, and vanishing values of all other fields. This means that the path integral reduces to an ordinary integral over scalar zero modes. This phenomenon may be seen as a variant of the cancellations between contributions from fermionic and bosonic degrees of freedom that frequently occur in supersymmetric field theories, leaving behind only contributions from states of zero energy.

The results obtained by localisation [Pe07, GOP, HH] have the following structure:

- Partition functions:

$$\mathbf{Z}(m; \tau; \epsilon_1, \epsilon_2) := \langle 1 \rangle_{S^4}^S = \int da |\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2)|^2, \quad (2.15)$$

where $\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2)$ are the instanton partition functions briefly discussed in Section 1.5. More details can be found in [V:5].

- Wilson or 't Hooft loop expectation values:

$$\langle \mathcal{L} \rangle_{S^4}^S = \int da (\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2))^* \mathcal{D}_{\mathcal{L}} \cdot \mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2), \quad (2.16)$$

where $\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2)$ are the instanton partition functions described in Section 1.5, and $\mathcal{D}_{\mathcal{L}}$ is a difference operator acting on the scalar zero mode variables collectively referred to by the notation a . The difference operators \mathcal{D} are pure multiplication operators $\mathcal{D}_{\mathcal{L}} = 2 \cosh(2\pi a/\epsilon_i)$ if \mathcal{L} is a Wilson loop supported on \mathcal{C}_i . These results are reviewed in [V:6].

The integral over a in (2.15), (2.16) is the integration over the scalar zero modes. One may interpret these results as reduction to an effective quantum mechanics of these zero modes. From this point of view one would interpret the instanton partition function $\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2)$ as the wave-function $\Psi_{\tau}(a)$ of a state $|\tau\rangle_0$ in the zero-mode sub-sector defined by the path integral over field configurations on the lower half-ellipsoid $S_{\epsilon_1, \epsilon_2}^{4,-} := \{ (x_0, \dots, x_4) \in S_{\epsilon_1, \epsilon_2}^4; x_0 < 0 \}$. The expectation value (2.16) can then be represented as

$$\langle \mathcal{L} \rangle_{S^4}^S = \langle \tau | \mathcal{L}_0 | \tau \rangle_0, \quad (2.17)$$

where \mathcal{L}_0 denotes the projection of the operator representing \mathcal{L} to the zero mode sub-sector. This point of view is further discussed in [V:11].

Although the dynamics of the zero mode sub-sector is protected by supersymmetry, it captures very important non-perturbative information about the rest of the theory. Dualities between different UV-descriptions of the gauge theory must be reflected in the zero mode dynamics, and can therefore be tested with the help of localisation calculations. But the definition of the full theory must be compatible with these results, which is ultimately a consequence of unbroken supersymmetry. One may view the zero-mode dynamics as a kind of skeleton of the SUSY field theory. Whatever the QFT-“flesh” may be, it must fit to the skeleton, and exhibit same dualities, for example.

The localisation method has furthermore recently been used to obtain exact results on some correlation functions in $\mathcal{N} = 2$ supersymmetric QCD [BNP].

2.2 Relation to conformal field theory

In [AGT] it was observed that the results for partition functions of some four-dimensional supersymmetric gauge theories that can be calculated by the method of [Pe07] are in fact proportional to something known, namely the correlation functions of the two-dimensional quantum field theory known as Liouville theory. Such correlation functions are formally defined by the path integral using the action

$$S_b^{\text{Liou}} = \frac{1}{4\pi} \int d^2z \left[(\partial_a \phi)^2 + 4\pi\mu e^{2b\phi} \right]. \quad (2.18)$$

Liouville theory has been extensively studied in the past motivated by the relations to two-dimensional quantum gravity and noncritical string theory discovered by Polyakov. It is known to be conformally invariant, as suggested by the early investigation [CT], and established by the construction given in [Te01]. Conformal symmetry implies that the correlation functions can be represented in a holomorphically factorized form. As a typical example let us consider

$$\left\langle e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \right\rangle_b^{\text{Liou}} = \int_{\mathbb{R}^+} \frac{dp}{2\pi} C_{21}(p) C_{43}(-p) \left| \mathcal{F}_p \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (q) \right|^2, \quad (2.19)$$

where the conformal blocks $\mathcal{F}_p \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (q)$ can be represented by power series of the form

$$\mathcal{F}_p \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (q) = q^{\frac{Q^2}{4} + p^2 - \alpha_1(Q - \alpha_1) - \alpha_2(Q - \alpha_2)} \left(1 + \sum_{k=1}^{\infty} q^k \mathcal{F}_p^{(k)} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] \right), \quad (2.20)$$

having coefficients $\mathcal{F}_p^{(k)} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right]$ completely defined by conformal symmetry [BPZ].⁸ Explicit formulae for the coefficient functions $C_{ij}(p) \equiv C(\alpha_i, \alpha_j, \frac{Q}{2} + ip)$, $Q = b + b^{-1}$, have been conjectured in [DOt, ZZ], and nontrivial checks for this conjecture were presented in [ZZ]. A derivation of all these results follows from the free-field construction of Liouville theory given in [Te01].

In order to describe an example for the relations discovered in [AGT] let us temporarily restrict attention to the $\mathcal{N} = 2$ supersymmetric gauge theory often referred to as $N_f = 4$ -theory. This theory has field content consisting of an $SU(2)$ -vector multiplet coupled to four massive hypermultiplets in the fundamental representation of the gauge group. The relation discovered in [AGT] can be written as

$$\mathcal{Z}(a, m; \tau; \epsilon_1, \epsilon_2) \propto N_{21}(p) N_{43}(p) \mathcal{F}_p \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (q), \quad (2.21)$$

where $|N_{ij}(p)|^2 = C_{ij}(p)$. The factors of proportionality dropped in (2.21) are explicitly known, and turn out to be inessential. The parameters are identified, respectively, as

$$b^2 = \frac{\epsilon_1}{\epsilon_2}, \quad \hbar^2 = \epsilon_1 \epsilon_2, \quad q = e^{2\pi i \tau}, \quad (2.22a)$$

$$p = \frac{a}{\hbar}, \quad \alpha_r = \frac{Q}{2} + i \frac{m_r}{\hbar}, \quad Q := b + b^{-1}. \quad (2.22b)$$

⁸A concise description of the definition of the conformal blocks can be found in [V:11, Section 2.5].

In order to prove (2.21) one needs to show that the coefficients $\mathcal{Z}^{(k)}(a, m; \epsilon_1, \epsilon_2)$ in (1.12) are equal to $\mathcal{F}_p^{(k)} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right]$. This was done in [AGT] up order q^{11} . A proof of this equality for all values of k is now available [AFLT].

It furthermore follows easily from (2.21) that the partition function $\mathbf{Z}(m; \tau; \epsilon_1, \epsilon_2)$ defined in (2.15) can be represented up to multiplication with an inessential, explicitly known function as

$$\mathbf{Z}(m; \tau; \epsilon_1, \epsilon_2) \propto \left\langle e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \right\rangle_b^{\text{Liou}}. \quad (2.23)$$

The relations between certain $\mathcal{N} = 2$ supersymmetric gauge theories and Liouville theory are most clearly formulated in terms of the normalized expectation values of loop-observables

$$\langle\langle \mathcal{L} \rangle\rangle_{S^4}^S := \frac{\langle \mathcal{L} \rangle_{S^4}^S}{\langle 1 \rangle_{S^4}^S}. \quad (2.24)$$

To this aim let us note that the counterparts of the loop observables within Liouville theory will be certain nonlocal observables of the form

$$\mathbf{L}_\gamma := \text{tr} \left[\mathcal{P} \exp \left(\int_\gamma \mathcal{A}_y \right) \right], \quad (2.25)$$

where γ is a simple closed curve on $\mathbb{C} \setminus \{0, q, 1\}$, and \mathcal{A} is the flat connection

$$\mathcal{A} := \begin{pmatrix} -\frac{b}{2}\partial_z\phi & 0 \\ \mu e^{b\phi} & \frac{b}{2}\partial_z\phi \end{pmatrix} dz + \begin{pmatrix} \frac{b}{2}\partial_{\bar{z}}\phi & \mu e^{b\phi} \\ 0 & -\frac{b}{2}\partial_{\bar{z}}\phi \end{pmatrix} d\bar{z}. \quad (2.26)$$

Flatness of \mathcal{A} follows from the equation of motion. Let us furthermore define normalized expectation values in Liouville theory schematically as

$$\langle\langle \mathcal{O} \rangle\rangle_b^{\text{Liou}} := \frac{\langle \mathcal{O} e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \rangle_b^{\text{Liou}}}{\langle e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \rangle_b^{\text{Liou}}}. \quad (2.27)$$

We then have

$$\langle\langle \mathcal{W} \rangle\rangle_{S^4}^S = \langle\langle \mathbf{L}_{\gamma_s} \rangle\rangle_b^{\text{Liou}}, \quad \langle\langle \mathcal{T} \rangle\rangle_{S^4}^S = \langle\langle \mathbf{L}_{\gamma_t} \rangle\rangle_b^{\text{Liou}}, \quad (2.28)$$

where γ_s and γ_t are the simple closed curves encircling the pairs of points $0, q$ and $1, q$ on $\mathbb{C} \setminus \{0, q, 1\}$, respectively. A more detailed discussion can be found in [V:6] and [V:11].

2.3 Relation to topological quantum field theory

The localisation method is also applicable in the case when the manifold M^4 has the form $M^3 \times S^1$ with supersymmetric boundary conditions for the fermions on the S^1 . In this case the partition function coincides with a quantity called index [Ro, KMMR], a trace $\text{tr}(-1)^F \prod_i \mu_i^{C_i} e^{-\beta\{\mathcal{Q}, \mathcal{Q}^\dagger\}}$ over the Hilbert space of the theory on $M^3 \times \mathbb{R}$, with F being the

fermion number operator, \mathcal{Q} being one of the supersymmetry generators, and C_i being operators commuting with \mathcal{Q} . The index depends on parameters μ_i called fugacities. It has originally been used to perform nontrivial checks of existing duality conjectures on field theories with $\mathcal{N} = 1$ supersymmetry [DOs, SpV]. We will in the following restrict attention to cases where the field theories have $\mathcal{N} = 2$ supersymmetry which are more closely related to the rest of the material discussed in this special volume.

As before one may use the localisation method to express the path integral for such manifolds as an integral over the zero modes of certain fields, with integrands obtainable by simple one loop computations. This partition function can alternatively be computed by counting with signs and weights certain protected operators in a given theory. If, for example, one takes $M^3 = S^3$, the partition function of an $\mathcal{N} = 2$ gauge theory with gauge group G and N_f fundamental hypermultiplets takes the following form,

$$I(\mathbf{b}; p, q, t) = \oint [da]_G I_V(\mathbf{a}; p, q, t) \prod_{\ell=1}^{N_f} I_H(\mathbf{a}, \mathbf{b}; p, q, t), \quad (2.29)$$

where $[da]_G$ is the invariant Haar measure, we are using the notation $\{p, q, t, \mathbf{b}\}$ for the relevant fugacities, and I_V and I_H are contributions coming from free vector multiplets and hypermultiplets, respectively. The integral over \mathbf{a} is roughly over the zero mode of the component of the gauge field in the S^1 direction. For more details see the article [V:8]. It is important to note that the supersymmetric partition functions on $M^3 \times S^1$ are independent of the coupling constants by an argument going back to [W88]. Nevertheless, they are in general intricate functions of the fugacities and encode a lot of information about the protected spectrum of the theory.

There exists a relationship between the supersymmetric partition function on $M^3 \times S^1$, the supersymmetric index, on the one hand, and a topological field theory in two dimensions on the other hand [GPRR] which is somewhat analogous to the relation of the S^4 partition function to Liouville theory discussed above. Let us consider the example discussed above, $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with $N_f = 4$. The supersymmetric index of this theory can be represented in the form (2.29) noted above. In the particular case when the fugacities are chosen to satisfy $t = q$, this index is equal [GRRY11] to a four point correlation function in a topological quantum field theory (TQFT) which can be regarded as a one-parameter deformation of two-dimensional Yang-Mills theory with gauge group $SU(2)$ [AOSV],

$$I(b_1, b_2, b_3, b_4; p, q, t = q) = \prod_{\ell=1}^4 \mathcal{K}(b_\ell; q) \sum_{\mathcal{R}=0}^{\infty} C_{\mathcal{R}}^2 \prod_{\ell=1}^4 \chi_{\mathcal{R}}(b_\ell). \quad (2.30)$$

Here $\chi_{\mathcal{R}}(x)$ is the character of a representation \mathcal{R} of $SU(2)$. The parameters b_i are fugacities for the $\prod_{i=1}^4 SU(2)_i$ maximal subgroup of the $SO(8)$ flavor symmetry group of the theory. This relation can be generalized to a large class of $\mathcal{N} = 2$ theories and to indices depending on more general sets of fugacities [GRRY13, GRR].

3. What are the exact results good for ?

In the following we will briefly describe a few applications of the results outlined above that have deepened our insights into supersymmetric field theories considerably.

3.1 Quantitative verification of electric-magnetic duality conjectures

The verification of the conjectures of Seiberg and Witten by the works [N, NO03, NY, BE] leads in particular to a verification of the electric-magnetic duality conjectures about the low energy effective theories that were underlying the approach taken by Seiberg and Witten.⁹

Verification of UV duality relations like the Montonen-Olive duality seems hopeless in general (see, however, [VW]). However, in the cases where exact results on expectation values are available, as briefly described in Subsection 2.2, one can do better.

In the case of the $N_f = 4$ theory, for example, one expects to find weakly coupled Lagrangian descriptions when the UV gauge coupling q is near 0, 1 or infinity [SW2]. Let us denote the actions representing the expansions around these three values as S_s , S_t and S_u , respectively. A particularly important prediction of the S-duality conjectures is the exchange of the roles of Wilson- and 't Hooft loops,

$$\langle\langle \mathcal{W} \rangle\rangle_{S_s} = \langle\langle \mathcal{T} \rangle\rangle_{S_t}, \quad \langle\langle \mathcal{T} \rangle\rangle_{S_s} = \langle\langle \mathcal{W} \rangle\rangle_{S_t}. \quad (3.31)$$

In order to check (3.31) we may combine the results (2.16) of the localisation computations with the relation (2.21) discovered in [AGT]. From the study of the Liouville theory one knows that the conformal blocks satisfy relations such as

$$\mathcal{F}_p \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (q) = \int dp' F_{p,p'} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] \mathcal{F}_{p'} \left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 \end{smallmatrix} \right] (1-q), \quad (3.32)$$

which had been established in [Te01]. These relations may now be re-interpreted as describing a resummation of the instanton expansion defined by action S_s (the left hand side of (3.32)) into an instanton expansion defined by the dual action S_t . This resummation gets represented as an integral transformation with kernel $F_{p,p'}$. Using (2.16), (2.21), (3.32) and certain identities satisfied by the kernel $F_{p,p'} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right]$ established in [TV13], one may now verify explicitly that the S-duality relations (3.31) are indeed satisfied.

⁹The IR duality conjectures can be used to describe the moduli space of vacua as manifold covered by charts with local coordinates a_r, a_r^D . The transition functions between different charts define a Riemann-Hilbert problem. The solution to this problem defines the function $\mathcal{F}(a)$. It was shown in [N, NO03, NY, BE] that the series expansion of $\mathcal{F}(a)$ around one of the singular points on the moduli space of vacua satisfies (1.13). Taken together, one obtains a highly nontrivial check of the IR-duality conjectures underlying Seiberg-Witten theory.

In other words: Conformal field theory provides the techniques necessary to resum the instanton expansion defined from a given action in terms of the instanton expansions defined from a dual action. At least for the class of theories at hand, these results confirm in particular the long-standing expectations that the instantons play a crucial role for producing the cross-over between weakly-coupled descriptions related by electric-magnetic dualities.

The electric-magnetic dualities can be also checked using the supersymmetric index. Although the index does not depend on the coupling constants, in different duality frames it is given by different matrix integrals. Duality implies that these two matrix integrals evaluate to the same expression. In the relation of the index to TQFT discussed above, invariance under duality transformations in many cases follows from the associativity property of the TQFT.

3.2 Precision tests of AdS-CFT duality

Another famous set of duality conjectures concerns the behaviour of supersymmetric gauge theories in the limit where the rank of the gauge group(s) tends to infinity [Ma], see [AGMOO] for a review. In this limit one expects to find a dual description in terms of the perturbative expansion of string theory on a background that is equal or asymptotic to five-dimensional Anti-de Sitter space. This duality predicts in some cases representations for the leading strong-coupling behaviour of some gauge-theoretical observables in terms of geometric quantities in supergravity theories.

Some impressive quantitative checks of these duality conjectures are known in the case of maximal supersymmetry $\mathcal{N} = 4$ based on the (conjectured) integrability of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with infinite rank of the gauge group [Bei]. Performing similar checks for theories with less supersymmetry is much harder. It is therefore worth noting that the localisation calculations of partition functions and Wilson loop expectation values described above have been used to carry out quantitative checks of AdS-CFT type duality conjectures for some gauge theories with $\mathcal{N} = 2$ supersymmetry [BRZ, BEFP].

It seems quite possible that the exact results described above can be used to carry out many further precision tests of the AdS-CFT duality for $\mathcal{N} = 2$ supersymmetric field theories. The relevant backgrounds for string theory are not always known, but when they are known, one may use the results obtained by localisation to check these generalised AdS-CFT duality conjectures. Another result in this direction was recently reported in [MP14b].

The exact results can furthermore be used to study the phase structure of these gauge theories in the planar limit as function of the 't Hooft coupling. A surprisingly rich structure is found in [RZ13a, RZ13b]. It seems that the full physical content of most of the data provided by the localisation calculations remains to be properly understood.

3.3 Evidence for the existence of six-dimensional theories with (2,0)-supersymmetry

Low-energy limits of string theory can often be identified with conventional quantum field theories. The string theorist's toolkit contains a large choice of objects to play with, the most popular being compactifications and branes. One sometimes expects the existence of a low-energy limit with a certain amount of supersymmetry, but there is no known quantum field theory the limit could correspond to. Such a line of reasoning has led to the prediction that there exists a very interesting class of interacting quantum field theories with six-dimensional (2, 0)-superconformal invariance [W95a, St, W95b]. These theories have attracted a lot of attention over the last two decades, but not even the field content, not to speak of a Lagrangian, are known for these hypothetical theories, see [Sei, W09] for reviews of what is known.

Nevertheless, the mere existence of such theories leads to highly non-trivial predictions, many of which have been verified directly. One could, for example, study the six-dimensional (2, 0)-theories on manifolds of the form $M^4 \times C$, where C is a Riemann surface [Ga09, GMN2] (some aspects are reviewed in [V:1, V:2]). If C has small area, one expects that the theory has an effective description in terms of a quantum field theory on M^4 . The resulting quantum field theory \mathcal{G}_C is expected to depend only on the choice of a hyperbolic metric on C [ABBR], or equivalently (via the uniformisation theorem) on the choice of a complex structure on C . The $N_f = 4$ -theory with four flavours mentioned above, for example, corresponds to $C = C_{0,4}$, which may be represented as Riemann sphere with four marked points at $0, 1, q, \infty$. One may use q as parameter for the complex structure of $C_{0,4}$. When q is near $0, 1, \infty$, respectively, it is natural to decompose $C_{0,4}$ into two pairs of pants by cutting along contours surrounding the pairs of marked points $(0, q)$, $(q, 1)$ and (q, ∞) , respectively. It turns out that q can be identified with the function $e^{2\pi i \tau}$ of the complexified gauge coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ of the four-dimensional theory. The limits where q approaches $0, 1$ and ∞ are geometrically very similar, but $q \rightarrow 0$ corresponds to small gauge coupling, while $q \rightarrow 1$ would correspond to a strong coupling limit. Note, on the other hand that the marked points at 0 and 1 , for example, can be exchanged by a conformal mapping. This already suggests that there might exist a dual description having a complexified gauge coupling τ' such that $q' = e^{2\pi i \tau'}$ vanishes when $q \rightarrow 1$. The results described above provide a rather non-trivial quantitative check for this prediction.

Playing with the choice of C , and with the choice of the Lie algebra \mathfrak{g} one can generate a large class of interesting four-dimensional quantum field theories, and predict many non-trivial results about their physics [Ga09, GMN2]. The class of theories obtained in this way is often called class \mathcal{S} . Arguments of this type can be refined sufficiently to predict correspondences between four-dimensional gauge theories on M^4 and two-dimensional conformal field theories on C generalising the relations discovered in [AGT], see [Y12, CJ14]. In the resulting generalisations of the relation (2.23) one will find the correlation functions of the conformal Toda

theory associated to \mathfrak{g} on the Riemann surface C , in general. Considering the cases where $M^4 = M^3 \times S^1$, one may use similar arguments to predict that the partition functions are related to correlation functions in a TQFT on C , generalising the example noted in Section 2.3. Such correlation function only depend on the topology of C , corresponding to the fact that the partition functions on $M^4 = M^3 \times S^1$ are independent of exactly marginal coupling constants. This will be discussed in more detail in [V:8].

Other compactifications are also interesting, like $M^3 \times C^3$ or $M^2 \times C^4$, where C^3 and C^4 are compact three- and four-dimensional manifolds. Compactifying on C^3 or C^4 one gets interesting quantum field theories on three- or two-dimensional manifolds M^3 or M^2 , respectively. The origin from the six-dimensional $(2, 0)$ -theory may again be used to predict various nontrivial properties of the resulting quantum field theories, including relations between three-dimensional field theories on M^3 and complex Chern-Simons theory on C^3 [Y13, LY, CJ13]. Such relations are further discussed in [V:10].

The six-dimensional $(2, 0)$ -theories are for $d = 2, 3, 4$ -dimensional quantum field theory therefore something like “Eierlegende Wollmilchsäue”, mythical beasts capable of supplying us with eggs, wool, milk and meat at the same time. The steadily growing number of highly nontrivial checks that the predictions following from its existence have passed increase our confidence that such six-dimensional theories actually exist. Their existence supplies us with a vantage point from which we may get a better view on interesting parts of the landscape of supersymmetric quantum field theories.

3.4 Towards understanding non-Lagrangian theories

There are many cases where strong-coupling limits of supersymmetric field theories are expected to exist and to have a quantum field-theoretical nature, but no Lagrangian description of the resulting theories is known [AD, APSW, AS]. The existence of non-Lagrangian theories is an interesting phenomenon by itself. Certain non-Lagrangian quantum field theories are expected to serve as elementary building blocks for the family of quantum field theories obtained by compactifying $(2, 0)$ -theories [Ga09, CD].

The origin from a six-dimensional theory allows us to make quantitative predictions on some physical quantities of such non-Lagrangian theories including the prepotential giving us the low-energy effective action, and the supersymmetric index giving us the protected spectrum of the theory.

The results described in this special volume open the exciting perspective to go much further in the study of some non-Lagrangian theories. If the relation with two-dimensional conformal field theories continues to hold in the cases without known Lagrangian descriptions, one may,

for example compute the partition functions and certain finite-volume expectation values of loop operators in such theories. First steps in this direction were made in [BMT, GT, KMST].

3.5 Interplay between (topological) string theory and gauge theory

Superstring theory compactified on Calabi-Yau manifolds has two “topological” relatives called the A- and the B-model respectively. The “topological” relatives are much simpler than the full superstring theories, but they capture important information about the full theories like the coefficients of certain terms in the corresponding space-time effective actions governing the low-energy physics. The A- and the B-model are not independent but related by mirror symmetry.

The definition of the B-model topological string theory can be extended to so-called local Calabi-Yau Y , defined (locally) by equations of the form

$$zw - P(u, v) = 0, \quad (3.33)$$

with $P(u, v)$ being a polynomial. Superstring theories on such local Calabi-Yau manifolds are expected to have decoupling limits in which they are effectively represented by four-dimensional gauge theories¹⁰. Describing four-dimensional gauge theory as decoupling limits of superstring theory is called geometric engineering [KLMVW, KKV, KMV]. String-theoretic arguments [N, LMN] predict that the instanton partition function (for $\epsilon_1 + \epsilon_2 = 0$) coincides with the topological string partition function \mathcal{Z}^{top} of the B-model on Y , schematically

$$\mathcal{Z}^{\text{inst}} = \lim_{\beta \rightarrow 0} \mathcal{Z}^{\text{top}}, \quad (3.34)$$

where β is related to one of the parameters for the complex structures on Y . This prediction has been verified in various examples [IK02, IK03, EK, HIV]. It opens channels for the transport of information and insights from topological string theory to gauge theory and back. Interesting perspectives include:

- Results from topological string theory may help to understand 4d gauge theories even better, possibly including non-Lagrangian ones. To give an example, let us note that the topological vertex [AKMV, IKV] gives powerful tools for the calculation of topological string partition functions. These results give us predictions for the (yet undefined) instanton partition functions of non-Lagrangian theories, and may thereby provide a starting point for future studies of the physics of such theories. First steps in this direction were made in [KPW, BMPTY, HKN, MP14b].

¹⁰This limit is easier to define in the A-model, but the definition can be translated to the B-model using mirror symmetry.

- Exact results on supersymmetric gauge theories may feed back to topological string theory. As an example let us mention the development of the refined topological string, a one-parameter deformation of the usual topological string theory that appears to exist for certain local Calabi-Yau manifolds, capturing nontrivial additional information. The proposal was initially motivated by the observation that the instanton partition functions can be defined for more general values of the parameters ϵ_1, ϵ_2 than the case $\epsilon_1 + \epsilon_2 = 0$ corresponding to the usual topological string via geometric engineering [IKV, KW, HK, HKK]. There is growing evidence that such a deformation of the topological string has a world sheet realisation [AFHNZa, AFHNZb], and that the refinement fits well into the conjectured web of topological string/gauge theory dualities [AS12a, AS12b, CKK, NO14]. The relation with the holomorphic anomaly equation is reviewed in [V:13].
- As another interesting direction that deserves further investigations let us note that the topological string partition functions \mathcal{Z}^{top} can be interpreted as particular wave-functions in the quantum theory obtained by the quantisation of the moduli space of complex structures on Calabi-Yau manifolds, as first pointed out in [W93], see [ST] and references therein for further developments along these lines. By using the holomorphic anomaly equation one may construct \mathcal{Z}^{top} as formal series in the topological string coupling constant λ , identified with Planck's constant \hbar in the quantisation of the moduli spaces of complex structures. However, it is not known how to define \mathcal{Z}^{top} non-perturbatively in λ .

On the other hand it was pointed out above that the instanton partition functions are naturally interpreted as wave-functions in some effective zero mode quantum mechanics to which the gauge theories in question can be reduced by the localisation method. It seems likely that the effective zero mode quantum mechanics to which the gauge theories localise simply coincide with the quantum mechanics obtained from the quantisation of the moduli spaces of complex structures which appear in the geometric engineering of the gauge theories under considerations. These moduli spaces are closely related to the moduli spaces of flat connections on Riemann surfaces for the A_1 theories of class \mathcal{S} . The quantisation of these moduli spaces is understood for some range of values of ϵ_1, ϵ_2 [V:11]. Interpreting the results obtained thereby in terms of (refined) topological string theory may give us important insights on how to construct \mathcal{Z}^{top} non-perturbatively, at least for many local Calabi-Yau-manifolds.

4. What is going to be discussed in this volume ?

Let us now give an overview of the material covered in this volume.

The first chapter “*Families of $N = 2$ field theories*” [V:1] by D. Gaiotto describes how large

families of field theories with $\mathcal{N} = 2$ supersymmetry can be described by means of Lagrangian formulations, or by compactification from the six-dimensional theory with $(2, 0)$ supersymmetry on spaces of the form $M^4 \times C$, with C being a Riemann surface. The class of theories that can be obtained in this way is called class \mathcal{S} . This description allows us to relate key aspects of the four-dimensional physics of class \mathcal{S} theories to geometric structures on C .

The next chapter in our volume is titled “*Hitchin systems in $\mathcal{N} = 2$ field theory*” by A. Neitzke [V:2]. The space of vacua of class \mathcal{S} theories on $\mathbb{R}^3 \times S^1$ can be identified as the moduli space of solutions to the self-duality equations in two dimensions on Riemann surfaces studied by Hitchin. This fact plays a fundamental role for recent studies of the spectrum of BPS states in class \mathcal{S} theories, and it is related to the integrable structure underlying Seiberg-Witten theory of theories of class \mathcal{S} . This article reviews important aspects of the role of the Hitchin system for the infrared physics of class \mathcal{S} theories.

In the following chapter “*A review on instanton counting and W -algebras*” by Y. Tachikawa [V:3], it is explained how to compute the instanton partition functions. The results can be written as sums over bases for the equivariant cohomology of instanton moduli spaces. The known results relating the symmetries of these spaces to the symmetries of conformal field theory are reviewed.

The Chapter 4 “ *β -deformed matrix models and the 2d/4d correspondence*” by K. Maruyoshi [V:4] describes a very useful mathematical representation of the results of the localisation computations as integrals having a form familiar from the study of matrix models. Techniques from the study of matrix models can be employed to extract important information on the instanton partition functions in various limits and special cases.

The Chapter 5 “*Localization for $\mathcal{N} = 2$ Supersymmetric Gauge Theories in Four Dimensions*” by V. Pestun [V:5] describes the techniques necessary to apply the localisation method to field theories on curved backgrounds like S^4 , and how some of the results on partition functions outlined in Section 2.1 have been obtained.

In our Chapter 6 “*Line operators in supersymmetric gauge theories and the 2d-4d relation*” by T. Okuda [V:6] it is discussed how to use localisation techniques for the calculation of expectation values of Wilson and ’t Hooft line operators. The results establish direct connections between supersymmetric line operators in gauge theories and the Verlinde line operators known from conformal field theory. Similar results can be used to strongly support connections to the quantum theories obtained from the quantisation of the Hitchin moduli spaces.

Chapter 7 “*Surface Operators*” by S. Gukov [V:7] discusses a very interesting class of observables localised on surfaces that attracts steadily growing attention. In the correspondence to conformal field theory some of these observables get related to a class of fields in two dimensions called degenerate fields. These fields satisfy differential equations that can be used

to extract a lot of information on the correlation functions. Understanding the origin of these differential equations within gauge theory may help explaining the AGT-correspondence itself.

There are further interesting quantities probing aspect of the non-perturbative physics of theories of class \mathcal{S} . The Chapter 8 “*The superconformal index of theories of class \mathcal{S}* ” by L. Rastelli, S. Razamat [V:8] reviews the superconformal index. It is often simpler to calculate than instanton partition functions, but nevertheless allows one to perform many nontrivial checks of conjectured dualities. It turns out to admit a representation in terms of a new type of topological field theory associated to the Riemann surfaces C parameterising the class \mathcal{S} theories.

The correspondence between four-dimensional supersymmetric gauge theories and two-dimensional conformal field theories discovered in [AGT] has a very interesting relative, a correspondence between three-dimensional gauge theories and three-dimensional Chern-Simons theories with complex gauge group. It is related to the the correspondence of [AGT], but of interest in its own right. In order to see relations with the AGT-correspondences one may consider four-dimensional field theories of class \mathcal{S} on half-spaces separated by three-dimensional defects. The partition functions of the three-dimensional gauge theories on the defect turns out to be calculable by means of localisation, and the results have a deep meaning within conformal field theory or within the quantum theory of Hitchin moduli spaces. How to apply the localisation method to (some of) the three-dimensional gauge theories that appear in this correspondence is explained in the Chapter 9 “*A review on SUSY gauge theories on S^3* ” by K. Hosomichi [V:9]. The correspondences between three-dimensional gauge theories and three-dimensional Chern-Simons theory with complex gauge group are the subject of Chapter 10 “*3d Superconformal Theories from Three-Manifolds*” by T. Dimofte [V:10].

Chapter 11 “*Supersymmetric gauge theories, quantization of $\mathcal{M}_{\text{flat}}$, and Liouville theory*” by the author [V:11] describes an approach to understanding the AGT-correspondence by establishing a triangle of relations between the zero mode quantum mechanics obtained by localisation of class \mathcal{S} theories, the quantum theory obtained by quantisation of Hitchin moduli spaces, and conformal field theory. This triangle offers an explanation for the relations discovered in [AGT].

Some aspects of the string-theoretical origin of these results are discussed in the final Chapters of our volume.

Chapter 12 by M. Aganagic and S. Shakirov, “*Topological strings and 2d/4d correspondence*” [V:12], describes one way to understand an important part of the AGT-correspondence in terms of a triality between four-dimensional gauge theory, the two-dimensional theory of its vortices, and conformal field theory. This triality is related to, and inspired by known large N dualities of the topological string. It leads to a proof of some cases of the AGT-correspondence, and most importantly, of a generalisation of this correspondence to certain five-dimensional gauge theories.

In the final Chapter 13, “*B-Model Approaches to Instanton Counting*”, D. Krefl, J. Walcher [V:13] discuss the relation between the instanton partition functions and the partition function of the topological string from the perspective of the B-model. The instanton partition functions provide solutions to the holomorphic anomaly equations characterising the partition functions of the topological string.

5. What is missing ?

This collection of articles can only review a small part of the exciting recent progress on $N = 2$ supersymmetric field theories. Many important developments in this field could not be covered here even if they are related to the material discussed in our collection of articles in various ways. In the following we want to indicate some of the developments that appear to have particularly close connections to the subjects discussed in this volume.

5.1 BPS spectrum, moduli spaces of vacua and Hitchin systems

BPS states are states in the Hilbert space of a supersymmetric field theory which are forming distinguished “small” representations of the supersymmetry algebra, a feature which excludes various ways of “mixing” with generic states of the spectrum that would exist otherwise. Gaiotto, Moore and Neitzke have initiated a vast program aimed at the study of the spectrum of BPS-states in the $\mathcal{N} = 2$ supersymmetric gauge theories \mathcal{G}_C of class \mathcal{S} [GMN1, GMN2, GMN3], see the article [V:2] for a review of some aspects. To this aim it has turned out to be useful to consider at intermediate steps of the analysis a compactification of the theories \mathcal{G}_C to space-times of the form $\mathbb{R}^3 \times S^1$. The moduli space of vacua of the compactified theory is “twice as large” compared to the one of \mathcal{G}_C on \mathbb{R}^4 , and it can be identified with Hitchin’s moduli space of solutions to the self-duality equations on Riemann surfaces [Hi].

The list of beautiful results that has been obtained along these lines includes:

- A new algorithm for computing the spectrum of BPS states which has a nontrivial, but piecewise constant dependence on the point on the Coulomb-branch of the moduli space of vacua of \mathcal{G}_C on \mathbb{R}^4 . The spectrum of BPS states may change along certain “walls” in the moduli space of vacua. Knowing the spectrum on one side of the wall one may compute how it looks like on the other side using the so-called wall-crossing formulae. Similar formulae were first proposed in the work of Kontsevich and Soibelman on Donaldson-Thomas invariants.
- Considering the gauge theory \mathcal{G}_C compactified on $\mathbb{R}^3 \times S^1$ one may study natural line operators including supersymmetric versions of the Wilson- or ’t Hooft loop observables. Such

observables can be constructed using either the fields of the UV Lagrangian, or alternatively those of a Wilsonian IR effective action. The vacuum expectation values of such line operators furnish coordinates on the moduli space \mathcal{M} of vacua of \mathcal{G}_C on $\mathbb{R}^3 \times S^1$ which turn out to coincide with natural sets of coordinates for Hitchin's moduli spaces. The coordinates associated to observables defined in the IR reveal the structure of Hitchin's moduli spaces as a cluster algebra, closely related to the phenomenon of wall-crossing in the spectrum of BPS-states. Considering the observables constructed from the fields in the UV-Lagrangian one gets coordinates describing the Hitchin moduli spaces as algebraic varieties. The relation between these sets of coordinates is the renormalisation group (RG) flow, here protected by supersymmetry, and therefore sometimes calculable [GMN3].

Even if the main focus of this direction of research is the spectrum of BPS-states, it turns out to be deeply related to the relations discovered in [AGT], as is briefly discussed in [V:11].

5.2 Relations to integrable models

It has been observed some time ago that the description of the prepotentials characterising the low-energy physics of $\mathcal{N} = 2$ supersymmetric field theories provided by Seiberg-Witten theory is closely connected to the mathematics of integrable systems [GKMMM, MW, DW]. There are arguments indicating that such relations to integrable models are generic consequences of $\mathcal{N} = 2$ supersymmetry: $\mathcal{N} = 2$ supersymmetry implies that the Coulomb branch of vacua carries a geometric structure called special geometry. Under certain integrality conditions related to the quantisation of electric and magnetic charges of BPS states one may canonically construct an integrable system in action-angle variables from the data characterising the special geometry of the Coulomb branch [Fr].

The connections between four-dimensional field theories with $\mathcal{N} = 2$ -supersymmetry and integrable models have been amplified enormously in a recent series of papers starting with [NSc].¹¹ It was observed that a *partial* Omega-deformation of many $\mathcal{N} = 2$ field theories localised only on one of the half-planes spanning \mathbb{R}^4 is related to the quantum integrable model obtained by quantising the classical integrable model related to Seiberg-Witten theory. The Omega-deformation effectively localises the fluctuations to the origin of the half plane. This can be used to argue that the low-energy physics can be effectively represented by a two-dimensional theory with $(2, 2)$ -supersymmetry [NSc] living on the half-plane in \mathbb{R}^4 orthogonal to the support of the Omega-deformation. The supersymmetric vacua of the four-dimensional theory are determined by the twisted superpotential of the effective two-dimensional theory which can be calculated

¹¹This paper is part of a program initiated in [NSa, NSb] investigating even more general connections between field theories with $\mathcal{N} = 2$ -supersymmetry and integrable models.

by taking the relevant limit of the instanton partition functions. This was used in [NSc] to argue that the supersymmetric vacua are in one-to-one correspondence with the eigenstates of the quantum integrable model obtained by quantising the integrable model corresponding to the Seiberg-Witten theory of the four-dimensional gauge theory under consideration.

This line of thought has not only lead to many new exact results on large families of four-dimensional $\mathcal{N} = 2$ gauge theories [NRS, NP, NPS], it has also created a new paradigm for the solution of algebraically integrable models. More specifically

- For gauge theories \mathcal{G}_C of class \mathcal{S} an elegant description for the two-dimensional superpotential characterising the low-energy physics in the presence of a partial Omega-deformation was given in [NRS] in terms of the mathematics of certain flat connections called opers living on the Riemann C specifying the gauge theory \mathcal{G}_C .
- In [NP] the instanton calculus was generalised to a large class of $\mathcal{N} = 2$ gauge theories \mathcal{G}_Γ parameterised by certain diagrams Γ called quivers. A generalisation of the techniques from [NO03] allowed the authors to determine Seiberg-Witten type descriptions of the low-energy physics for all these theories, and to identify the integrable models whose solution theory allows one to calculate the corresponding prepotentials.
- The subsequent work [NPS] generalised the results of [NP] to the cases where one has a one-parametric Omega-deformation preserving two-dimensional supersymmetry. The results of [NPS] imply a general correspondence between certain supersymmetric observables and the generating functions of conserved quantities in the models obtained by quantising the integrable models describing the generalisations of Seiberg-Witten-theory relevant for the gauge theories \mathcal{G}_Γ .

The relations between these developments and the relations discovered in [AGT] deserve further studies. One of the existing relations for A_1 -theories of class \mathcal{S} is briefly discussed in the article [V:11]. These results suggest that the AGT-correspondence and many related developments can ultimately be understood as consequences of the integrable structure in $\mathcal{N} = 2$ supersymmetric field theories. This point of view is also supported by the relations between the quantisation of Hitchin moduli spaces and conformal field theory described in [Te10].

5.3 Other approaches to the AGT-correspondence

In this special volume we collect some of the basic results related to the AGT-correspondence. The family of results on this subject is rapidly growing, and many important developments have occurred during the preparation of this volume. The approaches to proving or deriving the AGT-correspondences and some generalisations include

- Representation-theoretic proofs [AFLT, FL, BBFLT] that W-algebra conformal blocks can be represented in terms of instanton partition functions. This boils down to proving existence of a basis for W-algebra modules in which the matrix elements of chiral vertex operators coincide with the so-called bifundamental contributions representing the main building blocks of instanton partition functions.
- Another approach [MMS, MS] establishes relations between the series expansions for the instanton partition functions and the expressions provided by the free field representation for the conformal blocks developed by Feigin and Fuchs, and Dotsenko and Fateev.
- Mathematical proofs [SchV, MaOk, BFN] that the cohomology of instanton moduli spaces naturally carries a structure as a W-algebra module. This leads to a proof of the versions of the AGT-correspondence relevant for pure $\mathcal{N} = 2$ supersymmetric gauge theories for all gauge groups of type A , D or E . The instanton partition functions get related to norms of Whittaker vectors in modules of W-algebras in these cases. For a physical explanation of this fact see [Tan]. Some aspects of this approach are described in [V:3].
- Physical arguments leading to the conclusion that the six-dimensional $(2, 0)$ -theory on certain compact four-manifolds or on four-manifolds M^4 with Omega-deformation can effectively be represented in terms of two-dimensional conformal field theory [Y12, CJ14], or as a $(2, 2)$ -supersymmetric sigma model with Hitchin target space [NW].
- Considerations of the geometric engineering of supersymmetric gauge theories within string theory have led to the suggestion that the instanton partition functions of the gauge theories from class \mathcal{S} should be related to the partition functions of chiral free fermion theories on suitable Riemann surfaces [N], see [ADKMV, DHSV, DHS] for related developments¹². It was proposed in [CNO] that the relevant theory of chiral free fermions is defined on the Riemann surface C specifying the gauge theories \mathcal{G}_C of class \mathcal{S} . These relations were called BPS-CFT correspondence in [CNO]. A mathematical link between BPS-CFT correspondence and the AGT-correspondence was exhibited in [ILTe].

5.4 Less supersymmetry

A very interesting direction of possible future research concerns possible generalisations of the results discussed here to theories with less ($\mathcal{N} = 1$) supersymmetry. Recent progress in this

¹²The relations between the topological vertex and free fermion theories discussed in [ADKMV] imply general relations between topological string partition functions of local Calabi-Yau manifolds, integrable models and theories of free fermions on certain Riemann surfaces; possible implications for four-dimensional gauge theories were discussed in [DHSV, DHS].

direction includes descriptions of the moduli spaces of vacua resembling the one provided by Seiberg-Witten theory for field theories with $\mathcal{N} = 2$ supersymmetry.

The rapid growth of the number of publications on this direction of research makes it difficult to offer a representative yet concise list of references on this subject here.

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