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SIMPLIFIED PRESENTATION OF HEAD TAIL TURBULENCE

by

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Contents

I Introduction:

A theory of "head tail turbulence" was developed in order to understand the transverse single bunch PETRA instability. (1,2) In this note a short review of the observed properties of the PETRA instability is given, the idea of the "head tail turbulence" model is described and the results are presented.

In addition a more "physical" interpretation of head tail "turbulence" is discussed.

II Observed properties of the transverse PETRA instability

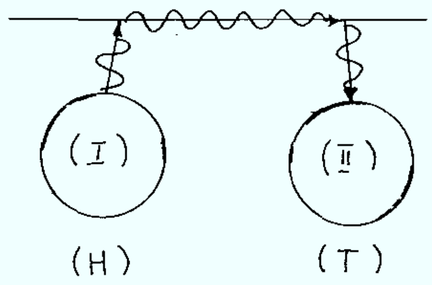
1.) The instability is a single bunch instability in the vertical plane.
2.) The instability occurs even at zero chromaticity.
3.) No dipole oscillations are excited.
4.) The strength of the instability depends on the number of r f cavities and the β_z-function in this area.
5.) The instability is weaker at higher synchrotronfrequencies.
6.) The instability is weaker at a larger bunchlength

III "Physical" approach to head tail turbulence

We consider a simplified model of a bunch consisting only of two rigid collectives (I) and (II) performing betatron- and synchrotron oscillations (fig. 1).[3]

The two collectives "interact" via the vacuum chamber. Due to causality collective (I) acts on collective (II) only if collective (I) is at the head (H) and collective (II) is at the tail of the bunch.
During a period of the synchrotron oscillation the positions of collective (I) and collective (II) are exchanged.

One can now think of a beam break-up within the bunch with an additional "long time" exchange between acting and reacting parts, leading to a successive exponential growth.



$f_s \mathrel{\hat{=}}$ synchrotron frequency

fig. 1

simplified model of a bunch

When collective (I) is "leading", the equations of motion can be written as:

$$
\begin{aligned}
(\mathrm{I}):&\quad \ddot{x} + \omega_\beta^2 x + 2\delta_\beta \dot{x} = 0 \\
(\mathrm{II}):&\quad \ddot{y} + \omega_\beta^2 y + 2\delta_\beta \dot{y} = k\cdot x
\end{aligned}
\qquad (1)
$$

where x,y are the transverse coordinates of the collectives, ω_β denotes the betatron circular frequency and δ_β is the radiation damping constant; kx is the force acting on (II) due to the deviation x of (I) (k is real because the chromaticity is assumed to be zero).

When collective (II) is leading after a half period of the synchrotron ascillation, equs. (1) approximately change into

$$(\mathrm{I}):\quad \ddot{x} + \omega_\beta^2 x + 2\delta_\beta \dot{x} = k\,y$$
$$(\mathrm{II}):\quad \ddot{y} + \omega_\beta^2 y + 2\delta_\beta \dot{y} = 0 \tag{2}$$

Due to the successive exchange of action and reaction, equs. (1), (2) can be treated as equations of external excitation.

For a general linear oscillator excited by an external force on resonance, the equation of motion reads

$$\ddot{z} + \omega^2 z + 2\delta\dot{z} = F(t)e^{i\omega t} + C.C. \tag{3}$$

If $Z \equiv 0$ at $t \leq 0$, and the force $F(t)$ jumps to its final (real) value F at $t = o$, the amplitude $\hat{Z}$ of the slution (3) is given by

$$\hat{Z}(t) = \frac{F}{2i\delta\omega}\left(1 - e^{-\delta t}\right), \tag{4}$$

where $\frac{F}{2i\omega\delta}$ is the asymptotie amplitude of forced motion.
For $\delta t \ll 1$ from (4) follows

$$\hat{Z}(t) \longrightarrow \frac{F}{2i\omega}\,t \tag{5}$$

According to (5) we obtain <u>recursion formulae</u> for the <u>successive amplitudes</u> in (1) and (2)

$$\begin{aligned} \hat{y}_{n+1} &= y_n + \frac{k}{2i\omega_\beta}\cdot\frac{1}{2f_s}\cdot x_n\,,^{*)} \\ \hat{x}_{n+1} &= x_n + \frac{k}{2i\omega_\beta}\cdot\frac{1}{2f_s}\cdot y_{n+1}\,,^{*)} \end{aligned} \tag{6}$$

putting $t = \frac{1}{2f_s}$.

We rewrite the system (6) in matrix form.

*) the radiation damping has been neglected in the first term on the r.h.s of equs. (6).

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{y}_{n+1} \end{pmatrix} = \underline{M}\begin{pmatrix} \hat{x}_n \\ y_n \end{pmatrix} \tag{7a}$$

with

$$\underline{M} = \begin{pmatrix} 1 - A^2 & -iA \\ -iA & 1 \end{pmatrix}, \quad A = \frac{k}{4\omega_\beta f_s} \tag{7b}$$

In order to investigate the properties of $\underline{M}^n$, we look for eigenvalues of $\underline{M}$

$$|\underline{M} - \lambda| = 0\,, \tag{8}$$

and we find:

$$\lambda = \tfrac{1}{2}\,\mathrm{Tr}\,\underline{M} \pm \sqrt{\left(\tfrac{1}{2}\mathrm{Tr}\,\underline{M}\right)^2 - 1} \tag{9}$$

because $\mathrm{Det}\underline{M} = 1$.
We consider two cases:

1.) $\left|\tfrac{1}{2}\mathrm{Tr}\,\underline{M}\right| \leq 1$

2.) $\left|\tfrac{1}{2}\mathrm{Tr}\,\underline{M}\right| > 1$

1.) In this case we write

$$\tfrac{1}{2}\,\mathrm{Tr}\,\underline{M} = \cos\mu \tag{10}$$

and therefore we can write:

$$\underline{M} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix} \tag{11}$$

which describes a real frequency shift.

This is plausible since for a short time interval $\frac{1}{2f_s}$ equs. (1) and (2) turn over in a system of coupled oscillators, leading only to a real frequency split.

2.) In this case we write

$$-\frac{1}{2}T_r\,\underline{M} = \cosh\mu \qquad (12)$$

$$\sqrt{\left(\tfrac{1}{2}T_r\underline{M}\right)^2-1} = \sinh\mu$$

with

$$\mu = \log\left\{\left|\tfrac{1}{2}T_r\underline{M}\right| + \sqrt{\left(\tfrac{1}{2}T_r\underline{M}\right)^2-1}\right\} > 0\,, \qquad (13)$$

so that one obtains

$$\underline{M} = -\begin{pmatrix} e^{\mu} & 0 \\ 0 & e^{-\mu} \end{pmatrix},$$

which leads to an exponential growth.

The condition $\left|\frac{1}{2}T_r\underline{M}\right| > 1$ implies

$$A > 2 \qquad (14)$$

or

$$\frac{k}{2\omega_\beta} > 2\,f_s \qquad (15)$$

Since k describes the "interaction" equ. (15) is just a condition similar to the condition of "head tail turbulence" derived in ref (1): If the driving term $\left(\frac{k}{2\omega_\beta}\right)$ exceeds the synchrotron frequency, an instability occurs (head tail turbulence).

IV "Theoretical" approach

A single head tail mode $\langle m\rangle$ is connected with a density function $f_m(\vartheta)$ describing the distribution of the transverse dipole moment along the longitudinal coordinate ϑ.

In the frequency domain $f_m(\vartheta)$ is a standing wave pattern composed of two travelling waves travelling in opposite directions:

$$\langle m\rangle \longleftrightarrow f_m(\vartheta)\,,$$

$$f_m(\vartheta) \longrightarrow \sum_{p>0}\left\{h_m(p)\,e^{ip\vartheta - i\omega t} + h_m(-p)\,e^{-ip\vartheta - i\omega t}\right\} \qquad (16)$$

$\omega \mathrel{\hat{=}}$ betatron frequency
$h_m(p) \mathrel{\hat{=}}$ spectrum function
$m \mathrel{\hat{=}}$ mode number

The deflecting forces produced by the interaction have different signs for the two different waves due to the different signs of the corresponding magnetic fields obeying Masewell's equations. Therefore the standing wave combination (16) is stable in the case of vanishing chromaticity.

For high intensities, however, different head tail modes are coupled, so that an unstable travelling wave can arise:

$$\alpha\langle m\rangle + \beta\langle m'\rangle \longrightarrow \sum_{p>0} h(p)\,e^{ip\vartheta - i\omega t} \qquad (17)$$

Since adjacent head tail mode frequencies differ by the synchrotron frequency f_s (i.e. coefficients α, β have different time dependence), the travelling wave (17) will be destroyd ("damped") in a time of the order of the synchrotron period. If the growth rate (related to driving terms) of the travelling wave exceeds the synchrotron frequency, an instability occurs. (1)

Applying the mathematical frame work of Vlasow equation theory for head tail mode coupling, the growth rate of the instability can be written as:

$$\delta = \frac{1}{2}\sqrt{4\,|M(J)|^2 - \omega_s^2}\;;\quad |M(J)| > \omega_s \qquad (18)$$

M (J) is a matrix element proportional to

1. bunch current
2. number of driving objects (transverse impedance)
3. β_z-function at the place of the driving objects (this is plausible because in a strong focussing machine any Q-shift is proportional to β-function)
4. M(J) depends on the particle density in the bunch (bunchlength)

V Conclusion

While the "physical" approach may serve as a physical picture, the "theoretical" approach allows to derive quantitative results and it describes the internal transverse motion of the bunch in terms of travelling waves (turbulent "flood-waves").

The quantitative formulation together will the observed scaling laws for bunchlength and higher order mode losses as functions of the machine parameters allows to derive a scaling law for the threshold current of head tail turbulence. (1)

This sealing law is presented here in a very rough form:

$$I \sim \frac{Q_s \cdot \gamma \cdot \sigma_E}{n \cdot \beta_z} \qquad (19)$$

$Q_s \hat{=}$ longitudinal tune
$\sigma_E \hat{=}$ relative energy spread
$\gamma \hat{=}$ energy in rest mass units
$n \hat{=}$ number of driving objects

References

1. R.D. Kohaupt, DESY 80/22 (1980)
2. R.D. Kohaupt, XIth International Conference on High Energy Accelerators CERN, Genf (1980) (to be published)
3. M. Sands, SLAC-TN-69-8, (1969)