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Integrable amplitude deformations for $\mathcal{N} = 4$ super Yang-Mills and ABJM theory

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We study Yangian-invariant deformations of scattering amplitudes in 4d $\mathcal{N} = 4$ super Yang-Mills theory and 3d $\mathcal{N} = 6$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. In particular, we obtain the deformed Graßmannian integral for 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, both in momentum and momentum-twistor space. For 3d ABJM theory, we initiate the study of deformed scattering amplitudes. We investigate general deformations of on-shell diagrams, and find the deformed Graßmannian integral for this theory. We furthermore introduce the algebraic R-matrix construction of deformed Yangian invariants for ABJM theory.

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I. INTRODUCTION

Recently it has become clear that the physics of scattering amplitudes contains a plethora of interesting mathematical structures and unexpected symmetries.[1] The prime examples for investigating these phenomena are $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 6$ super Chern-Simons (ABJM) [2] theory in four and three dimensions, respectively. Both of these theories are believed to be equivalent to an AdS/CFT-dual string theory, and both are believed to be completely *integrable* in the planar limit.

In the context of scattering amplitudes, integrability is realized as a Yangian symmetry acting on the external legs of the supersymmetric amplitude [3]. Equivalently, the Yangian can be formulated as the combination of superconformal and dual superconformal symmetry, where—at least in four dimensions—the latter arises from the duality between amplitudes and Wilson loops. The Yangian symmetry is highly restrictive and, when combined with locality, completely fixes the tree-level scattering matrix [4,5].

Lately, it has been noticed that the tree-level S-matrix of $\mathcal{N} = 4$ SYM theory can be identified with the maximally length-changing contributions of the dilatation operator of the same theory [6]. The simplest example of this map is the four-point amplitude, which serves as an integral kernel for the celebrated one-loop dilatation operator alias a super spin-chain Hamiltonian. Like the spin-chain Hamiltonian, the four-point scattering amplitude can be obtained from an R-matrix which depends on a spectral parameter z [7,8]. Remarkably, this implies the existence of a deformation of the tree-level four-point amplitude in the parameter z. Importantly, this deformation is still Yangian invariant. In fact, it is necessary to consider a corresponding deformation of the Yangian generators known as the evaluation representation. This representation is more general than the previously considered representation and thus allows for more general invariant functions of the kinematical scattering data. Interestingly, if the evaluation parameters are all real, the deformed Yangian generators preserve the positive Graßmannian.

It turns out that such deformed invariants also exist for higher multiplicities. The most natural approaches to construct these invariants are closely related to the on-shell methods of Arkani-Hamed *et al.* (see e.g. [9]). In four dimensions, in particular a diagrammatic approach has been studied [7,8,10], as well as an R-matrix construction of Yangian invariants, similar in spirit to the algebraic Bethe ansatz [11–13].

As these invariants are functions of the external data on which the S-matrix is defined, a natural question is how they are related to the scattering amplitudes. Preliminary attempts to relate these invariants to the Britto-Cachazo-Feng-Witten (BCFW) [14] building blocks of scattering amplitudes via a uniform set of deformation parameters, however, appear to break down when going beyond six points and the maximally-helicity-violating (MHV) level [10]. More precisely, attempts to simultaneously deform all contributing BCFW terms in a consistent fashion have failed so far.

To circumvent the above difficulties, instead of deforming the individual BCFW contributions, one might consider embedding the BCFW terms into a parent integral

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[1] For example, see [1] for a recent review.

with some unspecified contour, and which depends on the deformation parameters. As one turns off the deformation, we are allowed to choose the contour such that the integral reduces to the individual BCFW terms. In this way, as the deformation parameters are introduced at the level of the parent integral, there is *a priori* no inconsistency. Luckily such an integral already exists in the form of the Graßmannian integral [15], and the first task is to introduce deformations such that Yangian invariance is preserved.

In the present paper, we study deformed scattering amplitudes in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory, and in three-dimensional $\mathcal{N}=6$ super Chern-Simons theory (ABJM). The purpose of this work is the following:

(i) 4d: We present a deformed and Yangian-invariant Graßmannian integral and discuss its role for further investigations of amplitude deformations. We review and summarize the recent progress on deformed scattering amplitudes in $\mathcal{N}=4$ SYM theory in a compact form and highlight the connections among different approaches.

(ii) 3d: In four dimensions, the deformation parameters are to some extent associated with central charges or deformed helicities of the external particles. The $\mathfrak{osp}(6|4)$ algebra of ABJM theory does not contain a central charge and the considered three-dimensional particles do not carry helicity degrees of freedom. Thus it is interesting to ask whether or not integrable deformations for ABJM theory exist as well. Indeed, we find that the four-point amplitude allows for a one-parameter deformation which is invariant under the evaluation representation of the Yangian algebra $Y[\mathfrak{osp}(6|4)]$. This deformed four-point vertex furnishes the building block for invariants with higher multiplicities, which we construct along the lines of the on-shell diagram methods of [9,16]. We then propose a deformation of the orthogonal Graßmannian integral introduced in [17] and show that it is consistent with the previous investigations. Finally, we also introduce an algebraic R-matrix construction of deformed Yangian invariants for the three-dimensional theory.

This paper is organized as follows. In Sec. II we review the construction of deformed Yangian invariants in $\mathcal{N}=4$ SYM theory: In particular, we present the deformed Graßmannian integral in Sec. II B, whose Yangian invariance follows from the on-shell diagram formalism discussed in Sec. II A or the direct proof in Appendix B. We also obtain the deformed momentum-twistor version of the Graßmannian. The study of deformed scattering amplitudes in three-dimensional ABJM theory is initiated in Sec. III: We demonstrate Yangian invariance of the deformed four-point amplitude in Sec. III A. We then propose a deformed orthogonal Graßmannian integral and show its Yangian symmetry in Sec. III C. We explain how to build deformed Yangian-invariant on-shell diagrams in Sec. III B, and introduce an algebraic R-matrix construction of these invariants in Sec. III D. Finally, we comment on differences and similarities between the four- and three-dimensional case and point out interesting directions for the future in Sec. IV.

II. INTEGRABLE DEFORMATIONS IN $\mathcal{N}=4$ SYM THEORY

Recently, it has been found that on-shell diagrams allow for interesting deformations that maintain the complete Yangian invariance [7,8,10]. Here, we will briefly summarize these ideas in preparation for the ABJM case below, and comment on a number of features of the deformations.

The Yangian level-one generators generically take the form[2]

$$\hat{\mathfrak{J}}^a = f^a{}_{bc}\sum_{\substack{i,j=1\\ i<j}}^{n}\mathfrak{J}_i^b\mathfrak{J}_j^c + \sum_{i=1}^{n}u_i\mathfrak{J}_i^a, \tag{2.1}$$

where $\mathfrak{J}_i^a$ are the local level-zero generators acting on the leg i, and $f^a{}_{bc}$ are the structure constants of the level-zero algebra. The evaluation parameters u_i are set to zero in the undeformed case. For the superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ in twistor variables,[3] the level-zero and level-one generators take the form [3]

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i=1}^{n}\mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{B}}, \qquad \mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{B}} = \mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{B}}} - (\text{trace}), \tag{2.2}$$

$$\hat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}(-1)^{\mathcal{C}}[\mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{C}}\mathfrak{J}_j^{\mathcal{C}}{}_{\mathcal{B}} - (i\leftrightarrow j)] + \sum_i u_i\mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{B}}, \tag{2.3}$$

where $\mathcal{A},\mathcal{B}$ are fundamental $\mathfrak{su}(2,2|4)$ indices. The central charge operators read

$$\mathfrak{C}_i = -\mathcal{Z}_i^{\mathcal{C}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{C}}}. \tag{2.4}$$

A. Deformed on-shell diagrams

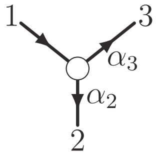
Every on-shell diagram is either a BCFW term of a tree amplitude or loop integrand, or a leading singularity of a loop amplitude [9]. In the following, we review the known integrable deformations of general on-shell diagrams.

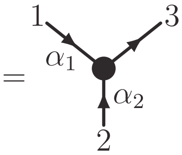
[2]More about Yangian symmetry in the present context can be found in [3,18,19]. For general introductions, see [20].

[3]For a definition of the twistor variables $\mathcal{Z}_i$, see [21]. In (2, 2) signature, $\mathcal{Z}_i^{\mathcal{A}} = (\tilde{\mu}_i^{\dot\alpha}, \tilde{\lambda}_i^{\dot\alpha}|\eta_i^A)$, where $\tilde{\mu}_i$ is the Fourier transform of the momentum spinor λ_i. Here, $p_i^\mu = \sigma^\mu_{\alpha\dot\alpha}\lambda_i^\alpha\tilde{\lambda}_i^{\dot\alpha}$, and the anticommuting spinor η_i^A, with $A=1,\ldots,4$, parametrizes the $\mathcal{N}=4$ on-shell superfield [22].

1. Three-vertices

The basic building blocks for the deformed on-shell diagrams are the three-point vertices





$$\hat{\mathcal{A}}_3^{\circ} = \int \frac{d\alpha_2}{\alpha_2^{1+a_2}} \frac{d\alpha_3}{\alpha_3^{1+a_3}} \delta^{4|4}(C_\circ \cdot \mathcal{Z}), \quad \hat{\mathcal{A}}_3^{\bullet} = \int \frac{d\alpha_1}{\alpha_1^{1+a_1}} \frac{d\alpha_2}{\alpha_2^{1+a_2}} \delta^{8|8}(C_\bullet \cdot \mathcal{Z}), \tag{2.5}$$

where

$$C_\circ = (\,1 \quad \alpha_2 \quad \alpha_3\,), \qquad C_\bullet = \begin{pmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{pmatrix}, \tag{2.6}$$

and $\mathcal{Z}_i^A$ are twistor variables that parametrize the external states. Unlike the undeformed vertices (with $a_i = 0$), these vertices have nonvanishing eigenvalues c_i under the action of the "local" central charges (2.4), where

$$\begin{aligned} \hat{\mathcal{A}}_3^{\circ}: &\quad c_1 = a_2 + a_3 \equiv a_1, \qquad c_2 = -a_2, \qquad c_3 = -a_3, \\ \hat{\mathcal{A}}_3^{\bullet}: &\quad c_1 = a_1, \qquad c_2 = a_2, \qquad c_3 = -a_1 - a_2 \equiv -a_3. \end{aligned} \tag{2.7}$$

They are invariant under the Yangian with evaluation parameters u_i, where [10][4]

$$\begin{aligned} \hat{\mathcal{A}}_3^{\circ}: &\quad c_1 = u_3 - u_2, \qquad c_2 = u_1 - u_3, \qquad c_3 = u_2 - u_1, \\ \hat{\mathcal{A}}_3^{\bullet}: &\quad c_1 = u_2 - u_3, \qquad c_2 = u_3 - u_1, \qquad c_3 = u_1 - u_2. \end{aligned} \tag{2.8}$$

Converting the twistors back to spinor-helicity variables, the three-vertices evaluate to the deformed amplitudes

$$\begin{aligned} \hat{\mathcal{A}}_3^{\circ} &= \frac{\delta^4(P)\delta^4(\tilde{Q})}{[12]^{1+a_3}[23]^{1-a_2-a_3}[31]^{1+a_2}}, \\ \hat{\mathcal{A}}_3^{\bullet} &= \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle^{1-a_1-a_2}\langle 23\rangle^{1+a_1}\langle 31\rangle^{1+a_2}}. \end{aligned} \tag{2.9}$$

Here, $[ij] \equiv \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}}$, $\langle ij\rangle \equiv \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$, $P \equiv \sum_{i=1}^n \lambda_i\tilde{\lambda}_i$, $Q \equiv \sum_{i=1}^n \lambda_i\eta_i$, and $\tilde{Q} \equiv ([12]\eta_3 + [23]\eta_1 + [31]\eta_2)$.

2. Gluing

All bigger on-shell diagrams $\mathcal{Y}$ can be built by iterating two gluing operations: Taking products,

$$(\mathcal{Y}_1, \mathcal{Y}_2) \mapsto \mathcal{Y}_1\mathcal{Y}_2, \tag{2.10}$$

and fusing lines,

$$\begin{aligned} &\mathcal{Y}(\mathcal{Z}_1, \dots, \mathcal{Z}_n, \mathcal{Z}_I, \mathcal{Z}_J) \\ &\quad \mapsto \int d^{3|4}\mathcal{Z}_I \mathcal{Y}(\mathcal{Z}_1, \dots, \mathcal{Z}_n, \mathcal{Z}_I, \mathcal{Z}_J)|_{\mathcal{Z}_J = \mathcal{Z}_I^-}, \end{aligned} \tag{2.11}$$

where $\mathcal{Z}_I^-$ is the twistor of line I with inverse momentum. Yangian invariance is preserved under both of these operations. In addition to the twistor, each external line of a deformed diagram carries two labels: A central charge c_i and an evaluation parameter u_i. When fusing lines of deformed diagrams, Yangian invariance requires that [10]

$$c_I = -c_J, \qquad u_I = u_J. \tag{2.12}$$

Successively applying these two operations generates all Yangian-invariant deformed on-shell diagrams. Combining (2.12) with the invariance conditions (2.8) for the three-vertices, one finds that the external central charges c_i and the evaluation parameters u_i of any invariant diagram must obey [10]

$$u_i^+ = u_{\sigma(i)}^-, \tag{2.13}$$

where

$$u_i^\pm \equiv u_i \pm c_i, \tag{2.14}$$

and σ is the permutation associated to the diagram. The permutation is obtained as follows: Starting at the external line i, follow a path through the diagram, turning left/right at each white/black vertex. The external line at the end of the path will be $\sigma(i)$.[5]

3. General deformed diagrams

Every deformed on-shell diagram can be written as

$$\hat{\mathcal{Y}}(1, \dots, n) = \int \prod_{j=1}^{n_F - 1} \frac{d\alpha_j}{\alpha_j^{1+a_j}} \delta^{4k|4k}(C \cdot \mathcal{Z}), \tag{2.15}$$

[4]An overall shift of all evaluation parameters u_i is generated by the level-zero symmetry and is thus trivial; hence, all relevant parameters are captured by the differences of consecutive evaluation parameters.

[5]While every on-shell diagram has a unique permutation associated to it, the converse is only true for "reduced" diagrams. See [9] for more details.

where n_{F} is the number of faces of the diagram, α_j are a minimal number of edge variables,[6] and C is the matrix constructed from the edge variables by "boundary measurement" as explained in [9]. The gluing conditions (2.12) together with the identifications (2.7) for the three-vertices imply that the deformation parameter a_j in the exponent of an edge variable α_j equals the central charge on the respective line up to a sign:

$$c_i = \pm a_i. \tag{2.16}$$

Here the sign is plus/minus if the arrow on the line points in the same/opposite direction as the permutation path.

The constraints (2.13) impose n conditions on the $2n$ central charges and evaluation parameters; hence, it is clear that *every* diagram admits $n-1$ independent nontrivial deformation parameters (an additional parameter is the trivial uniform shift of all u_i).

4. R-matrix construction

Alternatively, (deformed) on-shell amplitudes can be constructed by acting with a chain of R-matrices on a suitable "vacuum state," where the action of the R-matrices exactly corresponds to the insertion of a (deformed) BCFW bridge [11,12]. Each R-matrix/BCFW bridge contributes an adjacent transposition σ_i to the (decorated) permutation σ associated with the final diagram. Following the procedure of [9], we can associate a canonical decomposition into transpositions σ_i,

$$\sigma = \sigma_\ell \ldots \sigma_1. \tag{2.17}$$

The deformed amplitude (2.15) can then be written as

$$\hat{\mathcal{Y}}(\vec{u}) = R_{\sigma_\ell}(a_\ell)\ldots R_{\sigma_1}(a_1)\delta^{4k|4k}(C_{\mathrm{vac}}\cdot\mathcal{Z}). \tag{2.18}$$

Here C_{vac} is a suitable "vacuum matrix," which is a $k\times n$ matrix with k unit columns and $(n-k)$ zero columns. For example, for $n=6$, $k=3$, a valid choice is

$$C_{\mathrm{vac}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.19}$$

The operator $R_{\sigma=(ij)}$ is defined by [11]

$$R_{\sigma=(ij)}(a)f(\mathcal{Z}) = \int \frac{d\alpha}{\alpha^{1+a}} f(\mathcal{Z})\Big|_{\mathcal{Z}_i\to\mathcal{Z}_i+\alpha\mathcal{Z}_j}. \tag{2.20}$$

This R-operator can be identified with (an integral kernel for) an R-matrix; the shift of $\mathcal{Z}$ in the definition is nothing but the BCFW shift. We can prove (2.18) by induction with respect to the number of BCFW bridges.

This algebraic formulation is another way of demonstrating the Yangian invariance of the amplitude (at the level of on-shell diagrams), and the integrable structures behind it. In Sec. III D we provide a similar discussion for the ABJM theory.

B. Deformed Graßmannian integral

In this section we present the deformed Graßmannian integral[7] for $\mathcal{N}=4$ SYM theory as a special case of the above on-shell diagrams. Being embedded into the on-shell-diagram formalism already implies the Yangian symmetry of the deformed integral. We additionally demonstrate the Yangian invariance of the Graßmannian integral explicitly in Appendix B.

1. The Graßmannian integral

A special class of diagrams are the "top cells" [15]. These are diagrams of maximal dimension (maximal number of integration variables). Their name stems from the fact that all lower-dimensional on-shell diagrams are realized as (iterated) boundaries of top-cell diagrams. They can be classified by the number n of external lines and the helicity

$$k = 2n_{\mathrm{b}} + n_{\mathrm{w}} - n_{\mathrm{i}}, \tag{2.21}$$

where $n_{\mathrm{b/w}}$ is the number of black/white vertices, and n_{i} is the number of internal lines. For each n and k, there is a unique top-cell diagram. It is the reduced diagram with the maximal number of faces, $n_{\mathrm{F}} = k(n-k)+1$. Every boundary measurement on the top-cell diagram equals a gauge-fixed version of the Graßmannian integral of [15],

$$\mathcal{G}_{n,k}(\mathcal{Z}_1,\ldots,\mathcal{Z}_n) = \int \frac{d^{k\cdot n}C}{|\mathrm{GL}(k)|}\frac{1}{M_1^{1+b_1}\ldots M_n^{1+b_n}}\delta^{4k|4k}(C\cdot\mathcal{Z}), \tag{2.22}$$

where $M_i = |i,\ldots,i+k-1|$ is the ith minor of C. The integrand is invariant under $C\mapsto \mathrm{GL}(k)\cdot C$, and $|\mathrm{GL}(k)|$ is the volume of the gauge group. The permutation associated to the top cell simply is a k-fold cyclic shift, $\sigma\colon\{1,\ldots,n\}\mapsto\{n-k+1,\ldots,n,1,\ldots,n-k\}$. Noting that $c_i = -(b_{i-k+1}+\cdots+b_i)$, the invariance conditions (2.13) imply

$$b_i = \frac{1}{2}(u_i^- - u_{i-1}^-) = \frac{1}{2}(u_{i-k}^+ - u_{i-k-1}^+), \tag{2.23}$$

[6]The edge variables are essentially BCFW shift parameters. Fixing a GL(1)-gauge redundancy at each vertex, their number can always be reduced to $n_{\mathrm{F}}-1$.

[7]Note that the deformed Graßmannian formula as well as its momentum-twistor version discussed below, were independently obtained in [34].

and hence $\sum_i b_i = 0$, which ensures $\mathrm{GL}(k)$ invariance. The Yangian invariance of (2.22), with the parameters set to (2.23), can also be shown by directly acting with the Yangian generators, see Appendix B.

2. Singularities and residues

In the undeformed case, lower-dimensional on-shell diagrams are obtained from the top cell (2.22) by localizing some of the integrations on residues. The n-point, helicity k top-cell diagram is defined in terms of $k(n-k)$ integrations, of which $2n-4$ can be performed trivially, using the bosonic delta functions. Iteratively localizing all remaining integrations on a suitable combination of residues gives the tree-level amplitude $\mathcal{A}_{n,k}$. In terms of edge variables, taking a residue amounts to setting one edge variable to zero, and the residue is given by the on-shell diagram with the corresponding edge removed. Hence tree-level amplitudes are given by summing a certain set of on-shell diagrams.

In the presence of generic deformations, the integrations can no longer be performed on residues. From the perspective of the gauge-fixed integral (2.15), one possibility to proceed is to just evaluate the integral on the same contour as in the undeformed case. This requires us to set the deformation parameters a_j to zero on the respective edge variables, as otherwise the contour would not be closed, due to branch cuts. In this case, the result of the integration is a sum over the same set of on-shell diagrams as in the undeformed case, where now each diagram is deformed. However, setting some of the parameters a_j to zero reduces the space of deformation moduli. In fact, it was noted in [10] that for generic tree amplitudes, the constraints imposed by this procedure rule out all deformations. In other words, the integral can only be localized by residues on a standard tree contour when *all* moduli a_j are set to zero. Setting the exponent of an edge variable α to unity and localizing the integration by residue on $\alpha = 0$ amounts to “undoing” a deformed BCFW bridge (R-matrix insertion). The deformation parameters of the diagram obtained in this way from the top cell will by construction satisfy (2.13) for σ being the k-fold cyclic shift of the top cell; but they will also satisfy (2.13) for all intermediate permutations σ' that lead from σ to the permutation of the final diagram. Therefore, localizing the Graßmannian integral generates only a subspace of all admitted deformations for all lower-dimensional diagrams.

Another perspective on the incompatibility of BCFW and deformations is provided by the Graßmannian integral in its original, un-gauge-fixed form (2.22). In the undeformed case, the tree-level amplitudes are given by the residues as the integral localizes on the zeros of the minors. Thus an amplitude is identified with the locus of zeros for a collection of minors. To further admit this localization, the exponents of these minors should be undeformed. As the number of BCFW terms increases, eventually this collection of minors covers the whole set, and thus no deformation is allowed. Indeed from [15], we see that in the seven-point NMHV case, the collection of minors involved in the localization covers six of them, and since the sum of b_i's must vanish, there are no admissible deformations left.

An alternative and perhaps more promising treatment for the deformed Graßmannian integral (2.22) would be to leave the deformation parameters generic and to evaluate the integral by other means on an appropriate contour. We will comment on this idea in Sec. IV below.

3. Note on positivity

There exists a remarkable relation between on-shell diagrams and the positroid stratification of the Graßmannian [9]. The positroid stratification is the classification of all distinct linear dependencies of consecutive columns in the C-matrix, and it turns out that there is a one-to-one correspondence between inequivalent on-shell diagrams and inequivalent cells in the stratification. An interesting property that can be associated with these cells is that there exist parametrizations such that all non-vanishing minors are positive. It has been noted that the Yangian generators generate diffeomorphisms that act on the Graßmannian in such a way that positivity is preserved [9]. One may ask if the deformed Yangian generators can still be understood as positivity-preserving diffeomorphism. This is indeed the case: The level-one generators are deformed by terms $u_i \mathcal{Z}_i \partial/\partial \mathcal{Z}_i$, which, when acting on the delta functions $\delta^{4k|4k}(C\cdot\mathcal{Z})$, translate into $u_i \sum_a C_{ai}\partial/\partial C_{ai}$, which is nothing but a little group scaling that simply rescales the ith column of the matrix C. Thus, as long as all evaluation parameters u_i are real, positivity of the cell is preserved.

C. Relation to deformed momentum-twistor invariants

Everything that has been stated above for on-shell diagrams in twistor variables $\mathcal{Z}$ is equally true for on-shell diagrams in momentum-twistor variables $\mathcal{W}$: Replacing $\mathcal{Z}_i$ with $\mathcal{W}_i$ in the expressions (2.15),(2.5),(2.11),(2.22), the resulting momentum-twistor diagrams are invariant under the momentum-twistor Yangian[8] with generators [23]

$$\hat{\mathfrak{J}}^a = f^a{}_{bc} \sum_{\substack{i,j=1\\ i<j}}^{n} \mathfrak{J}_i^b \mathfrak{J}_j^c + \sum_{i=1}^{n} v_i \mathfrak{J}_i^a, \tag{2.25}$$

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i=1}^{n} \mathfrak{J}_i{}^{\mathcal{A}}{}_{\mathcal{B}}, \qquad \mathfrak{J}_i{}^{\mathcal{A}}{}_{\mathcal{B}} = \mathcal{W}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{B}}} - (\mathrm{trace}),$$
$$\mathfrak{C}_i = \mathcal{W}_i^{\mathcal{C}} \frac{\partial}{\partial \mathcal{W}_i^{\mathcal{C}}}. \tag{2.26}$$

[8] Here, $\mathcal{W}_i \equiv (\lambda_i^\alpha, \mu_i^{\dot\alpha} | \chi_i^I)$, with

$$\mu_i^{\dot\alpha} \equiv \epsilon_{\alpha\beta} y_i^{\dot\alpha\alpha} \lambda_i^\beta, \qquad \chi_i \equiv \epsilon_{\alpha\beta}\theta_i^\alpha \lambda_i^\beta, \tag{2.24}$$

where the dual coordinates (y_i, θ_i) are defined through $y_i - y_{i+1} = p_i$ and $\theta_i - \theta_{i+1} = \lambda_i \eta_i$.

On the other hand, in the undeformed case it is known that for any invariant $\mathcal{Y}(\mathcal{W})$ of the momentum-twistor Yangian, the expression

$$\frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\mathcal{Y}(\mathcal{W}), \tag{2.27}$$

when transformed to twistor variables $\mathcal{Z}$, is an invariant of the twistor-variable Yangian, and vice versa [23,24].[9] It turns out that a similar statement holds in the deformed case. This can be seen as follows: An explicit procedure for reducing any on-shell twistor diagram to a corresponding momentum-twistor diagram is given in Sec. 8.3 of [9]. The deformation only affects the integration measure, and hence the reduction procedure applies in exactly the same way to the deformed diagrams. Under the reduction, the minors of the matrix C transform to minors of the reduced matrix $\tilde{C}$ as

$$\det C|_{(i,\ldots,i+k-1)} = \langle i,i+1\rangle\langle i+1,i+2\rangle\ldots \times\langle i+k-2,i+k-1\rangle \det\tilde{C}|_{(i+1,\ldots,i+k-2)}. \tag{2.28}$$

Hence the only modification in the deformed case is a deformation of the MHV tree prefactor in (2.27). In particular, for the deformed top cell (2.22), one finds

$$\hat{\mathcal{G}}_{n,k}(\mathcal{Z})\overset{\mathcal{Z}\to\mathcal{W}}{\to}\frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle^{1+(u_1^- - u_{n-k+2}^-)/2}\ldots\langle n1\rangle^{1+(u_n^- - u_{n-k+1}^-)/2}} \times\hat{\mathcal{G}}_{n,k-2}(\mathcal{W}), \tag{2.29}$$

$$\hat{\mathcal{G}}_{n,k}(\mathcal{W}) \equiv \int\frac{d^{k\cdot n}\tilde{C}}{|\mathrm{GL}(k)|}\frac{1}{\tilde{M}_1^{1+b_n}\tilde{M}_2^{1+b_1}\ldots\tilde{M}_n^{1+b_{n-1}}}\delta^{4k|4k}(\tilde{C}\cdot\mathcal{W}), \tag{2.30}$$

where the $\tilde{M}_i$ are the minors of the reduced matrix $\tilde{C}$. At the same time, we know that the momentum-twistor top cell $\hat{\mathcal{G}}_{n,k-2}(\mathcal{W})$ by itself is invariant under the deformed momentum-twistor Yangian (2.26),(2.25) once we identify

$$\hat{b}_i \equiv b_{i-1} = \frac{1}{2}(v_i^- - v_{i-1}^-), \qquad v_i^- = v_i - c_i^{\mathrm{dual}}, \tag{2.31}$$

according to (2.23). Here, $c_i^{\mathrm{dual}} = c_{i-1}$ are the eigenvalues of $\hat{\mathcal{G}}_{n,k-2}(\mathcal{W})$ under the local central charges $\mathfrak{C}_i$ in (2.26). It follows that the deformed Graßmannian integral is invariant, both under the deformed original-twistor Yangian (2.3),(2.2) and the deformed momentum-twistor Yangian (2.25),(2.26), once one identifies

$$u_i^- - u_{i-1}^- = v_{i+1}^- - v_i^-, \tag{2.32}$$

where the prefactor in (2.29) has to be taken into account in the invariance statement. The relation (2.29) between invariants of the twistor Yangian and the momentum-twistor Yangian generalizes to all deformed on-shell diagrams that can be obtained from the deformed Graßmannian formula by localization on residues as explained above:

$$\hat{\mathcal{Y}}_{n,k}(\mathcal{Z})\overset{\mathcal{Z}\to\mathcal{W}}{\to}\frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle^{1+(u_1^- - u_{n-k+2}^-)/2}\ldots\langle n1\rangle^{1+(u_n^- - u_{n-k+1}^-)/2}} \times\hat{\mathcal{Y}}_{n,k-2}(\mathcal{W}). \tag{2.33}$$

Here the parameters u_i^- now satisfy additional constraints imposed by setting the appropriate moduli b_i to zero for the purpose of localizing the integrations. It would be interesting to understand whether the equivalence (2.33) extends also to deformed diagrams that *cannot* be obtained by localizing the top cell, i.e. whose deformation parameters u_i^- are unconstrained. Also, it would be interesting to check whether the deformed Yangians (2.3),(2.2) and (2.25), (2.26) still map to each other as in the undeformed case.

D. Examples

Let us summarize the construction of invariant deformed diagrams, and close the discussion of $\mathcal{N}=4$ SYM deformations by commenting on some interesting examples, including curious deformations for MHV amplitudes as well as an explicit exposition of the deformed six-point NMHV Graßmannian integral.

1. Summary of construction

Working out the admissible deformations for any given single diagram works as follows [10]. First pick a perfect orientation and a set of $(n_\mathrm{F}-1)$ edge variables. The candidate invariant then is (2.15), and the invariance constraints on the external central charges $c_{1,\ldots,n}$, evaluation parameters $u_{1,\ldots,n}$, and edge variable parameters $a_{1,\ldots,n_\mathrm{F}-1}$ are the following: For each left-right path[10] from site i to site j, set $u_i^+ = u_j^-$, which also must equal the *internal* parameter $u_{\mathrm{int}} \pm c_{\mathrm{int}}$ on each labeled edge along the path. Here the sign depends on the direction of the edge, and the internal central charge c_{int} equals the edge parameter a_ℓ on that edge, as follows from (2.7).

For diagrams that appear in BCFW decompositions of tree-level amplitudes, the number of integrations $(n_\mathrm{F}-1)$ equals the number $(2n-4)$ of bosonic delta functions. In this case, the expression for the diagram in terms of spinor-helicity variables can be worked out by solving the delta-function constraints for the edge variables, taking into

[9] The underlying reason is that the Yangian of the ordinary superconformal algebra with generators (2.2),(2.3) and the Yangian of the dual superconformal algebra with generators (2.25),(2.26) in fact can be mapped to each other [3,23].

[10] Turn left at each white vertex ($\overline{\mathrm{MHV}}$), turn right at each black vertex (MHV).

account the resulting Jacobi factor. In the case of MHV tree amplitudes (which are single top-cell diagrams), the minors (ij) of the matrix C equal the spinor brackets $\langle ij\rangle$, and the edge variables are given in terms of these minors by the function bridgeToMinors of the Mathematica package given in [25]. Comparing the measure $(\alpha_1\ldots\alpha_{2n-4})^{-1}$ to the Parke-Taylor MHV denominator, one can directly infer the Jacobi factor from the delta functions and write down the deformed amplitude.

2. MHV amplitudes

For MHV amplitudes, the explicit deformations can be worked out and analyzed using [25] as explained above. Let us briefly discuss these deformations (see also [8]).

For *odd multiplicities*, all c_i can be expressed in terms of the u_i, which remain free. The c_i among themselves only satisfy the single equation $\sum_{i=1}^{n} c_i = 0$. When all u are set to zero, no deformation remains. Conversely, there is no deformation with all $c_i = 0$.

For *even multiplicities*, invariance requires that the sums of even/odd c_i vanish separately:

$$\sum_{\substack{i=1\\ i\,\text{odd}}}^{n} c_i = 0, \qquad \sum_{\substack{i=2\\ i\,\text{even}}}^{n} c_i = 0. \tag{2.34}$$

These cases admit a one-parameter family of solutions where all c_i vanish, namely

$$(u_1, \ldots, u_n) = (z, -z, \ldots, z, -z), \tag{2.35}$$

in this case the deformed amplitudes take the form

$$\mathcal{A}_n^{\text{MHV}}(z) = \mathcal{A}_n^{\text{MHV}}\left(\frac{\langle 12\rangle\langle 34\rangle\ldots\langle n-1n\rangle}{\langle 23\rangle\langle 45\rangle\ldots\langle n1\rangle}\right)^{z}. \tag{2.36}$$

Note that cyclic shifts of these deformations amount to flipping the sign of z, such that the deformed amplitude is invariant under two-site cyclic shifts.

When $n = (6 \bmod 4)$, still all c_i can be expressed in terms of the u_i, which remain unconstrained; and there is no deformation where all u_i vanish. However when $n = (4 \bmod 4)$, then not only do the c_i need to satisfy (2.34), but also the even and odd u_i need to satisfy separate equations:

$$\sum_{\substack{i=1\\ i\,\text{odd}}}^{n} (-1)^{(i-1)/2} u_i = 0, \qquad \sum_{\substack{i=2\\ i\,\text{even}}}^{n} (-1)^{i/2} u_i = 0. \tag{2.37}$$

As a consequence, not all c_i can be expressed in terms of u_i, and in addition to the solutions (2.35), there is a two-parameter family of solutions with all $u_i = 0$, namely

$$(c_1, \ldots, c_n) = (c_\text{o}, c_\text{e}, -c_\text{o}, -c_\text{e}, \ldots, c_\text{o}, c_\text{e}, -c_\text{o}, -c_\text{e}). \tag{2.38}$$

These deformations take the form

$$\mathcal{A}_n^{\text{MHV}}(c^+, c^-) = \mathcal{A}_n^{\text{MHV}}\left(\frac{\langle 12\rangle\langle 56\rangle\ldots\langle n-3, n-2\rangle}{\langle 34\rangle\langle 78\rangle\ldots\langle n-1, n\rangle}\right)^{c^+} \times \left(\frac{\langle 45\rangle\langle 89\rangle\ldots\langle n1\rangle}{\langle 23\rangle\langle 67\rangle\ldots\langle n-2, n-1\rangle}\right)^{c^-}, \tag{2.39}$$

where $c^\pm \equiv (c_\text{o} \pm c_\text{e})/2$. Note that this implies that MHV amplitudes with $n = (4 \bmod 4)$ allow for a deformation without deforming the Yangian generators.

3. General cells

The different types of deformations for MHV amplitudes discussed above can be understood as follows: For a given n-point diagram with associated permutation σ, let P_σ be the finest partition of $\{1, \ldots, n\}$ such that σ only permutes labels within individual parts of the partition. Summing up the invariance conditions (2.13) for each part of the partition results in the constraints

$$0 = \sum_{j\in p} c_j \quad \text{for all } p \in P_\sigma, \tag{2.40}$$

where p denotes any single part of P_σ. For parts with an even number of elements one finds the further conditions

$$0 = \sum_{j\in p} (-1)^{i_p(j)} u_j \quad \text{for all } p \in P_\sigma \quad \text{with} \quad |p|\text{even}, \tag{2.41}$$

where $i_p(j)$ denotes the position of j in p, i.e. the sign alternates. For p with $|p|$ odd, one cannot sum the invariance relations in a way that all c_j drop out.

For MHV amplitudes, the permutation is a shift by two sites. When n is odd, the corresponding partition is trivial, and there are no nontrivial constraints solely among u's or c's. When n is even, the partition simply consists of a part that contains only odd sites and a part that contains only even sites; hence the c_i will satisfy the relations (2.34). The relations (2.37) solely among u's will only hold when $n = (4 \bmod 4)$, in which case each of the two parts contains an even number of elements. This kind of analysis straightforwardly generalizes to all top-cell diagrams.

The most extreme examples of partitions arise when the permutation satisfies $\sigma(i) = j \Leftrightarrow \sigma(j) = i$. In this case, each part of the partition has only two elements, and the invariance conditions simply become $c_i = -c_j$, $u_i = u_j$ for each $p = \{i, j\}$. The simplest example of this type is given by the four-point amplitude. As will become clear in Sec. III B, this case is relevant for ABJM theory [16,26].

4. Remark

Curiously, one type of deformation was already mentioned in the literature long before the idea of general invariance-preserving deformations was proposed in [7,8]: Section 6 of [27] discusses the possibility of deformations with nonvanishing central charges c_i. In particular, a specific deformation of the six-point NMHV top cell was identified, which has alternating exponents on the minors in the Graßmannian integral. This specific deformation is not compatible with a conventional BCFW decomposition though.

5. Six-point NMHV integral

In the NMHV case ($k = 3$), the momentum-twistor-space Graßmannian is $G(1, n)$. For $n = 6$, after gauge fixing the GL(1) and using the bosonic delta functions to localize four of the integrations, one obtains a one-dimensional integral. The deformed momentum-twistor integral then takes the form

$$\hat{\mathcal{G}}_{6,1}(\mathcal{W}) = \int dc \frac{\delta^4(\chi_1 + \sum_{i=2}^5 a_i^*(c - c_i^*)\chi_i + c\chi_6)}{(c - c_2^*)^{1+b_2}(c - c_3^*)^{1+b_3}(c - c_4^*)^{1+b_4}(c - c_5^*)^{1+b_5} c^{1+b_6} \prod_{i=2}^5 (a_i^*)^{1+b_i}}, \tag{2.42}$$

where $\mathcal{W}_i = (W_i, \chi_i)$ and

$$\begin{aligned} c_2^* &= -\frac{\langle 3451\rangle}{\langle 3456\rangle}, & c_3^* &= -\frac{\langle 4512\rangle}{\langle 4562\rangle}, \\ c_4^* &= -\frac{\langle 5123\rangle}{\langle 5623\rangle}, & c_5^* &= -\frac{\langle 1234\rangle}{\langle 6234\rangle}, \\ a_2^* &= +\frac{\langle 3456\rangle}{\langle 2345\rangle}, & a_3^* &= +\frac{\langle 4562\rangle}{\langle 2345\rangle}, \\ a_4^* &= +\frac{\langle 5623\rangle}{\langle 2345\rangle}, & a_5^* &= +\frac{\langle 6234\rangle}{\langle 2345\rangle}, \end{aligned} \tag{2.43}$$

with $\langle 1234\rangle \equiv \epsilon_{ABCD} W_1^A W_2^B W_3^C W_4^D$.

III. INTEGRABLE DEFORMATIONS IN ABJM THEORY

In this section we initiate the investigation of integrable deformations of scattering amplitudes in ABJM theory. Leading singularities of the $\mathcal{N} = 6$ superconformal Chern-Simons matter theory, also known as ABJM theory, are invariant under the undeformed $\mathfrak{osp}(6|4)$ Yangian algebra. These leading singularities are equivalent to the residues of an integral formula: An integral over the space of k null planes in a $2k$-dimensional space, whose integration contour localizes on the zeros of consecutive minors of the respective $(k \times 2k)$ matrix.[11] Tree-level amplitudes again are given by linear combinations of these leading singularities in which all unphysical poles cancel.

At four points, there is only one leading singularity and thus, without loss of generality, we will consider the most general possible deformation of the four-point amplitude that is consistent with the level-zero generators, i.e. the $\mathfrak{osp}(6|4)$ superconformal symmetry. The deformation will generically break the invariance under the original level-one generators. However, symmetry is preserved if we deform the level-one generators appropriately, that is if we use the evaluation representation of the Yangian with nonvanishing evaluation parameters.

The resulting four-point deformation will serve as a template from which we construct a deformation of the orthogonal Graßmannian integral and the corresponding deformed symmetry generators. It will also serve as the fundamental building block for constructing more general deformed Yangian invariants, which are deformations of the residues of the undeformed Graßmannian integral.

A. Deformed four-point amplitude

Super-Poincaré invariance requires the four-point amplitude to be proportional to the (super)momentum conserving delta functions. Dilatation invariance constrains the proportionality function to be a degree -2 polynomial of $\langle ij\rangle$.[12] We thus make the following natural ansatz for the deformed four-point amplitude[13]:

[11] Conformal three-dimensional Chern-Simons matter theories have nontrivial S-matrix elements for even multiplicity only. This is because only the matter fields carry physical degrees of freedom and dimensional analysis forbids cubic couplings among the matter fields.

[12] Here, $\langle ij\rangle \equiv \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$, where the spinors λ_i parametrize the three-dimensional massless momenta as $p_i^\mu = \sigma^\mu_{\alpha\beta}\lambda_i^\alpha\lambda_i^\beta$ for symmetric matrices σ^μ.

[13] There are two superfields in ABJM theory that take the form [18]

$$\Phi(\Lambda) = \phi^4(\lambda) + \eta^A\psi_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\phi^C(\lambda) + \frac{1}{6}\epsilon_{ABC}\eta^A\eta^B\eta^C\psi_4(\lambda), \tag{3.1}$$

$$\bar{\Phi}(\Lambda) = \bar{\psi}^4(\lambda) + \eta^A\bar{\phi}_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\bar{\psi}^C(\lambda) + \frac{1}{6}\epsilon_{ABC}\eta^A\eta^B\eta^C\bar{\phi}_4(\lambda). \tag{3.2}$$

Here Φ is bosonic while $\bar{\Phi}$ is fermionic.

$$\mathcal{A}_4(\bar{\Phi}_1,\Phi_2,\bar{\Phi}_3,\Phi_4)(z') = \frac{\delta^3(P)\delta^6(Q)}{\langle 12\rangle^{1+z'}\langle 23\rangle^{1-z'}} \equiv \delta^3(P)\delta^6(Q)f(\lambda). \tag{3.3}$$

Following [18], it is straightforward to see that this is invariant under the superconformal boost (level-zero) generators, and thus under the full $\mathfrak{osp}(6|4)$ level-zero symmetry algebra.

1. Invariance under the level-one momentum generator

For Yangian invariance, we only need to show the invariance under the level-one momentum generator; invariance under all other level-one generators then follows from the commutation relations and the level-zero invariance. The level-one momentum generator in the evaluation representation takes the form [18]

$$\hat{\mathfrak{P}}^{\alpha\beta} = \sum_{1\le j<k\le n}\frac{1}{2}[(\mathfrak{L}_{j\gamma}^{(\alpha} + \delta_\gamma^{(\alpha}\mathfrak{D}_j)\mathfrak{P}_k^{\gamma\beta)} - \mathfrak{Q}_j^{(\alpha A}\mathfrak{Q}_{k\ A}^{\beta)} - (j\leftrightarrow k)] + \sum_k u_k\mathfrak{P}_k^{\alpha\beta}. \tag{3.4}$$

The single-site generators are given in Appendix A. Using the transformation properties of the delta functions, the symmetry equation simplifies to[14]

$$\hat{\mathfrak{P}}^{\alpha\beta}\mathcal{A}_4(z') = \delta^3(P)\delta^6(Q)\left(\sum_{1\le j<k\le 4}\left[\frac{1}{2}\mathfrak{P}_k^{\gamma(\beta}\left(\lambda_j^{\alpha)}\partial_{j\gamma} + \frac{1}{2}\delta_\gamma^{\alpha)}\right)f(\lambda) - (j\leftrightarrow k)\right] + \sum_{k=1}^4 u_k\mathfrak{P}_k^{\alpha\beta}f(\lambda)\right). \tag{3.5}$$

We know from [18] that the undeformed amplitude is invariant under $\hat{\mathfrak{P}}|_{u_k=0}$, and thus only the terms proportional to z' remain when acting with $\hat{\mathfrak{P}}|_{u_k=0}$ on the deformed amplitude. These terms are generated when the bosonic derivatives act on the denominator of $\mathcal{A}_4(z')$. We can collect the terms from the first term in the square bracket as follows:

$$U_k^{\alpha\delta} \equiv \sum_{j=1}^{k-1}\lambda_j^\alpha\partial_j^\delta f(\lambda) = \begin{cases} 0+\cdots, & k=1,\\ -z'\frac{\lambda_1^\alpha\lambda_2^\delta}{\langle 12\rangle}f(\lambda)+\cdots, & k=2,\\ \left(-z'\frac{\lambda_1^{[\alpha}\lambda_2^{\delta]}}{\langle 12\rangle} + z'\frac{\lambda_2^\alpha\lambda_3^\delta}{\langle 23\rangle}\right)f(\lambda)+\cdots, & k=3,\\ \left(-z'\frac{\lambda_1^{[\alpha}\lambda_2^{\delta]}}{\langle 12\rangle} + z'\frac{\lambda_2^{[\alpha}\lambda_3^{\delta]}}{\langle 23\rangle}\right)f(\lambda)+\cdots, & k=4. \end{cases} \tag{3.6}$$

Here we only display the terms proportional to z' since the rest is known to cancel in the undeformed limit. Evaluating the symmetric and antisymmetric contributions from $U_k^{\alpha\delta}$ separately, we find[15]

$$\frac{1}{2}\sum_{k=1}^4\mathfrak{P}_k^{\gamma\beta}\varepsilon_{\gamma\delta}U_k^{(\alpha\delta)} + (\alpha\leftrightarrow\beta) = +z'[\mathfrak{P}_2^{\alpha\beta} - \mathfrak{P}_3^{\alpha\beta}]f(\lambda), \qquad \frac{1}{2}\sum_{k=1}^4\mathfrak{P}_k^{\gamma\beta}\varepsilon_{\gamma\delta}U_k^{[\alpha\delta]} + (\alpha\leftrightarrow\beta) = -z'[\mathfrak{P}_2^{\alpha\beta} + \mathfrak{P}_3^{\alpha\beta}]f(\lambda). \tag{3.7}$$

Repeating the analysis for the term with $(j\leftrightarrow k)$ in (3.5), we find (we relabel the summation indices)

$$\bar{U}_k^{\alpha\delta} \equiv \sum_{j=k+1}^4\lambda_j^\alpha\partial_j^\delta f(\lambda) = \begin{cases} z'\left(\frac{\lambda_2^\alpha\lambda_1^\delta}{\langle 12\rangle} + \frac{\lambda_2^{[\alpha}\lambda_3^{\delta]}}{\langle 23\rangle}\right)f(\lambda)+\cdots, & k=1,\\ -z'\frac{\lambda_3^\alpha\lambda_2^\delta}{\langle 23\rangle}f(\lambda)+\cdots, & k=2,\\ 0+\cdots, & k=3,4, \end{cases} \tag{3.8}$$

and thus

[14] Here we use the notation $X^{(\alpha\beta)} = X^{\alpha\beta} + X^{\beta\alpha}$ and $X^{[\alpha\beta]} = X^{\alpha\beta} - X^{\beta\alpha}$.

[15] We use that $\varepsilon_{\gamma\delta}\varepsilon^{\alpha\delta} = -\delta_\gamma^\alpha$ and $\lambda_j^{[\alpha}\lambda_k^{\beta]} = -\varepsilon^{\alpha\beta}\langle jk\rangle$, where we define $\varepsilon_{12} = 1 = -\varepsilon^{12}$.

$$-\frac{1}{2}\sum_{k=1}^{4}\mathfrak{P}_k^{\gamma\beta}\varepsilon_{\gamma\delta}\bar{U}_k^{(\alpha\delta)}+(\alpha\leftrightarrow\beta)=+z'[-\mathfrak{P}_1^{\alpha\beta}+\mathfrak{P}_2^{\alpha\beta}]f(\lambda),\qquad -\frac{1}{2}\sum_{k=1}^{4}\mathfrak{P}_k^{\gamma\beta}\varepsilon_{\gamma\delta}\bar{U}_k^{[\alpha\delta]}+(\alpha\leftrightarrow\beta)=-z'[+\mathfrak{P}_1^{\alpha\beta}+\mathfrak{P}_2^{\alpha\beta}]f(\lambda). \tag{3.9}$$

Combining the results from (3.7),(3.9), we finally arrive at

$$\hat{\mathfrak{P}}^{\alpha\beta}\mathcal{A}_4(z')=\delta^3(P)\delta^6(Q)\left[\mathfrak{P}_1^{\alpha\beta}(u_1-z')+\mathfrak{P}_2^{\alpha\beta}u_2+\mathfrak{P}_3^{\alpha\beta}(u_3-z')+\mathfrak{P}_4^{\alpha\beta}u_4\right]f(\lambda). \tag{3.10}$$

Hence, requiring this expression to be proportional to the total momentum acting on the deformed amplitude (i.e. to vanish), we find the following constraints on the parameters following from invariance under the level-one momentum generator:

$$u_1-z'=u_2=u_3-z'=u_4=\text{const}. \tag{3.11}$$

Alternatively, these constraints can be expressed as

$$u_k-u_{k-1}=(-1)^{k-1}z',\qquad k=1,\ldots,4. \tag{3.12}$$

In conclusion, the deformed four-point amplitude in (3.3) is invariant under the evaluation representation of the Yangian generators of $Y[\mathfrak{osp}(6|4)]$, provided that the level-one generators are deformed as in (3.4) with the parameters u_i related to z' via (3.12).

Note that the deformation (3.3) changes the weight of $\mathcal{A}_4(\bar{\Phi},\Phi,\bar{\Phi},\Phi)$ under $\exp(i\pi\lambda_i\cdot\partial/\partial\lambda_i)$ for $i=1,3$, but not for $i=2,4$. That is, the deformation deforms the phase of the fermionic legs, but preserves the phase of the bosonic legs. For further comments on this, see the discussion around (3.49) below.

In the next two sections we will discuss deformed invariants at higher multiplicities. First we will construct bigger deformed on-shell diagrams by gluing four-point vertices. Next we show the invariance of the deformed Graßmannian integral explicitly.

B. Gluing invariants

All ABJM on-shell diagrams can be constructed by iteratively gluing four-point vertices together [9,16]. Along the lines of the four-dimensional case [10] reviewed in Sec. II A above, the gluing procedure can be split into two steps that need to be iterated: Taking products of diagrams, and fusing lines. In the following we will show that the gluing procedure indeed preserves the Yangian invariance also in the deformed case, provided that the deformation parameters are identified appropriately. For showing invariance, we will use the completely general form (2.1) of the n-point Yangian level-one generators.

1. Products

Given two diagrams $\mathcal{Y}_1(1,\ldots,m)$ and $\mathcal{Y}_2(m+1,\ldots,n)$ that are invariant under the m-point and $(n-m)$-point Yangian algebras with evaluation parameters $\{u_1,\ldots,u_m\}$ and $\{u_{m+1},\ldots,u_n\}$, the product

$$\mathcal{Y}'(1,\ldots,n)=\mathcal{Y}_1(1,\ldots,m)\mathcal{Y}_2(m+1,\ldots,n) \tag{3.13}$$

is invariant under the n-point Yangian algebra with evaluation parameters $\{u_1,\ldots,u_n\}$:

$$\hat{\mathfrak{J}}^a\mathcal{Y}'=(\hat{\mathfrak{J}}^a\mathcal{Y}_1)\mathcal{Y}_2+\mathcal{Y}_1(\hat{\mathfrak{J}}^a\mathcal{Y}_2)+f^a{}_{bc}(\mathfrak{J}^a\mathcal{Y}_1)(\mathfrak{J}^b\mathcal{Y}_2)=0. \tag{3.14}$$

2. Fusion

From any invariant $(n+2)$-point diagram $\mathcal{Y}(1,\ldots,n,n+1,n+2)$, one can construct an n-point diagram by fusing two adjacent external lines,

$$\mathcal{Y}'(1,\ldots,n)=\int d^{2|3}\Lambda d^{2|3}\Lambda'\delta^{2|3}(\Lambda-i\Lambda')\times\mathcal{Y}(1,\ldots,n,\Lambda,\Lambda'). \tag{3.15}$$

Here we will use the kinematical variables $\Lambda^{\mathcal{A}}=(\lambda^\alpha,\eta^A)$ with $\alpha=1,2$ and $A=1,2,3$.[16] The diagram $\mathcal{Y}'$ will be invariant under the n-point Yangian algebra with evaluation parameters $\{u_1,\ldots,u_n\}$ provided that

$$u_{n+1}=u_{n+2}. \tag{3.16}$$

Both the level-zero and the level-one invariance of $\mathcal{Y}'$ can be shown straightforwardly, using the $(n+2)$-point invariance of $\mathcal{Y}$ and the fact that

$$\int d^{2|3}\Lambda d^{2|3}\Lambda'\delta^{2|3}(\Lambda-i\Lambda')(\mathfrak{J}_\Lambda^a+\mathfrak{J}_{\Lambda'}^a)f(\Lambda,\Lambda')=0 \tag{3.17}$$

for any function f. The latter can be verified directly with the explicit $\mathfrak{osp}(6|4)$ generators given in Appendix A. Using the invariance of $\mathcal{Y}$, the action of the level-one generator on $\mathcal{Y}'$ can be written as

[16]Some care needs to be taken in the definition of the on-shell integration over $d^{2|3}\Lambda$, see e.g. [28]. Throughout this work, such integrations will always be localized on delta functions.

$$\hat{\mathfrak{J}}^a_{1\ldots n}\mathcal{Y}'(1,\ldots,n) = -f^a{}_{bc}\int d^{2|3}\Lambda d^{2|3}\Lambda' \delta^{2|3}(\Lambda - i\Lambda') \times \left(\sum_{i=1}^{n} \mathfrak{J}^b_i(\mathfrak{J}^c_\Lambda + \mathfrak{J}^c_{\Lambda'}) + \mathfrak{J}^b_\Lambda \mathfrak{J}^c_{\Lambda'}\right) \times \mathcal{Y}(1,\ldots,n,\Lambda,\Lambda'). \tag{3.18}$$

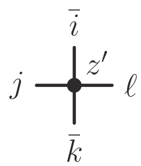
The first term in the parentheses vanishes due to (3.17). Again using (3.17), the second term can be rewritten as $\mathfrak{J}^b_\Lambda\mathfrak{J}^c_\Lambda \simeq \frac{1}{2}[\mathfrak{J}^b_\Lambda, \mathfrak{J}^c_\Lambda] = \frac{1}{2}f^{bc}{}_d\mathfrak{J}^d$. Hence, this term is proportional to $f^a{}_{bc}f^{bc}{}_d$, which vanishes for $\mathfrak{osp}(6|4)$, as it does for $\mathfrak{psu}(2,2|4)$, since the dual Coxeter number is zero.

The above procedure of taking products and fusing lines allows us to fuse two legs from different diagrams, or two legs sitting on the same diagram. Note, however, that the two fused legs have to correspond to different multiplets, i.e. to fields Φ and $\bar{\Phi}$, and that they must be adjacent.

3. Four-vertex

We have seen above that the deformed Yangian-invariant four-vertex reads[17]

$$\mathcal{A}_4(\bar{\Phi}_i, \Phi_j, \bar{\Phi}_k, \Phi_\ell)(z') =$$



$$= \frac{\delta^3(P)\,\delta^6(Q)}{\langle ij\rangle^{1+z'}\langle jk\rangle^{1-z'}} = \frac{\delta^3(P)\,\delta^6(Q)}{\langle kl\rangle^{1+z'}\langle li\rangle^{1-z'}}, \tag{3.19}$$

where the Yangian evaluation parameters $u_{i,j,k,\ell}$ need to satisfy

$$u_i = u_k, \qquad u_j = u_\ell, \qquad z' = u_i - u_j. \tag{3.20}$$

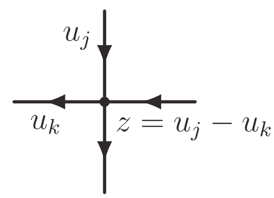
All bigger deformed on-shell diagrams can be constructed from this four-vertex by applying the invariance-preserving operations described above. For this purpose, it is most useful to write the vertex (3.19) in a gauge-fixed integral form[18]

$$\mathcal{A}_4(z) = \int \frac{d\theta}{\sin(\theta)^{1+z}}\delta^{4|6}(C(\theta)\cdot\Lambda), \tag{3.21}$$

where the C-matrix $C(\theta)$ is given by [16]

$$C(\theta) = \begin{pmatrix} 1 & 0 & i\sin\theta & i\cos\theta \\ 0 & 1 & -i\cos\theta & i\sin\theta \end{pmatrix}. \tag{3.22}$$

It is easy to see that $z = -z'$ for this choice of C-matrix. In general, the relative sign between z and z' depends on which columns of $C(\theta)$ are set to unit vectors. Therefore, as we build up a general on-shell diagram, we need to keep track of which columns of $C(\theta)$ are set to unity. A convenient way of keeping track is to decorate the lines connected to each vertex with two incoming and two outgoing arrows, where the former indicates that these columns form the unit matrix. We will only consider the cases where the two incoming arrows are adjacent, which leads to a constraint on z included in the following figure:



$$. \tag{3.23}$$

The sign of z is determined by the following rule: Start with a line associated with the parameter u_j and compare its arrow to the clockwise neighboring line associated with u_k. If both arrows have the same orientation with respect to the vertex, then $z = u_k - u_j$; if the arrows have opposite orientations, then $z = u_j - u_k$. As we will see further below, the lines in (3.23) will be identified with the rapidity lines of integrable models, and the parameters u_i with rapidity parameters.

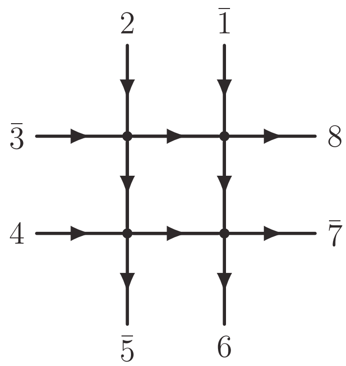
4. General deformed diagrams

Any reduced $2k$-point on-shell diagram of ABJM theory can be drawn as k straight lines that intersect,[19] where each intersection is a four-point on-shell vertex. Turning on the deformations, there is one deformation modulus z_i for each four-vertex, where i labels the vertices in the respective diagram. For the larger diagrams, the invariance conditions (3.23) for each four-vertex, and the invariance conditions (3.16) from fusing lines must be respected. It immediately follows that for every invariant $2k$-point diagram, there remains exactly one evaluation parameter for each of the k straight lines, as the evaluation parameters u_i on glued lines need to be identified. For example for the following diagram we have

[17]The last equality follows from $\langle ij\rangle = \pm\langle kl\rangle$, $\langle jk\rangle = \pm\langle li\rangle$ (with aligned signs) due to momentum conservation.

[18]The domain of integration has to be chosen such that the delta functions localize the integral on a single point. For real kinematics in Minkowski space, a valid choice for the integration domain is $[0,\pi)$.

[19]For reduced diagrams, any two lines intersect at most once.



$$u_1 = u_6\,,\quad u_2 = u_5\,,\quad u_3 = u_8\,,\quad u_4 = u_7\,.$$

Each vertex deformation modulus z_i is in turn determined to be the difference of the evaluation parameters on the lines that pass through the vertex according to (3.23). As an example, the z_i's in Fig. 1 are given by

$$\begin{aligned} z_1 &= u_2 - u_1, & z_2 &= u_3 - u_1, & z_3 &= u_4 - u_1, \\ z_4 &= u_3 - u_2, & z_5 &= u_4 - u_2, & z_6 &= u_4 - u_3. \end{aligned} \tag{3.24}$$

For generic diagrams, the conditions (3.23) not only determine the vertex moduli, but also induce constraints among them: For each closed loop, $\sum_i(\pm z_i) = 0$, where i enumerates the vertices along the loop, and where the sign depends on the relative directions of arrows along the loop at the respective vertex.

In summary, every $2k$-point diagram admits a $(k-1)$-parameter family of deformations, where the $(k-1)$ parameters are given by the evaluation parameters $u_{1\ldots k}$ on the k lines, modulo a trivial overall shift of all u_i's.

As described in [16], the vertex (3.21) provides "canonical coordinates," which means that gluing multiple such vertices produces no Jacobian from combining the delta functions; that is a general (deformed) diagram constructed in this way takes the simple form

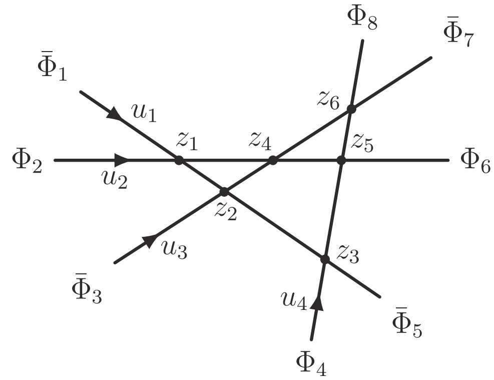


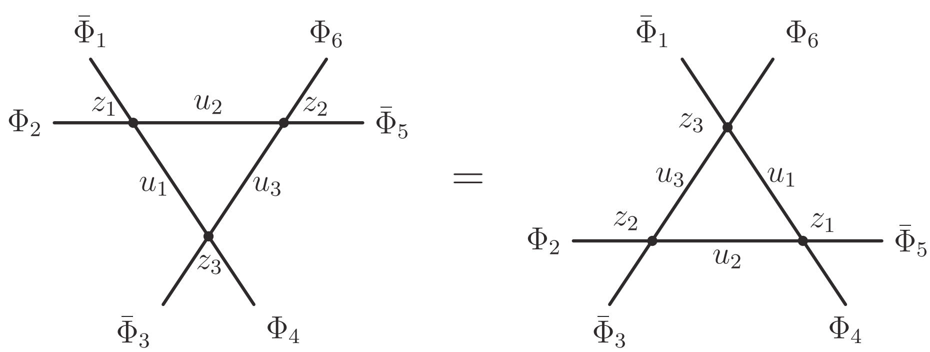
FIG. 1. An example of a deformed on-shell diagram with the invariance constraints given by (3.24).

$$\mathcal{Y}(\Lambda) = \int \frac{d\theta_1}{\sin(\theta_1)^{1+z_1}} \cdots \frac{d\theta_\ell}{\sin(\theta_\ell)^{1+z_\ell}} \delta^{2k|3k}(C(\theta_i)\cdot\Lambda), \tag{3.25}$$

where the orthogonal matrix $C(\theta_i)$ can be read off algorithmically from the diagram.

5. Deformed triangle move

Undeformed on-shell diagrams are invariant under triangle moves, which take one line past the intersection of two other lines [9,16]. The triangle move amounts to a change of integration variables in the Graßmannian integral that preserves the canonical form (3.25) in the undeformed case. Not surprisingly, this remains true without modifications in the deformed case:



. (3.26)

The triangle equality holds regardless of the orientations of the three lines, as long as the orientations are the same on both sides of the equation. Note that this result is consistent with the invariance constraints (3.23). In fact, it is not a coincidence that this diagrammatic equation looks very much like the Yang-Baxter equation, as will become more clear in Sec. III D below.

6. Invariance and permutations

Due to the triangle equality (3.26), every (deformed) reduced diagram is uniquely specified by a permutation σ that simply interchanges pairs of external legs. In other words, σ is composed of pairwise commuting transpositions, and σ^2 equals the identity permutation. The invariance equations for the evaluation parameters then take the rather trivial form

$$u_i = u_{\sigma(i)}. \tag{3.27}$$

In order to identify the vertex deformation moduli z_i, one needs to decorate the $2k$-point diagram with arrows such that each line [connecting legs i and $\sigma(i)$] carries a definite orientation. Then k columns of the C-matrix form the identity matrix, and each four-vertex is of the form (3.23), such that the z_i can be read off.

7. Deformed BCFW decomposition

Tree-level amplitudes in ABJM theory can be decomposed into a sum of BCFW terms [29], where each term is an on-shell diagram [16]. An interesting question is whether higher-point tree-level amplitudes can be consistently deformed by deforming each diagram in the sum, using the same evaluation parameters for each diagram. The six-point amplitude consists of a single triangle-shaped diagram as in (3.26), and thus allows for a two-parameter family of deformations. The diagrams for the eight- and ten-point amplitudes are given explicitly in [16]. The eight-point tree-level amplitude consists of two terms, in which the lines connect the eight points as $\{[15][27][36][48]\}$, $\{[14][26][37][58]\}$. Hence it allows for a one-parameter deformation in terms of u_1-u_2, where $u_1 = u_4 = u_5 = u_8$, and $u_2 = u_3 = u_6 = u_7$. The ten-point amplitude consists of the five terms:

$$\begin{aligned} &\{[14][27][39][58][6,10]\}, \qquad \{[14][26][38][59][7,10]\},\\ &\{[16][29][37][4,10][58]\},\\ &\{[17][29][36][48][5,10]\}, \qquad \{[15][28][36][49][7,10]\}. \end{aligned} \tag{3.28}$$

Combining the resulting invariance constraints enforces that all evaluation parameters must be equal, and hence there is no nontrivial deformation at ten points. In fact, the $(2p+4)$-point tree-level amplitude consists of $(2p)!/(p!(p+1)!)$ diagrams [16]. Since each diagram implies a different set of constraints, the number of constraints at higher points by far outweighs the number of parameters, and hence a consistent deformation of the BCFW decomposition beyond eight points cannot be expected.

8. Branches

Similar to the $\mathcal{N}=4$ SYM case, every ABJM on-shell diagram is an integral over a cell in the orthogonal Graßmannian [9,16]. Every cell in the orthogonal Graßmannian in fact consists of two distinct branches. The two branches can be distinguished by the ratios of non-overlapping minors

$$M_j/M_{j+k} = \mp 1. \tag{3.29}$$

In gluing on-shell diagrams, this subtlety is reflected in the matrix $C(\theta)$ of the four-point vertex (3.21), where

$$\mathrm{OG}_{2,\pm}\colon\ C(\theta) = \begin{pmatrix} 1 & 0 & \pm i\sin\theta & \pm i\cos\theta \\ 0 & 1 & -i\cos\theta & i\sin\theta \end{pmatrix}. \tag{3.30}$$

Note that these two C-matrices are not related by any coordinate transformation. While it may appear that there are 2^{n_v} distinct branches for a given on-shell diagram with n_v vertices, most of them are related by coordinate transformations, leaving only two distinct branches. Denoting the branches at each vertex by a sign, the branch of the final C-matrix is simply the product of all signs of the individual vertices. Since each branch is individually Yangian invariant, we restrict ourselves to diagrams built from the positive branch of the four-vertex. Generalizing to include the other branch is straightforward.

C. Deformed orthogonal Graßmannian

In this section we consider integrable deformations of the orthogonal Graßmannian integral of ABJM theory. As in four dimensions, the deformation under consideration is again a modification of the power of the minors, which are the only admissible deformations that maintain $\mathrm{GL}(k)$ invariance. We will first map the deformation parameters of the four-point Graßmannian to that of the four-point amplitude. Then we will show that for general multiplicities, the deformed Graßmannian is invariant under Yangian symmetry, provided that the deformation parameters obey a set of constraints that are a generalization of the four-point constraints (3.12).

1. Proposal for the deformed Graßmannian

We consider the following deformation of the orthogonal Graßmannian integral:

$$\mathcal{G}_{2k}(b_i) = \int \frac{d^{k\times 2k}C}{|\mathrm{GL}(k)|} \frac{\delta^{k(k+1)/2}(C\cdot C^{\mathsf{T}})\delta^{2k|3k}(C\cdot\Lambda)}{\prod_{i=1}^{k} M_i^{1+b_i}}. \tag{3.31}$$

The undeformed integral was originally proposed in [17]. Here $C\cdot C^{\mathsf{T}} \equiv \sum_i C_{ai}C_{bi}$ is a $k\times k$-symmetric matrix whose vanishing implies that the Graßmannian matrix C consists of k n-dimensional null vectors. We will denote the orthogonal Graßmannian $\mathrm{G}(k,2k)$ as OG_k. For the integral to be $\mathrm{GL}(k)$ invariant, the deformation parameters b_i must satisfy the relation $\sum_{i=1}^{k} b_i = 0$. When all b_i vanish, this reduces to the formula given in [17].

2. Relation to the four-point amplitude

For the simplest case of $k = 2$, the relation between z' in (3.3) and b_1 in (3.31) can be deduced by simply using the bosonic delta functions to localize the Graßmannian integration variables. More precisely, let us begin with the following integral, where we already used that $b_1 + b_2 = 0$ in (3.31):

$$\mathcal{G}_4(b_1, -b_1) = \int \frac{d^{2\times 4}C}{|\mathrm{GL}(2)|} \frac{\delta^3(C\cdot C^{\mathsf{T}})\delta^{4|6}(C\cdot\Lambda)}{M_1^{1+b_1} M_2^{1-b_1}}. \quad (3.32)$$

We work with the gauge

$$C = \begin{pmatrix} 1 & 0 & C_{13} & C_{14} \\ 0 & 1 & C_{23} & C_{24} \end{pmatrix}, \quad (3.33)$$

and the momentum delta function $\delta^4(C\cdot\lambda)$ gives [29]

$$\delta^4(C\cdot\lambda) = \frac{1}{\langle 34\rangle^2}\prod_{r,s}\delta^4(C_{r,s} - C^*_{r,s}),$$
$$\begin{pmatrix} C^*_{13} & C^*_{14} \\ C^*_{23} & C^*_{24} \end{pmatrix} = -\frac{1}{\langle 34\rangle}\begin{pmatrix} \langle 14\rangle & \langle 31\rangle \\ \langle 24\rangle & \langle 32\rangle \end{pmatrix}. \quad (3.34)$$

Substituting the solutions into (3.32), we find the following deformed amplitude:

$$\mathcal{A}_4(b_1) = \frac{\delta^3(P)\delta^6(Q)}{\langle 12\rangle^{1+b_1}\langle 23\rangle^{1-b_1}}. \quad (3.35)$$

Setting $b_1 = z'$, we see that the above deformation of the Graßmannian indeed induces the same deformed four-point amplitude as in (3.3).

3. Relation to deformed on-shell diagrams

For higher multiplicities, we first note that the deformation of the Graßmannian integral can be obtained from the deformed on-shell diagrams simply by considering the top-cell diagram. The $2k$-point top cell has dimension $k(k-1)/2$. All $2k$-point diagrams consist of k lines, and each four-vertex contributes one integration variable. Hence, in the top-cell diagram, each of the k lines has to cross each other line exactly once. Modulo triangle moves, this diagram is unique. A canonical representative is sketched in Fig. 2. Iteratively building up the top cell by gluing four-vertices (3.21), the top-cell integral will take the form (3.25), with $\ell = k(k-1)/2$. Comparing that form to the deformed Graßmannian integral formula (3.31), one could in principle read off the relation between b_i and z_i, and in turn express b_i in terms of the Yangian evaluation parameters u_i.

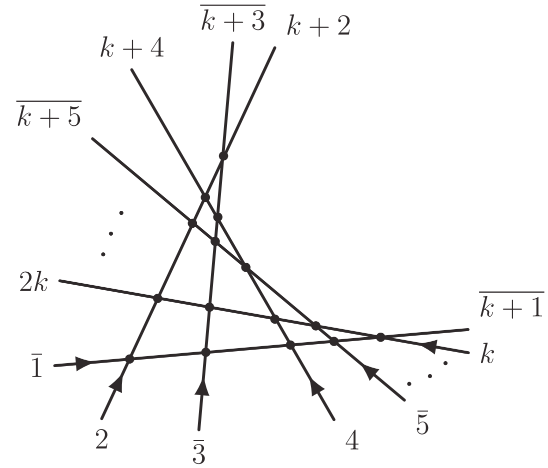


FIG. 2. The ABJM top-cell diagram.

In four dimensions, we saw that the invariance conditions for the top cell directly lead to the simple relation (2.23) between the exponents b_i and the parameters $u_i^{\pm}$, which are the natural deformation moduli from the on-shell diagram perspective. The relation followed from a direct identification of the central charges c_i in terms of the exponents b_i.

However, the symmetry algebra $\mathfrak{osp}(6|4)$ of ABJM theory does not admit a central extension, and thus there are no central charges that could be deformed.[20] Hence, the relation between the exponents b_i and the evaluation parameters u_i apparently cannot be deduced directly. Below, we will therefore derive the invariance relations by directly acting with the Yangian generators on the deformed Graßmannian integral.

4. Yangian invariance of the deformed Graßmannian

For the four-point example, we have deduced that the relation between the deformation parameter of the Graßmannian integral and the evaluation parameters is given by (3.12), with $b_1 = z'$. We now proceed to derive the general n-point relations.

For compactness we present the level-zero and level-one generators in Λ-space. For the invariance under the level-zero algebra, note that the level-zero generators take the form

$$\Lambda_i^{\mathcal{A}}\Lambda_i^{\mathcal{B}}, \qquad \Lambda_i^{\mathcal{A}}\frac{\partial}{\partial\Lambda_i^{\mathcal{B}}}, \qquad \frac{\partial}{\partial\Lambda_i^{\mathcal{A}}}\frac{\partial}{\partial\Lambda_i^{\mathcal{B}}}. \quad (3.36)$$

The invariance follows, respectively, from the momentum conservation, delta-function constraint $\delta^{2k|3k}(C\cdot\Lambda)$, and the orthogonality of C [17].

[20] More concretely, the scattering amplitudes and the Graßmannian integral for ABJM are not eigenstates of the local scaling operators $\Lambda_i\cdot\partial/\partial\Lambda_i$; see also the discussion around (3.49) below.

The undeformed level-one generators $\hat{\mathfrak{J}}^{\mathcal{AB}}$ with two upper indices of the same statistics ($\mathfrak{P}^{ab}$ and $\mathfrak{R}^{AB}$) can be written as [18]

$$\hat{\mathfrak{J}}^{(\mathcal{AB}]} = \left(\sum_{l<i} - \sum_{i<l}\right)\left((-1)^{|\mathcal{C}|}\Lambda_l^{(\mathcal{A}}\frac{\partial}{\partial\Lambda_l^{\mathcal{C}}}\Lambda_i^{\mathcal{C}}\Lambda_i^{\mathcal{B}]} + \frac{\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}}{2}\right), \tag{3.37}$$

where the indices $(\mathcal{A},\mathcal{B}]$ are understood to be (anti-)symmetrized if $\mathcal{A}$ and $\mathcal{B}$ denote indices of $\mathfrak{sp}(4)$ [$\mathfrak{su}(3)$]. Here we will simply consider the case where they have the same statistics, since invariance under all other generators follows from the $\mathfrak{osp}(6|4)$ algebra.[21] We first rewrite $(\sum_{l<i} - \sum_{i<l}) = 2\sum_{l<i} - \sum_{i,l} + \sum_{i=l}$. Then for $\hat{\mathfrak{J}}^{(\mathcal{AB}]}$, this amounts to

$$\hat{\mathfrak{J}}^{(\mathcal{AB}]} \simeq \left[2\left(\sum_{l<i}\Lambda_l^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}\Lambda_i^{\mathcal{C}}\frac{\partial}{\partial\Lambda_l^{\mathcal{C}}} + \frac{\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}}{2}\right) + \sum_i \Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}\Lambda_i^{\mathcal{C}}\frac{\partial}{\partial\Lambda_i^{\mathcal{C}}}\right], \tag{3.38}$$

up to terms proportional to level-zero generators. As the invariance of the undeformed orthogonal Graßmannian was not proved in the literature, we provide the details in Appendix C. The crucial step in the proof, as pointed out for the four-dimensional case in [23], is to realize that $\Lambda_l^{\mathcal{C}}\partial/\partial\Lambda_i^{\mathcal{C}}$ acts only on the delta functions and can be converted into a rotation generator acting on the Graßmannian variables:

$$\Lambda_i^{\mathcal{C}}\frac{\partial}{\partial\Lambda_l^{\mathcal{C}}} \longrightarrow O_l{}^i \equiv \sum_a C_{al}\frac{\partial}{\partial C_{ai}}. \tag{3.39}$$

Using integration by parts, the linear operator then acts on the integration measure. This operator simply replaces column i of the matrix C by column l, that is $O_l{}^i M_p = M_p^{i\to l}$, if $p \le i \le p+k-1$ and $l < p$, while its action vanishes otherwise. As we demonstrate in Appendix C, it is straightforward to show that

$$\sum_{l<i}\Lambda_i^{\mathcal{A}}\Lambda_l^{\mathcal{B}}O_l{}^i M_p = \sum_{l=1}^{p-1}\Lambda_l^{\mathcal{B}}\Lambda_l^{\mathcal{A}}M_p. \tag{3.40}$$

In other words, the minors transform covariantly under the operator $O_l{}^i$. Note that one must be careful as the integral formula has a $\mathrm{GL}(k)$ symmetry and is well defined only after gauge fixing. Thus, to prove the invariance of the integral, one should either introduce a gauge-fixing function, on which $O_i{}^l$ acts, or directly work with the gauge-fixed integral.

For consistency, we will proceed with the gauge-fixed integral with the columns 1 through k of the matrix C set to the unit matrix:

$$\begin{pmatrix} 1 & \cdots & 0 & C_{1,k+1} & \cdots & C_{1,2k} \\ 0 & \cdots & 0 & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & C_{k+1,2k} & \cdots & C_{k,2k} \end{pmatrix}. \tag{3.41}$$

We see that for $k \le i$ the operator $O_l{}^i$ is defined simply by replacing $C_{ai} \to C_{al}$ or $C_{ai} \to \delta_{al}$. However, for $i < k$ the operator requires careful treatment. For the four-dimensional case, this situation was discussed in detail in [23], where it was shown that for $i \le k$, $O_l{}^i$ should be replaced by $\mathcal{N}_i{}^l \equiv \sum_{r=k+1}^{2k} C_{lr}\frac{\partial}{\partial C_{ir}}$, which is nothing but a $\mathrm{GL}(k)$-rotation on the rows of the unfixed part C_{ai} of the gauge-fixed C-matrix. Thus $\sum_{i<l}\mathcal{N}_i{}^l M_p = 0$ for $k < p \le 2k$, whilst $\mathcal{N}_i{}^l M_p = -M_p^{l\to i}$ for $p \le k$.

Collecting these results, and noting that since the undeformed Graßmannian integral vanishes under $\hat{\mathfrak{J}}^{(\mathcal{AB}]}$, we can focus solely on the extra terms that are generated due to the additional exponents b_i of the measure. These additional terms are given by

$$\hat{\mathfrak{J}}^{(\mathcal{AB}]}\mathcal{G}_{2k}(b_i) = \mathcal{G}_{2k}(b_i)\sum_{j=1}^{2k}\left(\sum_{l=j+1}^{k}2b_l - b'_j\right)\Lambda_j^{(\mathcal{A}}\Lambda_j^{\mathcal{B}]}, \tag{3.42}$$

where b'_j is defined as

$$\left(\sum_{a=1}^{k}C_{aj}\frac{\partial}{\partial C_{aj}}\right)\frac{1}{\prod_{i=1}^{k}M_i(C)^{b_i}} = b'_j\frac{1}{\prod_{i=1}^{k}M_i(C)^{b_i}}. \tag{3.43}$$

Note that unlike in $\mathcal{N}=4$ SYM theory, the eigenvalue b'_j does not correspond to a central charge. We will further comment on this point below. In terms of the exponents b_i, the eigenvalue b'_j expands to

$$-b'_j = \begin{cases} b_1 + \cdots + b_j & j \le k \\ b_{j-k+1} + \cdots + b_k & j \ge k. \end{cases} \tag{3.44}$$

To retain Yangian invariance, it is necessary to deform the level-one generators by

$$\hat{\mathfrak{J}}^{(\mathcal{AB}]} \to \hat{\mathfrak{J}}^{(\mathcal{AB}]} + \sum_{j=1}^{2k}u_j\Lambda_j^{(\mathcal{A}}\Lambda_j^{\mathcal{B}]}, \tag{3.45}$$

where, for general $n = 2k$, the relation between the deformation parameters is given by

[21]The level-one generators $\hat{\mathfrak{J}}$ transform in the adjoint representation of the level-zero algebra.

$$\sum_{l=j+1}^{k} 2b_l + u_j^- = \text{constant}. \tag{3.46}$$

Here, we define $u_j^- = u_j - b'_j$, and the constant must be independent of j. This implies

$$\frac{1}{2}(u_j^- - u_{j-1}^-) = \begin{cases} b_j & \text{for } j \le k \\ 0 & \text{for } j > k. \end{cases} \tag{3.47}$$

Using (3.44), these conditions can be rewritten as

$$u_j = u_{j+k}, \qquad b_j = u_j - u_{j-1}, \qquad 1 \le j \le k. \tag{3.48}$$

In particular, this reproduces (3.27) for the permutation σ of the top cell, which is just a cyclic shift by k sites. Note that (3.46),(3.47) closely resemble the constraints (B8), (2.23) of the four-dimensional case. For four points, we have $b_1 = \frac{1}{2}(u_1^- - u_4^-)$ and $b_2 = \frac{1}{2}(u_2^- - u_1^-)$, which, combined with $b_1 + b_2 = 0$, implies $u_4 = u_2$ and $b_1 = u_1 - u_2$, in agreement with (3.12).

5. Little group and fermion number

In our discussion of the invariance of the deformed Graßmannian integral, we encountered the scaling operator $\mathfrak{f}_j = \Lambda_j^C \partial/\partial\Lambda_j^C$, which acts on the external scattering data as

$$\mathfrak{f}_j = \lambda_j^\alpha \frac{\partial}{\partial\lambda_j^\alpha} + \eta_j^A \frac{\partial}{\partial\eta_i^A}. \tag{3.49}$$

This operator generates the three-dimensional little group $\mathbb{Z}_2$: The exponentiated operator

$$\mathcal{F}_j \equiv \exp(i\pi\mathfrak{f}_j) \tag{3.50}$$

commutes with the whole $\mathfrak{osp}(6|4)$ algebra, and the amplitude transforms according to

$$\mathcal{F}_j \mathcal{A}(\bar{1}2\bar{3}\ldots 2k) = (-1)^j \mathcal{A}(\bar{1}2\bar{3}\ldots 2k). \tag{3.51}$$

As pointed out in [18], this equation looks similar to the local central charge constraint in $\mathcal{N}=4$ SYM theory. The group-like operator $\mathcal{F}_j$ measures the fermion number, i.e. whether the external leg j is a bosonic or fermionic superfield Φ or $\bar{\Phi}$, respectively.

An obvious question is how the deformed invariants behave under the operator $\mathcal{F}_j$. For the deformed four-point amplitude (3.3) the answer is simple and similar to the central charge constraint in four dimensions: While the local invariance under $\mathcal{F}_j$ is broken (only for the fermionic legs though), the global constraint given by $\prod_j \mathcal{F}_j \mathcal{A}_4 = \mathcal{A}_4$ is preserved. Here we consider the product of $\mathcal{F}_j$ due to the group-like structure of the operator $\mathcal{F}$ as opposed to $\mathfrak{f}$.

Note that the superfield in $\mathcal{N}=4$ SYM theory has a similar inconspicuous symmetry under the operator

$$\mathcal{F}_j^{4\text{d}} = \exp i\pi\left(\lambda_j^\alpha \frac{\partial}{\partial\lambda_j^\alpha} + \bar{\lambda}_j^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}_j^{\dot{\alpha}}} + \eta_j^A \frac{\partial}{\partial\eta_j^A}\right):$$

$$\mathcal{F}_j^{4\text{d}} \mathcal{A}_n = +\mathcal{A}_n. \tag{3.52}$$

This corresponds to the fact that $\Phi(-\Lambda) = \Phi(\Lambda)$, i.e. to the statement that the total number of spinors in all terms of the bosonic superfield is even, or even more simple: the bosonic superfield is bosonic. Note that $\mathcal{F}_j^{4\text{d}}$ is not generated by the central charge $\mathfrak{C}_i$. The breaking of the local fermion number operators $\mathcal{F}_j$ and $\mathcal{F}_j^{4\text{d}}$ in three and four dimensions, respectively, demonstrates the anyonic character of the above deformations.

6. The cells of the deformed Graßmannian and BCFW

The integral in (3.31) is a $k(k-1)/2$-dimensional integral representing the top cell, and it is invariant under the deformed Yangian. Here the evaluation parameters u_i of the level-one generators are constrained by (3.48). The bosonic delta function $\delta^{2k}(C \cdot \lambda)$ imposes $(2k-3)$ constraints, and thus the top cell has dimension $(k-2)(k-3)/2$. If some of the deformation parameters are turned off, then one can localize the top cell by residues on poles in the respective minor to obtain lower-dimensional cells, and thus obtain deformed descendant invariants. However, this does not yield the most general deformations of the lower-dimensional cells. As shown in Sec. III B, one can instead directly transform the lower-dimensional cells (on-shell diagrams), which leads to further deformations that do not form boundaries of the deformed top cell.

We can reconsider the question of consistent deformations for the BCFW terms of tree amplitudes from the perspective of the Graßmannian integral. Similar to the four-dimensional case, consistent deformations are only possible if the tree contour involves residues on fewer than $(k-1)$ minors. As soon as the tree contour includes poles from $(k-1)$ or all k minors, all exponents b_i need to be set to zero and no deformation remains. For six points, the amplitude is the top cell, and thus there is a consistent two-parameter deformation. For eight points, the BCFW terms are given by the sum of residues for M_1 and M_3, and thus only a one-parameter deformation remains. For ten points, as discussed in [16], the five BCFW terms are given by the zeros of

$$\{4,5,1\}, \qquad \{5,1,2\}, \qquad \{3,4,5\}, \qquad \{2,3,4\},$$
$$\{1,2,3\}, \tag{3.53}$$

where $\{i,j,k\}$ indicates the collection of minors that are necessary to localize the three-dimensional integral. As one can see, all five minors are involved in the localization, and no consistent deformation remains. It is to be expected that there will be no BCFW-preserving deformation for any amplitude beyond eight points. These results are consistent with the on-shell diagram analysis in Sec. III B above.

D. R-matrix construction for ABJM

In Sec. III B above, we have verified the invariance of the deformed four-point amplitude $\mathcal{A}_4(z)$ under the evaluation representation of the Yangian generators, and have obtained higher-point diagrams by successive gluing of the fundamental four-point invariant. In this section, we will identify $\mathcal{A}_4(z)$ with an integral kernel for the R-matrix $R_{jk}(z)$ of an integrable model, where z represents the spectral parameter.

1. R-Matrix

Let us define the action of the operator $R_{jk}(z)$ on a function $f(\Lambda)$ by[22]

$$(R_{jk}(z)\circ f)(\ldots,\Lambda_j,\Lambda_k,\ldots) \equiv \int d\Lambda_\sharp d\Lambda_\flat \mathcal{A}_4(z)(\Lambda_j,\Lambda_k,i\Lambda_\sharp,i\Lambda_\flat)f(\ldots,\Lambda_\flat,\Lambda_\sharp,\ldots). \tag{3.54}$$

An important property of this operator $R_{jk}(z)$ is that it preserves the Yangian invariance when applied to a Yangian-invariant function. To show this, first note that the expression (3.54) can be rewritten as

$$(R_{jk}(z)\circ f)(\ldots,\Lambda_j,\Lambda_k,\ldots) = \int d^{2|3}\Lambda_\sharp d^{2|3}\Lambda_\flat d^{2|3}\Lambda_\natural d^{2|3}\Lambda_\diamond \delta^{2|3}(\Lambda_\natural - i\Lambda_\sharp)\delta^{2|3}(\Lambda_\diamond - i\Lambda_\flat) \times \mathcal{A}_4(z)(\Lambda_j,\Lambda_k,\Lambda_\natural,\Lambda_\diamond)f(\ldots,\Lambda_\flat,\Lambda_\sharp,\ldots). \tag{3.55}$$

The expression (3.55) is a combination of the Yangian-preserving operations discussed in Sec. III B: We first take the product of two Yangian invariants, $\mathcal{A}_4(\Lambda_j,\Lambda_k,\Lambda_\natural,\Lambda_\diamond)$ and $f(\Lambda_\flat,\Lambda_\sharp)$, and then glue these objects by using the two delta-function identifications $\Lambda_\natural = i\Lambda_\sharp$ and $\Lambda_\diamond = i\Lambda_\flat$. This implies that $R_{jk}(z)$ as defined in (3.54) preserves the Yangian invariance. Recall that $\mathcal{A}_4(z)(\Lambda_j,\Lambda_k,\Lambda_\natural,\Lambda_\diamond)$ is invariant under the Yangian with evaluation parameters that satisfy $u_j = u_\natural$, $u_k = u_\diamond$, and $z = \pm(u_j - u_k)$ according to (3.23). In addition, the gluing conditions (3.16) require $u_\natural = u_\sharp$ and $u_\diamond = u_\flat$. Hence the action of R_{jk} permutes the evaluation parameters u_j and u_k: If $f(\ldots,\Lambda_j,\Lambda_k,\ldots)$ is Yangian invariant with $\vec{u} = (\ldots,u_j,u_k,\ldots)$, then $(R_{jk}\circ f)\times(\ldots,\Lambda_j,\Lambda_k,\ldots)$ is invariant with $\vec{u} = (\ldots,u_k,u_j,\ldots)$. In other words, we have

$$[\mathfrak{J}^a, R_{jk}(z)] = 0,$$
$$\hat{\mathfrak{J}}^a(\ldots,u_j,u_k,\ldots)R_{jk}(z) = R_{jk}(z)\hat{\mathfrak{J}}^a(\ldots,u_k,u_j,\ldots), \tag{3.56}$$

when acting on Yangian-invariant functions. Here the number of legs in the definition of the generators $\mathfrak{J}^a$ and $\hat{\mathfrak{J}}^a$ [cf. (2.1)] depends on the number of legs of the invariant acted on. However, since all other terms commute trivially, the Yangian generators $\mathfrak{J}^a$ and $\hat{\mathfrak{J}}^a$ in (3.56) reduce to the two-site Yangian generators with evaluation parameters (u_j,u_k). Note that invariance is only preserved when R_{jk} acts on adjacent legs of the invariant f.

For later purposes, let us simplify the definition of the R-matrix. Plugging in the definition of the four-point amplitude in (3.21), we obtain

$$(R_{jk}(z)\circ f)(\Lambda_j,\Lambda_k) = \int \frac{d\theta}{\sin(\theta)^{1+z}} \int d^{2|3}\Lambda_\sharp d^{2|3}\Lambda_\flat \delta^{4|6}(C(\theta) \times(\Lambda_j,\Lambda_k,i\Lambda_\sharp,i\Lambda_\flat))f(\Lambda_\flat,\Lambda_\sharp), \tag{3.57}$$

where the matrix $C(\theta)$ is defined as in (3.22). We can trivially solve the delta-function constraint for $\Lambda_\sharp,\Lambda_\flat$, giving rise to

$$(R_{jk}(z)\circ f)(\Lambda) \equiv \int \frac{d\theta}{\sin(\theta)^{1+z}} f(\Lambda)|_{\substack{\Lambda_j \to +\sin(\theta)\Lambda_k + \cos(\theta)\Lambda_j, \\ \Lambda_k \to -\cos(\theta)\Lambda_k + \sin(\theta)\Lambda_j}}. \tag{3.58}$$

2. RLL relation

The discussion in the previous sections can be nicely reformulated in the language of integrable models. To explain this, let us first define the L-operator $L_i(u)$ by

$$L_i(u) \equiv u\mathbf{1} + \sum_a \mathfrak{J}_i^a e_a, \tag{3.59}$$

where $\mathfrak{J}_i^a$ are the level-zero generators for the representation of the particle i, and e_a denotes the generators of the fundamental representation. Let us also define the monodromy operator by

$$T(u_0,\vec{u}) \equiv L_1\left(u_0 - \frac{1}{2}u_1\right)L_2\left(u_0 - \frac{1}{2}u_2\right)\ldots L_{2k}\left(u_0 - \frac{1}{2}u_{2k}\right). \tag{3.60}$$

By standard procedure,[23] expanding the monodromy yields the Yangian generators:

[22] Since OG_2 has two branches, we correspondingly have two different R-matrices $R^\pm(z)$. In the following we concentrate on one of the branches, say $R^+(z)$. Note that the actual undeformed 4-point amplitude is a linear combination of two contributions from two kinematical branches $R^\pm(z)$, each contribution being separately Yangian-invariant (see, however, the comments on the discussion of the collinear anomaly in Sec. IV).

[23] See e.g. [30].

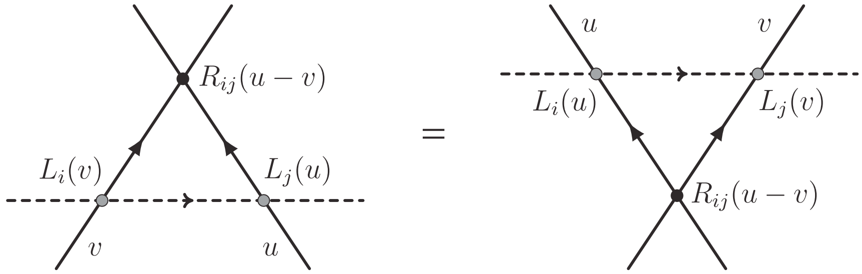


FIG. 3. The graphical representation of the RLL relation. The R-matrix is associated with an intersection of two undotted lines, while the L-matrix with that of an undotted line and a dotted line. The spectral parameters are associated with the particle lines, which will be identified with the rapidity lines of integrable models.

$$T(u_0,\vec{u}) = \sum_{n=0}^{2k} u_0^{2k-n}\mathfrak{J}^{(n-1)}(\vec{u}), \tag{3.61}$$

where $\mathfrak{J}^{(n)}(\vec{u})$ is (up to additive constants and combinations of lower-level generators) the level-n generator with evaluation parameters $\vec{u}$, namely[24]

$$\mathfrak{J}^{(-1)} = \mathbf{1}, \qquad \mathfrak{J}^{(0)}(\vec{u}) = \mathfrak{J}^a \boldsymbol{e}_a,$$
$$\mathfrak{J}^{(1)}(\vec{u}) = \frac{1}{2}\left(\hat{\mathfrak{J}}^a \boldsymbol{e}_a + \mathfrak{J}^a\boldsymbol{e}_a\mathfrak{J}^b\boldsymbol{e}_b - \alpha\mathfrak{J}^a\boldsymbol{e}_a + \frac{1}{2}\sum_{i<j}u_iu_j\mathbf{1}\right). \tag{3.62}$$

Here, the constant α stems from the single-site relation

$$\mathfrak{J}_i^a\boldsymbol{e}_a\mathfrak{J}_i^b\boldsymbol{e}_b = \alpha\mathfrak{J}_i^a\boldsymbol{e}_a. \tag{3.63}$$

This relation is representation-dependent, but holds for the fundamental representation. It ensures that the Yangian generators obey the Serre relations [18,31]. Now we can use (3.61) to encode (3.56) into the RLL relation

$$R_{ij}(u_j-u_i)L_i\left(u_0-\frac{1}{2}u_i\right)L_j\left(u_0-\frac{1}{2}u_j\right) = L_i\left(u_0-\frac{1}{2}u_j\right)L_j\left(u_0-\frac{1}{2}u_i\right)R_{ij}(u_j-u_i), \tag{3.64}$$

see Fig. 3. This equation is the so-called RLL relation, which is one version of the Yang-Baxter relation often found in integrable models [32]. As we have seen, this relation encodes the fact that $R_{ij}(z)$ preserves the Yangian invariance. The relation (3.64) holds when the operators act in the space of Yangian-invariant functions; this will be sufficient for the construction of Yangian invariants in the later part of this section.[25]

3. Yang-Baxter Equation

The relation (3.64) means that the R-operator is the intertwiner for the tensor product of representations of $Y[\mathfrak{osp}(6|4)]$. In particular, consistency with the associativity of the tensor product is guaranteed by the Yang-Baxter relation:

$$R_{ij}(w-v)R_{j\ell}(w-u)R_{ij}(v-u) = R_{j\ell}(v-u)R_{ij}(w-u)R_{j\ell}(w-v). \tag{3.65}$$

For our R-matrix (3.58), this can be shown to hold by direct computation: The θ-rotation in (3.58) is a rotation in the (Λ_i,Λ_j)-plane, and both sides of (3.65) give rise to a parametrization of the rotation group in terms of Euler angles, which are thus related by a coordinate transformation. One can verify explicitly that the product of measure factors in the integrals is kept invariant by the transformation.

This result shows that the R-operator (3.58) gives the R-matrix for a representation of $Y[\mathfrak{osp}(6|4)]$. It can be written in the Graßmannian integral form, a fact which we have not found in the literature. It would be nice to compare our expression for the R-matrix with the known expressions in the literature.

4. Yangian invariants

Having understood the four-vertex, the next task is to understand the more complicated on-shell diagrams obtained by fusing lines. In the language of R-matrices, this can be reformulated as the statement that higher-point invariants are obtained by iterated action with the R-matrix on vacuum delta functions. This is similar to the $\mathcal{N}=4$ case [see the discussion around (2.18)].[26]

[24] Recall the constraint $\sum_i u_i = 0$.

[25] We thank Carlo Meneghelli for helpful comments on the relation between (3.64) and (3.56).

[26] The connection to Yangians and integrable models is more direct for ABJM theory than for the $\mathcal{N}=4$ theory. For ABJM theory, the R-matrix (3.54) coming from the BCFW shift directly gives the R-matrix for (a representation of) the Yangian. By contrast the operator coming from the BCFW shift in $\mathcal{N}=4$ SYM is the 3-point amplitude, while the R-matrix for the Yangian corresponds to a 4-point amplitude, which is obtained by combining four BCFW operators.

To explain this, let us start with an on-shell diagram, i.e. a set of k lines connecting $2k$ points on a circle, such that no three lines intersect at the same point. By following each line, we obtain a permutation σ of order two, $\sigma^2 = 1$. That is, σ decomposes into k commuting transpositions of two elements: $\sigma = \sigma_k \ldots \sigma_1$, with each $\sigma_j = [a_j, b_j]$. Just as the diagram itself, also the permutation is kept invariant under triangle moves. While every diagram has a unique associated permutation, the converse is only true for reduced diagrams. Every permutation of order two uniquely specifies a reduced diagram (modulo triangle moves), but the same permutation is associated to an infinite number of inequivalent unreduced diagrams.[27]

Now, every on-shell diagram can be obtained from a k-line diagram without any four-vertex (a "vacuum diagram") by a sequence of BCFW bridges [9,16]. Let us first restrict to reduced diagrams, which are uniquely specified by their permutation. Starting from the "vacuum permutation"[28]

$$\sigma_{\mathrm{vac}} = [12][34]\ldots[2k-1, 2k], \tag{3.66}$$

we can arrive at any other permutation σ (representing a reduced on-shell diagram) by a sequence of BCFW bridges: Each BCFW bridge lets two adjacent legs intersect, and thus conjugates the associated permutation by a transposition. Hence

$$\sigma = \sigma_{\mathrm{R}} \sigma_{\mathrm{vac}} \sigma_{\mathrm{R}}^{-1}, \qquad \sigma_{\mathrm{R}} = [i_\ell, j_\ell]\ldots[i_1, j_1], \tag{3.67}$$

with each $[i_m, j_m]$ being a transposition of two *adjacent* elements.

Let us translate this into the language of integrable models. With the vacuum permutation σ_{vac} we associate an amplitude which is given by a product of delta functions

$$\Omega_{2k} \equiv \prod_{j=1}^{k} \delta^{2|3}(\Lambda_{2j-1} + i\Lambda_{2j}). \tag{3.68}$$

Next, each BCFW bridge is represented by the R-matrix acting on the two respective lines, as we have already seen. This means that a general Yangian invariant, corresponding to a general reduced on-shell diagram described by σ, can be obtained by acting with a chain of R-matrices on the vacuum amplitude:

$$\mathcal{Y}_\sigma(z_1, \ldots, z_\ell) = R_{\sigma_{\mathrm{R}}}(\vec{z})\Omega_{2k} = R_{i_\ell, j_\ell}(z_\ell)\ldots R_{i_1, j_1}(z_1)\Omega_{2k}, \tag{3.69}$$

where the sequence of transpositions $\sigma_{\mathrm{R}} = [i_\ell, j_\ell]\ldots[i_1, j_1]$ is defined by (3.67).

[27] Among all the diagrams associated to a given permutation, the reduced diagram is the one that has minimal degree (number of integration variables).

[28] The choice of vacuum is not unique, but one can restrict to this particular choice [16].

For a given permutation σ (of order two), both the choice of σ_{R} and its decomposition into adjacent transpositions are not unique. Two permutations σ_{R}, σ_{R}' lead to the same permutation σ if and only if $\sigma_{\mathrm{R}}' = \sigma_{\mathrm{R}}\sigma'$, where σ' is in the centralizer $\mathrm{C}(\sigma_{\mathrm{vac}})$. Hence the distinct permutations σ are in one-to-one correspondence with the elements of the coset $\mathrm{S}_n/\mathrm{C}(\sigma_{\mathrm{vac}})$. The ambiguity in the decomposition of σ_{R} into adjacent transpositions is due to the permutation group relation

$$[i, i+1][i+1, i+2][i, i+1] = [i+1, i+2][i, i+1][i+1, i+2]. \tag{3.70}$$

In terms of invariants (3.69), this identity amounts to the triangle move alias Yang-Baxter equation (3.65). Also the ambiguity in the choice of σ_{R} is due to this relation, in this case applied to the full permutation σ. Hence, for a given permutation σ, the invariant (3.69) is independent of the choice and decomposition of σ_{R}.

To summarize: For every reduced on-shell diagram, there is a decomposition of the associated permutation into adjacent transpositions that encodes the chain of R-matrices that need to act on the appropriate vacuum to reconstruct the diagram. Even though the decomposition into transpositions is ambiguous, the invariant is unique.

Unreduced on-shell diagrams are not uniquely specified by their associated permutation. Nevertheless, they are constructed just as reduced diagrams, by successively applying BCFW bridges. Hence also unreduced diagrams can be written as a chain of R-matrices that act on a vacuum amplitude,

$$\mathcal{Y}_{[i_\ell, j_\ell], \ldots, [i_1, j_1]}(z_1, \ldots, z_\ell) = R_{i_\ell, j_\ell}(z_\ell)\ldots R_{i_1, j_1}(z_1)\Omega_{2k}. \tag{3.71}$$

Here, the sequence of transpositions $[i_\ell, j_\ell], \ldots, [i_1, j_1]$ is sufficient to define the diagram, even though the resulting permutation $\sigma = [i_\ell, j_\ell]\ldots[i_1, j_1]$ is not.

Our claim here is that (3.69),(3.71) are indeed Yangian invariant if the spectral parameters z_i are constrained to obey the relation (3.23), that is

$$z_m = \pm(u_{j_m} - u_{i_m}). \tag{3.72}$$

Since the overall shift of u_i is irrelevant, only $(k-1)$ of these parameters are independent.

5. Yangian invariance

Let us prove the Yangian invariance of (3.69). Note that the invariance already follows from the fact that the action of an R-matrix is equivalent to gluing an invariant four-vertex to another invariant (Sec. III B). The purpose of this section is to recast the argument in a form closer to standard integrable models.

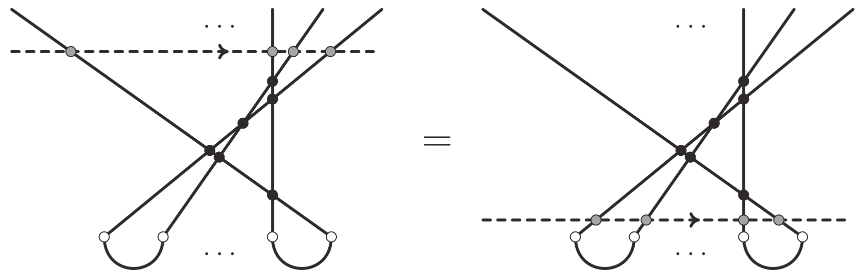


FIG. 4. Graphical proof of the Yangian invariance, c.f. (3.75). Using the RLL relations of Figure 3 multiple times, we can commute the product of L-operators (gray dots) through the product of R-matrices (black dots) when acting on the vacuum (3.68) (half circles). In the above example we have $k=2$, i.e. $2k=4$ external points, and the Yangian invariant (3.69) contains six R-matrices. The "…" represent additional lines that can be added in general.

Since the monodromy operator is the generating function of the Yangian generators, Yangian invariance is equivalent to the statement that (3.69) is an eigenfunction of the monodromy operator:

$$T(u_0,\vec{u})\mathcal{Y}_\sigma(\vec{z}) = \prod_{i=1}^{2k}\left(u_0 - \frac{1}{2}u_i\right)\mathcal{Y}_\sigma(\vec{z}). \tag{3.73}$$

where u_0 is arbitrary and $\vec{u}$ are fixed to be the evaluation parameters of the Yangian representation. On the left-hand side we can express $T(u_0,\vec{u})$ as a product of the L-operators, and then, due to (3.72), commute with the R-matrices with the help of the RLL relation (3.64):

$$L_1\left(u_0-\frac{1}{2}u_1\right)\ldots L_{2k}\left(u_0-\frac{1}{2}u_{2k}\right)R_{ab}(\vec{z}) = R_{ab}(\vec{z})L_1\left(u_0-\frac{1}{2}u_{[ab](1)}\right)\ldots L_{2k}\left(u_0-\frac{1}{2}u_{[ab](2k)}\right) \tag{3.74}$$

for any transposition $[ab]$ of two adjacent elements.[29] By induction we obtain (see Fig. 4 for a graphical representation)

$$\begin{aligned} T(u_0,\vec{u})\mathcal{Y}_\sigma(\vec{z}) &= T(u_0,\vec{u})R_{\sigma_{\mathrm{R}}}(\vec{z})\Omega_{2k} \\ &= L_1\left(u_0-\frac{1}{2}u_1\right)\ldots L_{2k}\left(u_0-\frac{1}{2}u_{2k}\right)R_{\sigma_{\mathrm{R}}}(\vec{z})\Omega_{2k} \\ &= R_{\sigma_{\mathrm{R}}}(\vec{z})L_1\left(u_0-\frac{1}{2}u_{\sigma_{\mathrm{R}}(1)}\right)\ldots L_{2k}\left(u_0-\frac{1}{2}u_{\sigma_{\mathrm{R}}(2k)}\right)\Omega_{2k}. \end{aligned} \tag{3.75}$$

Recalling the definition (3.68) of Ω_{2k}, this becomes

$$T(u_0,\vec{u})\mathcal{Y}_\sigma(\vec{z}) = R_{\sigma_{\mathrm{R}}}(\vec{z})\prod_{j=1}^{k}L_{2j-1}\left(u_0-\frac{1}{2}u_{\sigma_{\mathrm{R}}(2j-1)}\right)L_{2j}\left(u_0-\frac{1}{2}u_{\sigma_{\mathrm{R}}(2j)}\right)\delta^{2|3}(\Lambda_{2j-1}+i\Lambda_{2j}). \tag{3.76}$$

After some algebra, and using the relation (3.63), each factor in the product takes the form

$$L_i(u)L_j(v)\delta^{2|3}(\Lambda_i+i\Lambda_j) = \left[uv + (u\mathfrak{J}_j^a + v\mathfrak{J}_i^a)\boldsymbol{e}_a + \frac{1}{2}\mathfrak{J}^a\boldsymbol{e}_a\mathfrak{J}^b\boldsymbol{e}_b - \frac{1}{2}\alpha\mathfrak{J}^a\boldsymbol{e}_a + \frac{1}{2}\mathfrak{J}_i^a\mathfrak{J}_j^b f_{ab}{}^c\boldsymbol{e}_c\right]\delta^{2|3}(\Lambda_i+i\Lambda_j), \tag{3.77}$$

where $\mathfrak{J}^a = \mathfrak{J}_i^a + \mathfrak{J}_j^a$ are the two-site level-zero generators. Now the third and fourth term in the bracket vanish by (3.17), and the last term vanishes due to the argument below (3.18): It is proportional to the dual Coxeter number, which is zero for $\mathfrak{osp}(6|4)$. Finally, the vanishing of the second term requires $u=v$. Hence, (3.76) equals (3.73) if and only if

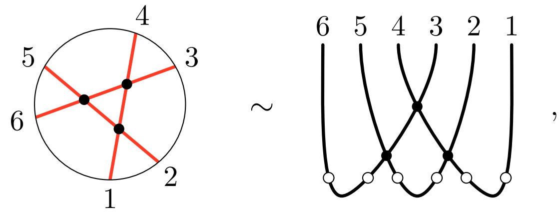
[29] Note that this argument does not work for the transposition $[2k,1]$, since the RLL relation does not apply in this case. However, one can always choose σ_{R} such that it does not act on the first and last legs (for any permutation σ). Hence at first sight, (3.75) only applies for such σ_{R}. However, different choices of σ_{R} are related to each other by triangle moves, and we know that triangle moves preserve diagrams, and hence also preserve Yangian invariance. Therefore, $R_{\sigma_{\mathrm{R}}}\Omega_{2k}$ is invariant for all σ_{R}. In fact, from the on-shell diagram point of view, the choice of first and last leg in the definition (3.60) of the monodromy matrix is arbitrary, and thus the invariance discussion should not depend upon this choice. Indeed one can show that the Yangian algebra is invariant under cyclic rotations of the chain of L-operators in (3.60) (for algebras with vanishing dual Coxeter number).

$$u_{\sigma_{\mathrm{R}}(2j-1)} = u_{\sigma_{\mathrm{R}}(2j)}, \qquad j = 1, \ldots, k. \tag{3.78}$$

Using (3.67),(3.66), one can easily see that these conditions are equivalent to the previously derived invariance constraints (3.27). In conclusion, Yangian invariance of (3.69) is recovered provided that the constraints (3.78) and (3.72) hold.

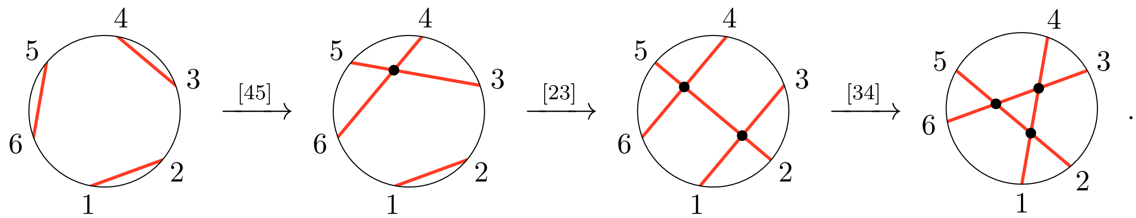
6. An example

Let us consider as an example the deformation of the six-point top cell. Like all other on-shell diagrams, it can be represented as a disk with intersecting lines that end at the boundary, e.g.



, (3.79)

where the second figure illustrates the relation to diagrams of the type shown in Fig. 4. This diagram corresponds to a permutation $\sigma = [14][25][36]$. As described above, we can build such a diagram from a vacuum diagram by applying successive transpositions:



. (3.80)

Above, each transposition labeled by $[ij]$ corresponds to applying an additional four-point vertex/R-matrix to the previous diagram. For the permutation, this means

$$\sigma = [34][23][45]\sigma_{\mathrm{vac}}[45][23][34], \qquad \sigma_{\mathrm{vac}} = [12][34][56]. \tag{3.81}$$

The vacuum amplitude (3.68) is simply given by

$$\Omega_6 = \delta^{2|3}(\Lambda_1 + i\Lambda_2)\delta^{2|3}(\Lambda_3 + i\Lambda_4)\delta^{2|3}(\Lambda_5 + i\Lambda_6). \tag{3.82}$$

We can then act iteratively with the R-operator (3.58) and the sequence in (3.80) translates into

$$R_{34}(z_3)R_{23}(z_2)R_{45}(z_1)\Omega_6. \tag{3.83}$$

The corresponding orthogonal Graßmannian C-matrix takes the form:

$$\begin{pmatrix} 1 & i\mathrm{c}_2 & i\mathrm{s}_2\mathrm{c}_3 & i\mathrm{s}_2\mathrm{s}_3 & 0 & 0 \\ 0 & \mathrm{s}_2 & -\mathrm{c}_2\mathrm{c}_3 + i\mathrm{c}_1\mathrm{s}_3 & -i\mathrm{c}_1\mathrm{c}_3 - \mathrm{c}_2\mathrm{s}_3 & i\mathrm{s}_1 & 0 \\ 0 & 0 & \mathrm{s}_1\mathrm{s}_3 & -\mathrm{s}_1\mathrm{c}_3 & -\mathrm{c}_1 & i \end{pmatrix}, \tag{3.84}$$

where $\mathrm{s}_j = \sin\theta_j$ and $\mathrm{c}_j = \cos\theta_j$. One can verify that this indeed corresponds to the top-cell diagram, as all consecutive minors are nonvanishing.

As we have seen above, diagrams that are equivalent under triangle moves simply correspond to using a different sequence of R-matrices that yields the same final permutation. In our example, the triangle move corresponds to the equivalence between (3.83) and

$$R_{61}(z_3)R_{23}(z_2)R_{45}(z_1)\Omega_6. \tag{3.85}$$

The corresponding matrix parametrizing the orthogonal Graßmannian is given by

$$\begin{pmatrix} -c_3 & ic_2 & is_2 & 0 & 0 & s_3 \\ 0 & s_2 & -c_2 & ic_1 & is_1 & 0 \\ -is_3 & 0 & 0 & s_1 & -c_1 & ic_3 \end{pmatrix}. \qquad (3.86)$$

Again one can verify that all adjacent minors are nonvanishing, and hence the top cell is recovered.

IV. DISCUSSION, CONCLUSIONS AND OUTLOOK

In this paper, we have considered integrable deformations of scattering amplitudes in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory and three-dimensional ABJM theory. We found a similar structure of deformed invariants for both theories, which incorporates the deformed Graßmannian integrals, the construction of deformed on-shell diagrams via gluing, as well as the algebraic R-matrix construction.

Interestingly, while part of the deformation parameters in four dimensions is associated with the violation of invariance under the local central charge generators of $\mathcal{N}=4$ SYM theory, there is no such central charge for the three-dimensional symmetry algebra $\mathfrak{osp}(6|4)$. Furthermore, the four-dimensional deformations remind of deformed helicities, but the external particles in ABJM do not carry helicity charges either. Nevertheless, we demonstrate that consistent deformations are possible in 3d ABJM theory, which can be attributed to the introduction of nontrivial evaluation parameters in the Yangian level-one generators. A local operator, similar to the central charge generator in four dimensions, is the $\mathbb{Z}_2$ phase of the three-dimensional little group in ABJM. We have briefly commented on this fermion number operator at the end of Sec. III B, whose breaking indicates the anyonic nature of the introduced deformations. We close with some comments on both cases, and discuss future directions.

Certainly the most pressing question is whether the deformations discussed here will be useful for computing loop amplitudes. Up to now, the deformations might mostly look like a mathematical curiosity. For instance, the famous BCFW decomposition cannot be deformed consistently, not even at tree level [10], cf. Table I.[30] Moreover, the deformed one-loop four-point amplitude in four dimensions generically integrates to zero: Only very special deformations give a nonvanishing result [10]. Still there are a few interesting approaches one might want to pursue. For example, noting that the four-point amplitude is maximally helicity violating, it is not excluded that suitable deformations will be useful for computing the ratio function of [33]. Perhaps most promising is the idea to give up on a BCFW-like decomposition, and to interpret the deformed top cell, or equivalently the deformed Graßmannian integral, as the complete deformed amplitude. The challenge is to find a suitable contour on which the deformed top cell integrates to a useful function. As proposed in [34], one could try to require that the integrated result is meromorphic in the deformation parameters. Interestingly, irrespective of the contour, the resulting function will have deformed helicities for the external legs. This points towards a possible connection with continuous-spin theories proposed recently [35].

TABLE I. Comparison of the deformation degrees of freedom. Each Yangian invariant n-leg diagram in four or three dimensions has $n-1$ or $\frac{n}{2}-1$ free parameters, respectively. Scattering amplitudes on the other hand may be BCFW composed of several diagrams. Requiring that the external data of all diagrams contributing to a certain amplitude is the same, this imposes stronger constraints, which in general result in less degrees of freedom. In four dimensions the numbers were checked explicitly up to $n=16$. The numbers in ABJM, and at higher n in four dimensions, result from the naive counting of degrees of freedom and constraints; for larger n the constraints outweigh the parameters and no degrees of freedom remain (beyond the MHV sector in 4d).

Yangian invariant	$\mathcal{N}=4$ SYM theory	$\mathcal{N}=6$ SCS theory
n-leg amplitude	MHV, …, $\overline{\text{MHV}}$	
3	2, 2	– – –
4	3	1
5	4, 4	– – –
6	5, 1, 5	2
7	6, 0, 0, 6	– – –
8	7, 0, 0, 0, 7	1
9	8, 0, 0, 0, 0, 8	– – –
$n\geq 10$	$n-1, 0, \ldots, 0, n-1$	0
n-leg diagram	$n-1$	$\frac{n}{2}-1$

Note that the form of the (deformed) Graßmannian integrals mainly follows from the symmetry structure of the underlying gauge theory. This suggests to identify a similar Graßmannian integral also in other theories, for instance in two and six dimensions, where much less is known about scattering amplitudes. In particular, for the two-dimensional theories with an AdS_3 string dual, it would be interesting to initiate the study of scattering amplitudes based on the symmetry algebras in analogy to the steps taken in [18] for three dimensions. Comparison should then allow us to write down a Graßmannian integral and to study amplitude-like symmetry invariants. This could be helpful to make progress on understanding the gauge-theory duals of these AdS_3 string theories.

Importantly, here—as in all previous considerations of the deformed scattering invariants—we have not

[30]In principle, one could consider deforming each term in the BCFW decomposition with a different set of central charges c_i, as long as the Yangian evaluation parameters u_i remain universal. Empiric case studies for higher multiplicities (up to $n=18$) and helicities (up to $k=5$) suggest that such deformations are admissible for generic amplitudes. However, while a plethora of deformation parameters remains unconstrained, the physical interpretation and the practical use of such deformations remains unclear.

considered the exact symmetry generators of $Y[\mathfrak{psu}(2,2|4)]$ and $Y[\mathfrak{osp}(6|4)]$. That is we have ignored the fact that at collinear momentum configurations the above symmetry generators do not annihilate the tree-level S-matrices, but have to be corrected due to the collinear anomaly [4,28,36,37]. It is known that these collinear contributions recursively relate amplitudes with different numbers of external legs to each other and it might be very enlightening to see whether these relations impose further constraints on the deformation parameters. In this context, it is interesting to note that the vacuum (3.68) of the algebraic R-matrix construction of invariants in ABJM theory is a product of two-point invariants that might be the necessary starting point to render the recursive symmetry in three dimensions exact, cf. the discussion in [28].

As mentioned in Sec. I, the study of deformed scattering amplitudes in four dimensions was motivated by the map between the one-loop dilatation operator and the four-point scattering amplitude of $\mathcal{N}=4$ SYM theory [6]. Construction of the amplitude form of the associated R-matrix then led to the introduction of a (spectral) deformation parameter [7,8]. In this paper we have introduced the deformation of scattering amplitudes in three dimensions. It would be interesting to see how the dilatation operator of ABJM theory can be constructed from deformed amplitudes or on-shell diagrams.

It would also be interesting to further explore the similarities with the integrable structures discussed in the context of 4d $\mathcal{N}=1$ quiver gauge theories [38]. Our discussion of ABJM scattering amplitudes suggests the existence of a new 3d duality associated with the triangle move, cf. (3.26).

Another notable question concerns the bonus symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM theory found in [39]. Is this symmetry still preserved and what is its role for the deformations of four-dimensional scattering amplitudes? Since the generator of this level-one symmetry is bilocal in both the ordinary and the dual conformal coordinates, and acts as a raising operator for the Yangian levels, this might also clarify the relation between the Yangian generators in twistor and momentum-twistor space discussed in Sec. II C, which remains an interesting open problem. Finally, studying these issues could shed light on the existence of a similar symmetry in ABJM theory.

Lastly, an important question is whether we can incorporate the above deformations into other approaches like the amplituhedron of [40]. Studying this question would be a good opportunity to elucidate the fate of Yangian symmetry in the amplituhedron. Another recent development to use integrability for the computation of scattering amplitudes is the nonperturbative flux-tube formulation introduced in [41]. Since the approaches of [40] and [41] seem to bring many advantages over the previous methods, combining them with the deformation might be the most useful step in order to continue to investigate the role of Yangian symmetry and the impact of integrability for amplitudes in planar supersymmetric gauge theories.

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Note added.—While this manuscript was in preparation, we found out that the deformed Graßmannian integrals for $\mathcal{N}=4$ SYM theory (both in momentum space and in momentum-twistor space) were independently obtained by L. Ferro, T. Łukowski, and M. Staudacher [34]. We would like to thank them for correspondence, and for discussions during the Strings 2014 conference.

APPENDIX A: EXPLICIT $\mathfrak{osp}(6|4)$ GENERATORS

For reference, here we list the level-zero generators of the $\mathfrak{osp}(6|4)$ algebra in the singleton representation as given in [18]:

$$\begin{aligned}
&\mathfrak{L}^{\alpha}{}_{\beta} = \lambda^\alpha\partial_\beta - \frac{1}{2}\delta^\alpha_\beta\lambda^\gamma\partial_\gamma, \qquad \mathfrak{P}^{\alpha\beta} = \lambda^\alpha\lambda^\beta,\\
&\mathfrak{D} = \frac{1}{2}\lambda^\alpha\partial_\alpha + \frac{1}{2}, \qquad \mathfrak{K}_{\alpha\beta} = \partial_\alpha\partial_\beta,\\
&\mathfrak{R}^{AB} = \eta^A\eta^B, \qquad \mathfrak{R}^{A}{}_{B} = \eta^A\partial_B - \frac{1}{2}\delta^A_B, \qquad \mathfrak{R}_{AB} = \partial_A\partial_B,\\
&\mathfrak{Q}^{\alpha A} = \lambda^\alpha\eta^A, \qquad \mathfrak{S}^A_\alpha = \eta^A\partial_\alpha,\\
&\mathfrak{Q}^\alpha_A = \lambda^\alpha\partial_A, \qquad \mathfrak{S}_{\alpha A} = \partial_\alpha\partial_A.
\end{aligned} \tag{A1}$$

See Appendices F and G of [18] for the construction of the level-one Yangian generators.

APPENDIX B: YANGIAN INVARIANCE OF THE 4D DEFORMED GRAßMANNIAN INTEGRAL

In this appendix, we check the Yangian invariance of the deformed Graßmannian formula (2.22). The discussion is parallel to the case of the ABJM theory discussed in Sec. III. We only need to prove the invariance under level-zero and level-one generators, since all other generators can be obtained from the commutation relations.

The invariance under the level-zero is unaffected by the deformation: It simply follows from the fact that the level-zero generators (2.2) are realized linearly in the twistor variables $\mathcal{Z}_i$, and thus annihilate the delta function $\delta^{4k|4k}(C\cdot\mathcal{Z})$ present in the Graßmannian integral.

The invariance under the level-one generator will be verified below following the methods of [23]. Let us first rewrite $\hat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ as

$$\hat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}=\left(2\sum_{i<j}-\sum_{i,j}+\sum_{i=j}\right)\left(\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_j^{\mathcal{B}}}\mathcal{Z}_j^{\mathcal{C}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{C}}}-\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{B}}}\right)+\sum_i u_i\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{B}}}. \tag{B1}$$

The sum $\sum_{i,j}$ gives a square of the level-zero generator and acts trivially on the Graßmannian formula. In the sum $\sum_{i=j}$ on the other hand, we find the central charge operator $\mathfrak{C}_i=-\mathcal{Z}_i^{\mathcal{C}}\partial/\partial\mathcal{Z}_i^{\mathcal{C}}$ (2.4), which yields the following expression when acting on the Graßmannian integral:

$$2\sum_{i<j}\left(\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_j^{\mathcal{B}}}\mathcal{Z}_j^{\mathcal{C}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{C}}}-\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{B}}}\right)+\sum_i(u_i-c_i)\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial\mathcal{Z}_i^{\mathcal{B}}}. \tag{B2}$$

Now the crucial observation is that the operator $\mathcal{Z}_j^{\mathcal{C}}\partial/\partial\mathcal{Z}_i^{\mathcal{C}}$, when acting on the delta functions, can be replaced by a $\mathrm{GL}(k)$-rotation on the rows of the matrix C [23].

To do this properly, we need to fix the $\mathrm{GL}(k)$-gauge ambiguity as[31]

$$C=\begin{pmatrix}1&\cdots&0&C_{1,k+1}&\cdots&C_{1,n}\\0&\cdots&0&\vdots&\vdots&\vdots\\0&\cdots&1&C_{k+1,n}&\cdots&C_{k,n}\end{pmatrix}. \tag{B3}$$

The operator $\mathcal{Z}_j^{\mathcal{C}}\partial/\partial\mathcal{Z}_i^{\mathcal{C}}$ then can be replaced by a $\mathrm{GL}(k)$-rotation on the row of the non-gauge-fixed part of the matrix C. It then follows that

$$\hat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{G}_{k,n}=\sum_b\int\frac{\prod_{a=1}^k\prod_{m=k+1}^n dC_{am}}{M_1^{1+b_1}\dots M_n^{1+b_n}}\times[\mathcal{N}_b^{\mathcal{A}}-\mathcal{V}_b^{\mathcal{A}}+\mathcal{U}_b^{\mathcal{A}}](\partial_{\mathcal{B}}\delta_b)\prod_{a\neq b}\delta_a, \tag{B4}$$

[31]For the deformed amplitude it is crucial to fix the $\mathrm{GL}(k)$-ambiguity to obtain correct identification of deformation parameters. This contrasts with the case of the undeformed case, where the formal analysis without fixing the $\mathrm{GL}(k)$-ambiguity also gives the Yangian invariance of the amplitude [23].

where $\delta_a\equiv\delta^{4|4}(\mathcal{Z}_a+\sum_{l=k+1}^n C_{al}\mathcal{Z}_l)$,

$$\mathcal{N}_b^{\mathcal{A}}\equiv2\sum_{i<j}\mathcal{N}_{ij}\mathcal{Z}_i^{\mathcal{A}}C_{bj},\qquad\mathcal{V}_b^{\mathcal{A}}\equiv2\sum_{i<j}\mathcal{Z}_i^{\mathcal{A}}C_{bi},\qquad\mathcal{U}_b^{\mathcal{A}}\equiv\sum_i u_i^-\mathcal{Z}_i^{\mathcal{A}}C_{bi}. \tag{B5}$$

Here, $u_i^-=u_i-c_i$ as before, and the operator $\mathcal{N}_{ij}$ is a gauge-fixed version of the operator $\sum_{a=1}^k C_{ai}\partial/\partial C_{aj}$ [23].

We can integrate by parts for the operator $\mathcal{N}_b^{\mathcal{A}}$. The operator annihilates the measure, but acts nontrivially on the minors $M_i(C)$. Generalizing the commutation relations of [23], we find 0

$$\left[\frac{1}{M_1^{1+b_1}\dots M_n^{1+b_n}},\mathcal{N}_b^{\mathcal{A}}\right]=\frac{1}{M_1^{1+b_1}\dots M_n^{1+b_n}}\sum_{i<j}(1+b_j)Z_i^{\mathcal{A}}C_{bi}, \tag{B6}$$

and therefore

$$\hat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{A}_{k,n}=\int\frac{\prod_{a=1}^k\prod_{m=k+1}^n dt_{am}}{M_1^{1+b_1}\dots M_n^{1+b_n}}\times\sum_b\left[2\sum_{i<j}b_jZ_i^{\mathcal{A}}C_{bi}+\sum_i u_i^-Z_i^{\mathcal{A}}C_{bi}\right](\partial_{\mathcal{B}}\delta_b)\times\prod_{a\neq b}\delta_a. \tag{B7}$$

Requiring that the coefficient of $Z_i^{\mathcal{A}}C_{bi}$ is a constant, we thus find the invariance constraints

$$\sum_{j=i+1}^n 2b_j+u_i^-=\mathrm{const} \tag{B8}$$

for all $i=1,\dots,n$, and where the constant is independent of i. In other words we have

$$b_i=\frac{1}{2}(u_i^--u_{i-1}^-), \tag{B9}$$

which agrees with the constraints in (2.23).

APPENDIX C: YANGIAN INVARIANCE OF THE 3D GRAßMANNIAN INTEGRAL

In this appendix, we prove the Yangian invariance of the (undeformed) orthogonal Graßmannian integral:

$$\mathcal{G}_{2k}=\int\frac{d^{k\times2k}C}{|\mathrm{GL}(k)|}\frac{\delta^{k(k+1)/2}(C\cdot C^{\mathsf{T}})\delta^{2k|3k}(C\cdot\Lambda)}{\prod_{i=1}^k M_i(C)}. \tag{C1}$$

To show that the above integral is invariant under the level-one generator in (3.37), we begin by rewriting again

$$(-)^{|\mathcal{C}|}\Lambda_l^{(\mathcal{A}}\frac{\partial}{\partial\Lambda_l^{\mathcal{C}}}\Lambda_i^{\mathcal{C}}\Lambda_i^{\mathcal{B}]}=\Lambda_l^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}\Lambda_i^{\mathcal{C}}\frac{\partial}{\partial\Lambda_l^{\mathcal{C}}}\equiv\Lambda_l^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}\mathcal{O}_i{}^l, \quad \text{(C2)}$$

where $\mathcal{O}_i{}^l$ is simply an $\mathrm{GL}(2k)$-rotation on the external data Λ_i, and we again can conveniently rewrite the action of the first term in (3.37) as

$$\sum_{l<i}(\Lambda_l^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}\mathcal{O}_i{}^l-\Lambda_i^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}\mathcal{O}_l{}^i)\delta^{2|3}(C\cdot\Lambda)$$
$$=\sum_{l<i}\Lambda_l^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}(O_l{}^i-O_i{}^l)\delta^{2|3}(C\cdot\Lambda), \quad \text{(C3)}$$

where $O_i{}^l$ is defined in (3.39), and in obtaining the last line we have used the fact that the indices of the level-one generators under consideration are (anti-)symmetrized. Since the operator in the square bracket is in fact a $\mathrm{O}(2k)$-rotation, after integration by parts it vanishes when acting on the $\mathrm{O}(2k)$ invariant constraint $\delta(C\cdot C^{\mathsf{T}})$. Thus the only contribution we receive after integration by parts is when the linear operator acts on the minors:

$$\sum_{l<i}O_i{}^lM_p=\sum_{l=p}^{p+k-1}\sum_{i=p+k}^{2k}M_p^{l\to i},$$
$$\sum_{l<i}O_l{}^iM_p=\sum_{l=1}^{p-1}\sum_{i=p}^{p+k-1}M_p^{i\to l}. \quad \text{(C4)}$$

Finally, since on the support of $\delta(C\cdot C^{\mathsf{T}})$, the matrix C is a collection of null k-planes in a $2k$-dimensional space, one can define a set of dual k-planes to construct $\hat{C}$ such that [17]

$$\hat{C}\cdot\hat{C}^{\mathsf{T}}=0, \qquad C\cdot\hat{C}^{\mathsf{T}}=\hat{C}\cdot C^{\mathsf{T}}=I_{k\times k}. \quad \text{(C5)}$$

Note that due to (C5), one can immediately deduce

$$C^{\mathsf{T}}\cdot\hat{C}+\hat{C}^{\mathsf{T}}\cdot C=I_{2k\times 2k}. \quad \text{(C6)}$$

This is a useful identity, since we can now rewrite

$$\Lambda_i^{\mathcal{A}}=\sum_{j=1}^{2k}\Lambda_j^{\mathcal{A}}\sum_a(C_{ja}\hat{C}_{ia}+\hat{C}_{ja}C_{ia}). \quad \text{(C7)}$$

On the support of $\delta^{2|3}(C\cdot\Lambda)$, the first term vanishes. Using this result, with $p\le k$, we find that

$$\sum_{l<i}\Lambda_i^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}O_i{}^lM_p=\sum_{l<i}\sum_{j=1}^{2k}\Lambda_i^{(\mathcal{A}}\Lambda_j^{\mathcal{B}]}\sum_a(\hat{C}_{ja}C_{la})O_i{}^lM_p$$
$$=\sum_{j=1}^{2k}\sum_{i=p+k}^{2k}\Lambda_i^{(\mathcal{A}}\Lambda_j^{\mathcal{B}]}\sum_a\hat{C}_{ja}\sum_{l=p}^{p+k-1}M_p^{l\to i}C_{la}$$
$$=\sum_{i=p+k}^{2k}\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}M_p, \quad \text{(C8)}$$

where a k-term Schouten identity was used in the last line, as well as the completeness relation in (C6). This leads to the following rewriting of the first term in (C3):

$$\sum_{l<i}\Lambda_i^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}O_i{}^l\frac{1}{\prod_{j=1}^kM_j}=-\frac{(\sum_{l=1}^k\sum_{i=l+k}^{2k}\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]})}{\prod_{j=1}^kM_j}$$
$$=\frac{1}{\prod_{j=1}^kM_j}\sum_{k\le l<i}\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}. \quad \text{(C9)}$$

Similarly, we find

$$\sum_{l<i}\Lambda_i^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}O_l{}^iM_p=\sum_{l<i}\sum_{j=1}^{2k}\Lambda_j^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}\sum_a\hat{C}_{aj}C_{ai}O_l{}^iM_p$$
$$=\sum_{l=1}^{p-1}\sum_{j=1}^{2k}\Lambda_j^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}\sum_a\hat{C}_{aj}\sum_{i=p}^{p+k-1}C_{ai}M_p^{i\to l}$$
$$=\sum_{l=1}^{p-1}\Lambda_l^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}M_p. \quad \text{(C10)}$$

Hence, for the second term in (C3) we now have

$$-\sum_{l<i}\Lambda_i^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}O_l{}^i\frac{1}{\prod_{j=1}^kM_j}=\frac{1}{\prod_{j=1}^kM_j}\sum_{l<i\le k}\Lambda_l^{(\mathcal{A}}\Lambda_l^{\mathcal{B}]}. \quad \text{(C11)}$$

Collecting these results, and using the level-zero constraint $\sum_i\Lambda_i^{\mathcal{A}}\Lambda_i^{\mathcal{B}}=0$, we find that (C9) and (C11) is exactly what is needed to cancel against the terms of the form $\Lambda_i\Lambda_i$ in (3.37). For example, for $n=2k=4$, the $\Lambda_i\Lambda_i$ terms in (3.37) are given by

$$\left(\sum_{l<i}-\sum_{i<l}\right)\frac{\Lambda_i^{(\mathcal{A}}\Lambda_i^{\mathcal{B}]}}{2}$$
$$=\frac{1}{2}(-3\Lambda_1^{(\mathcal{A}}\Lambda_1^{\mathcal{B}]}-\Lambda_2^{(\mathcal{A}}\Lambda_2^{\mathcal{B}]}+\Lambda_3^{(\mathcal{A}}\Lambda_3^{\mathcal{B}]}+3\Lambda_4^{(\mathcal{A}}\Lambda_4^{\mathcal{B}]})$$
$$=-3\Lambda_1^{(\mathcal{A}}\Lambda_1^{\mathcal{B}]}-2\Lambda_2^{(\mathcal{A}}\Lambda_2^{\mathcal{B}]}-\Lambda_3^{(\mathcal{A}}\Lambda_3^{\mathcal{B}]}. \quad \text{(C12)}$$

On the other hand (C9) and (C11) yield

$$(-\Lambda_3^{(\mathcal{A}}\Lambda_3^{\mathcal{B}]}-2\Lambda_4^{(\mathcal{A}}\Lambda_4^{\mathcal{B}]})+(\Lambda_1^{(\mathcal{A}}\Lambda_1^{\mathcal{B}]})$$
$$=3\Lambda_1^{(\mathcal{A}}\Lambda_1^{\mathcal{B}]}+2\Lambda_2^{(\mathcal{A}}\Lambda_2^{\mathcal{B}]}+\Lambda_3^{(\mathcal{A}}\Lambda_3^{\mathcal{B}]}. \quad \text{(C13)}$$

Indeed the above is what is necessary to cancel (C12). This completes the proof of the invariance of the orthogonal Graßmannian integral under the level-one generator in (3.37).

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