HYPERgeometric functions Differential REDuction (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: F_D and F_S Horn-type hypergeometric functions of three variables[☆]

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ARTICLE INFO

Article history:

Received 11 January 2014

Received in revised form

19 June 2014

Accepted 18 July 2014

Available online 28 July 2014

Keywords:

Hypergeometric functions

Differential reduction

Feynman diagrams

ABSTRACT

HYPERDIRE is a project devoted to the creation of a set of Mathematica based programs for the differential reduction of hypergeometric functions. The current version includes two parts: the first one, **FdFunction**, for manipulations with Appell hypergeometric functions F_D of r variables; and the second one, **FsFunction**, for manipulations with Lauricella–Saran hypergeometric functions F_S of three variables. Both functions are related with one-loop Feynman diagrams.

Program summary

Program title: HYPERDIRE

Catalogue identifier: AEPP_v3_0

Program summary URL: http://cpc.cs.qub.ac.uk/summaries/AEPP_v3_0.html

Program obtainable from: CPC Program Library, Queen's University, Belfast, N. Ireland

Licensing provisions: GNU General Public License

No. of lines in distributed program, including test data, etc.: 310

No. of bytes in distributed program, including test data, etc.: 7666

Distribution format: tar.gz

Programming language: Mathematica

Computer: All computers running Mathematica

Operating system: All operating systems running Mathematica

Classification: 4.4.

Catalogue identifier of previous version: AEPP_v1_0

Journal reference of previous version: Comput. Phys. Comm. 184 (2013) 2332

Does the new version supersede the previous version?: No. It is an extension to the previous program, which performs reductions of hypergeometric functions ${}_pF_p$, F_1 , F_2 , F_3 and F_4

Nature of problem: Reduction of hypergeometric functions F_D and F_S to the set of basis functions

Solution method: Differential reduction

Reasons for new version: New algorithms for the reduction of multivariable Lauricella functions, Horn functions, (hypergeometric functions F_D and F_S)

[☆] This paper and its associated computer program are available via the Computer Physics Communication homepage on ScienceDirect (<http://www.sciencedirect.com/science/journal/00104655>).

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Summary of revisions: HYPERDIRE is a project devoted to the creation of a set of Mathematica based programs for the differential reduction of hypergeometric functions. The current version includes two parts: the first one, FdFunction, for manipulations with Appell hypergeometric functions F_D of r variables; and the second one, FsFunction, for manipulations with Lauricella–Saran hypergeometric functions F_S of three variables. Both functions are related with one-loop Feynman diagrams.

Running time: Depends on the complexity of problem

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1. Introduction

The study of solutions of linear partial differential equations (PDEs) of a few variables in terms of multiple series, i.e., a multivariable generalization of Gauss hypergeometric function [1], was started a long time ago [2]. Following the Horn definition,¹ a multiple series is called Horn-type hypergeometric function [4], if around some point $\vec{z} = \vec{z}_0$, there are series representations

$$H(\vec{z}) = \sum_{\vec{m}} C(\vec{m}) (\vec{z} - \vec{z}_0)^{\vec{m}},$$

where $\vec{m}$ is a set of integers and the ratio of two coefficients can be represented as a ratio of two polynomials:

$$\frac{C(\vec{m} + \vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}, \quad (1)$$

where $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$, is the j th unit vector. The coefficients $C(\vec{m})$ of such a series are expressible as product/ratio of Gamma-functions (up to some factors irrelevant for our consideration) [5]:

$$C(\vec{m}) = \frac{\prod_{j=1}^p \Gamma\left(\sum_{a=1}^r \mu_{ja} m_a + \gamma_j\right)}{\prod_{k=1}^q \Gamma\left(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k\right)}, \quad (2)$$

where $\mu_{ja}, \nu_{kb}, \sigma_j, \gamma_j \in \mathbb{Z}$ and m_a are elements of $\vec{m}$.

The Horn-type hypergeometric function, Eq. (1), satisfies the following system of differential equations:

$$0 = D_j(\vec{z})H(\vec{z}) = \left[Q_j \left(\sum_{k=1}^r z_k \frac{\partial}{\partial z_k} \right) \frac{1}{z_j} - P_j \left(\sum_{k=1}^r z_k \frac{\partial}{\partial z_k} \right) \right] H(\vec{z}), \quad (3)$$

where $j = 1, \dots, r$. The degree of polynomials P_i and Q_i is p_i and q_i , respectively. The largest of these numbers, $r = \max\{p_i, q_i\}$, is called the order of the hypergeometric series.

Any Horn-type hypergeometric function is a function of two kind of variables, *continuous* variables, $z_1, z_2, \dots, z_r$ and *discrete* variables: $\{u_a\} := \{\gamma_k, \sigma_r\}$, where the latter can change by integer numbers and are often referred to as the *parameters* of the hypergeometric function. For any Horn-hypergeometric function, there are linear differential operators changing the value of the discrete variables by one unit:

$$R_K(\vec{z}) \frac{\partial^K}{\partial \vec{z}} H(\vec{J}; \vec{z}) = H(\vec{J} \pm e_K; \vec{z}), \quad (4)$$

where $R_K(\vec{z})$ are polynomial (rational) functions. In Refs. [6,7] it was shown that there is an algorithmic solution for the construction of inverse linear differential operators:

$$B_L(\vec{z}) \frac{\partial^L}{\partial \vec{z}} \left(R_K(\vec{z}) \frac{\partial^K}{\partial \vec{z}} \right) H(\vec{J}; \vec{z}) \equiv B_L(\vec{z}) \frac{\partial^L}{\partial \vec{z}} H(\vec{J} \pm e_K; \vec{z}) = H(\vec{J}; \vec{z}). \quad (5)$$

Applying the direct or inverse differential operators to the hypergeometric function, the value of parameters can be changed by an arbitrary integer number:

$$S(\vec{z})H(\vec{J} + \vec{m}; \vec{z}) = \sum_{j=0}^r S_j(\vec{z}) \frac{\partial^j}{\partial \vec{z}} H(\vec{J}; \vec{z}), \quad (6)$$

where $\vec{m}$ is a set of integers, S and S_j are polynomials and r is the holonomic rank (the number of linearly independent solutions) of the system of differential equations, Eq. (3). Additionally, the construction of inverse differential operators defined by Eq. (5) allows to

- (i) find a set of exceptional parameters for any hypergeometric function, and this set coincides with the condition of reducibility of the monodromy group of the corresponding hypergeometric functions (see discussion in [8] and Section 3.4 for F_D and F_S functions);

¹ The modern approach to hypergeometric functions has been presented in [3].

- (ii) convert the system of linear PDEs, Eq. (3), into Pfaff form for any hypergeometric functions, including functions with Puiseux monomials as one of the solution, see details in [9].

The interest of physicists in hypergeometric functions is related with

- (i) the necessity of an analytical evaluation of multiple series generated by multiple residues of Mellin–Barnes integrals [10];
- (ii) the restricted set of values of parameters of hypergeometric functions or multiple series, where the algorithms [11–14] are applicable;
- (iii) the complicated analytical structure of one-loop massive Feynman diagrams, where, nevertheless, a simple hypergeometric representation exists [15–17].

It was pointed out in [18] that the differential reduction algorithm, defined as a full system of differential operators, Eqs. (3)–(5), can be applied to the construction of analytical coefficients of the so-called ε -expansions of hypergeometric functions about any rational values of parameters via the direct solution of the linear systems of differential equations.

This is the motivation for creating a package for the manipulation of the parameters of Horn-type hypergeometric functions of several variables.

In the previous publications the algebraic reduction of ${}_2F_1$ functions has been considered [19], the program **pfq**, for the manipulation of hypergeometric functions, ${}_pF_p$ ($p \geq 1$) [8], the program **AppellF1F4**, for the manipulation of Appell hypergeometric functions, F_1 , F_2 , F_3 and F_4 [20], the program **Horn**, for the manipulation of Horn-hypergeometric functions of two variables (30 hypergeometric functions in addition to four Appell functions) [9].

The aim of this paper is to present a further extension of the Mathematica [21] based package **HYPERDIRE** for the differential reduction of the Horn-type hypergeometric function with arbitrary values of parameters to a set of basis functions. The current version consists of two parts: one, **FdFunction**, for the manipulation of Lauricella hypergeometric functions, F_D , of r variables, and the second one, **FsFunction**, for the manipulation with Lauricella–Saran hypergeometric functions F_S with three variables.

2. The structure of hypergeometric functions related with one-loop off-shell Feynman diagrams

A generic scalar one-loop N -point function is defined by the following integral in d space–time dimensions

$$I_{N;a_1,\dots,a_N}^{(d)} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{((l-p_1)^2 + m_{12}^2)^{a_1} \cdots ((l-(p_1+\cdots+p_{N-1}))^2 + m_{N-1,N}^2)^{a_{N-1}} (l^2 + m_{N,1}^2)^{a_N}}, \quad (7)$$

where l is the loop momentum to be integrated, p_i are the external momenta and $m_{i,j}^2$ the masses of the internal propagators, $i, j = 1, \dots, N$. Energy–momentum conservation enforces $\sum_i p_i = 0$.

2.1. Massive case

In accordance with the algorithm described in [22], one-loop N -point diagrams with all powers of propagators equal to unity, i.e., all $a_i = 1$ in Eq. (7), satisfy to the following difference equation

$$I_N^{(d)} = b_N(d) + \sum_{k=1}^N \left(\frac{\partial_k \Delta_N}{2\Delta_N} \right) \sum_{r=0}^{\infty} \left(\frac{d-N+1}{2} \right)_r \left(\frac{G_{N-1}}{\Delta_N} \right)^r \mathbf{k}^- I_N^{(d+2r)}, \quad (8)$$

where $I_N^{(d)} \equiv I_{N;a_1,\dots,a_N}^{(d)}$, $(a)_k$ is a Pochhammer symbol, $(a)_k = \Gamma(a+k)/\Gamma(a)$, the operator $\mathbf{k}^-$ shifts the value of index $a_k \rightarrow a_k - 1$ (in our case, $a_k \equiv 1 \Rightarrow \mathbf{k}^- a_k = 0$), d is the dimension of space–time, and $b_N(d)$ is an arbitrary periodic function depending on the kinematic invariants. Moreover, we are working in Euclidean space–time, which is the source of the sign “+” instead of “−” as it was defined in [22], G_N is a Gram determinant, Δ_N is a Cayley determinant for the N -point diagram and $\partial_k \Delta_N = \frac{\partial}{\partial m_k^2} \Delta_N$. For details we refer to [22,15]. Eq. (8) can be solved iteratively [22,15] and the result for the one-loop N -point diagram in an arbitrary dimension d can be written as linear combinations of the following hypergeometric functions:

$$\begin{aligned} I_{N \geq 2}^{(d)} &\sim \prod_{j=2}^N \left(\frac{\partial_{k_j} \Delta_j}{2\Delta_j} \right) \times \sum_{r_1, r_2, r_3, \dots, r_{N-1}=0}^{\infty} \left(-m_{N-1,N}^2 \frac{G_{N-1}}{\Delta_N} \right)^{r_{N-1}} \cdots \left(-m_{12}^2 \frac{G_1}{\Delta_2} \right)^{r_1} \\ &\times \frac{\Gamma\left(\frac{d-N+1}{2} + r_{N-1}\right)}{\Gamma\left(\frac{d-N+1}{2}\right)} \cdots \frac{\Gamma\left(\frac{d-4}{2} + r_4 + r_{N-1}\right)}{\Gamma\left(\frac{d-4}{2} + r_5 + r_{N-1}\right)} \frac{\Gamma\left(\frac{d-3}{2} + r_3 + r_4 + r_{N-1}\right)}{\Gamma\left(\frac{d-3}{2} + r_4 + r_{N-1}\right)} \\ &\times \frac{\Gamma\left(\frac{d-2}{2} + r_2 + r_3 + r_{N-1}\right)}{\Gamma\left(\frac{d-2}{2} + r_3 + r_{N-1}\right)} \frac{\Gamma\left(\frac{d-1}{2} + r_1 + r_2 + r_{N-1}\right)}{\Gamma\left(\frac{d-1}{2} + r_2 + r_{N-1}\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + r_1 + r_2 + r_{N-1}\right)} \\ &+ \sum_{j=3}^N b_j(d) c_j(d), \end{aligned} \quad (9)$$

where $m_{i,j}^2$ are some masses, cf., Eq. (7). In accordance with Proposition 1 of [23],² the holonomic rank of the product $b_j(d)c_j(d)$, is expected to be equal to the holonomic rank of the hypergeometric function, defined by Eq. (10). Dropping all irrelevant factors, the hypergeometric

² For completeness, we recall it here: A multiple Mellin–Barnes integrals can be presented as a linear combination of Horn-type hypergeometric functions about some point. Therefore, the holonomic rank of the corresponding system of linear differential equations related with the Mellin–Barnes integral is equal to the holonomic rank of any hypergeometric function in the corresponding hypergeometric representation. This statement has been proven only for one-fold Mellin–Barnes integrals.

function related with one-loop N -point off-shell massive Feynman diagram is (it has a simpler form in contrast to the results of [24]):

$$H_{N \geq 2}^{(d)} = \sum_{r_1, r_2, r_3, \dots, r_{N-1}=0}^{\infty} \frac{\Gamma\left(\frac{d-1}{2} + r_1 + r_2 + \dots + r_{N-1}\right)}{\Gamma\left(\frac{d}{2} + r_1 + r_2 + \dots + r_{N-1}\right)} \times \frac{\Gamma\left(\frac{d-2}{2} + r_2 + r_3 + \dots + r_{N-1}\right)}{\Gamma\left(\frac{d-1}{2} + r_2 + r_3 + \dots + r_{N-1}\right)} \frac{\Gamma\left(\frac{d-3}{2} + r_3 + r_4 + \dots + r_{N-1}\right)}{\Gamma\left(\frac{d-2}{2} + r_3 + r_4 + \dots + r_{N-1}\right)} \times \frac{\Gamma\left(\frac{d-4}{2} + r_4 + \dots + r_{N-1}\right)}{\Gamma\left(\frac{d-3}{2} + r_4 + \dots + r_{N-1}\right)} \dots \frac{\Gamma\left(\frac{d-N+1}{2} + r_{N-1}\right)}{\Gamma\left(\frac{d-N+2}{2} + r_{N-1}\right)} z_1^{r_1} z_2^{r_2} \dots z_{N-1}^{r_{N-1}}. \quad (10)$$

For the lowest values of $N = 2, 3, 4, 5$, Eq. (10) has the following form:

$$H_2^{(d)} = \sum_{r_1} \frac{\left(\frac{d-1}{2}\right)_{r_1}}{\left(\frac{d}{2}\right)_{r_1}} z_1^{r_1}, \quad (11)$$

$$H_3^{(d)} = \sum_{r_1, r_2} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2}}{\left(\frac{d}{2}\right)_{r_1+r_2}} \frac{\left(\frac{d-2}{2}\right)_{r_2}}{\left(\frac{d-1}{2}\right)_{r_2}} z_1^{r_1} z_2^{r_2}, \quad (12)$$

$$H_4^{(d)} = \sum_{r_1, r_2, r_3} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2+r_3}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3}} \frac{\left(\frac{d-2}{2}\right)_{r_2+r_3}}{\left(\frac{d-1}{2}\right)_{r_2+r_3}} \frac{\left(\frac{d-3}{2}\right)_{r_3}}{\left(\frac{d-2}{2}\right)_{r_3}} z_1^{r_1} z_2^{r_2} z_3^{r_3}, \quad (13)$$

$$H_5^{(d)} = \sum_{r_1, r_2, r_3, r_4} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2+r_3+r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} \frac{\left(\frac{d-2}{2}\right)_{r_2+r_3+r_4}}{\left(\frac{d-1}{2}\right)_{r_2+r_3+r_4}} \frac{\left(\frac{d-3}{2}\right)_{r_3+r_4}}{\left(\frac{d-2}{2}\right)_{r_3+r_4}} \frac{\left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d-3}{2}\right)_{r_4}} z_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4}. \quad (14)$$

In accordance with Eq. (3), the order of differential equations of hypergeometric functions Eq. (10), increases with the number of external legs:

$$\frac{P_N^j}{Q_N^j} = j,$$

where the index j is the same as the summation index r_j and $j+1$ is equal to the number of external legs of the Feynman diagrams, cf., Eq. (7). To reduce the order of differential equations of the hypergeometric function Eq. (10), we apply recursively the following transformation:

$$\sum_{r=0}^{\infty} \frac{\Gamma(A+r)}{\Gamma(B+r)} z^r = \frac{\Gamma(A)}{\Gamma(B)} {}_2F_1\left(\begin{matrix} 1, A \\ B \end{matrix} \middle| z\right) = \frac{1}{1-z} \frac{\Gamma(A)}{\Gamma(B)} {}_2F_1\left(\begin{matrix} 1, B-A \\ B \end{matrix} \middle| \frac{z}{z-1}\right) = \frac{1}{1-z} \frac{\Gamma(A)}{\Gamma(B-A)} \sum_{r=0}^{\infty} \left(\frac{z}{z-1}\right)^r \frac{\Gamma(B-A+r)}{\Gamma(B+r)}. \quad (15)$$

Let us introduce new variables:

$$x_i = -\frac{z_i}{1-z_i}, \quad z_i = -\frac{x_i}{1-x_i}, \quad 1-z_i = \frac{1}{1-x_i}. \quad (16)$$

The recursive application of the linear-fractional transformation, Eq. (15), to Eq. (10) gives rise to the following hypergeometric function:

$$\prod_{k=1}^{N-1} (1-z_k) H_{N \geq 2}^{(d)} = \sum_{r_1, r_2, r_3, r_4, \dots, r_{N-1}=0}^{\infty} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + r_1 + \dots + r_{N-1}\right)} \prod_{i=1}^{N-1} x_i^{r_i} \left[\frac{\Gamma\left(\frac{i}{2} + r_1 + \dots + r_i\right)}{\Gamma\left(\frac{i}{2} + r_1 + \dots + r_{i-1}\right)} \right]. \quad (17)$$

For completeness, we present explicitly the hypergeometric terms defined by Eq. (17) for the first few values of $N = 2, 3, 4, 5^3$:

$$H_2^{(d)} = \sum_{r_1} \frac{\left(\frac{1}{2}\right)_{r_1}}{\left(\frac{d}{2}\right)_{r_1}} x_1^{r_1}, \quad (18)$$

$$H_3^{(d)} = \sum_{r_1, r_2} \frac{\left(\frac{1}{2}\right)_{r_1}}{\left(\frac{d}{2}\right)_{r_1+r_2}} \frac{\left(\frac{d-2}{2}\right)_{r_2}}{\left(\frac{d-1}{2}\right)_{r_2}} x_1^{r_1} x_2^{r_2}, \quad (19)$$

$$= \sum_{r_1, r_2} \frac{\left(\frac{1}{2}\right)_{r_1}}{(1)_{r_1}} \frac{(1)_{r_1+r_2}}{\left(\frac{d}{2}\right)_{r_1+r_2}} x_1^{r_1} x_2^{r_2}, \quad (20)$$

³ For our discussion we drop all irrelevant factors, like $(1-z_i)^{\pm 1}$ and assume, wherever it does not cause any problems, that $\Gamma\left(\frac{d}{2} \pm k + \vec{r}\right) \equiv \left(\frac{d}{2} \pm k\right)_{\vec{r}}$ with k being integer.

$$H_4^{(d)} = \sum_{r_1, r_2, r_3} \frac{\left(\frac{1}{2}\right)_{r_1} \left(\frac{d-2}{2}\right)_{r_2+r_3} \left(\frac{d-3}{2}\right)_{r_3}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3} \left(\frac{d-2}{2}\right)_{r_3}} x_1^{r_1} z_2^{r_2} z_3^{r_3} \quad (21)$$

$$= \sum_{r_1, r_2, r_3} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{d-3}{2}\right)_{r_3}}{(1)_{r_1} \left(\frac{d}{2}\right)_{r_1+r_2+r_3}} x_1^{r_1} x_2^{r_2} z_3^{r_3} \quad (22)$$

$$\equiv \sum_{r_1, r_2, r_3} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{3}{2}\right)_{r_1+r_2+r_3}}{(1)_{r_1} \left(\frac{3}{2}\right)_{r_1+r_2} \left(\frac{d}{2}\right)_{r_1+r_2+r_3}} x_1^{r_1} x_2^{r_2} x_3^{r_3}, \quad (23)$$

where the x_i are defined in Eq. (16). As follows from Eqs. (19) and (20), the vertex diagrams are described by the Appell hypergeometric functions F_3 or F_1 [25]. For the pentagon ($N = 5$), the hypergeometric function has the following form:

$$H_5^{(d)} = \sum_{r_1, r_2, r_3, r_4} \frac{\left(\frac{1}{2}\right)_{r_1} \left(\frac{d-2}{2}\right)_{r_2+r_3+r_4} \left(\frac{d-3}{2}\right)_{r_3+r_4} \left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_4} \left(\frac{d-2}{2}\right)_{r_3+r_4} \left(\frac{d-3}{2}\right)_{r_4}} x_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4}, \quad (24)$$

$$= \sum_{r_1, r_2, r_3, r_4} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{d-3}{2}\right)_{r_3+r_4} \left(\frac{d-4}{2}\right)_{r_4}}{(1)_{r_1} \left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4} \left(\frac{d-3}{2}\right)_{r_4}} x_1^{r_1} x_2^{r_2} z_3^{r_3} z_4^{r_4}, \quad (25)$$

$$= \sum_{r_1, r_2, r_3, r_4} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{3}{2}\right)_{r_1+r_2+r_3} \left(\frac{d-4}{2}\right)_{r_4}}{(1)_{r_1} \left(\frac{3}{2}\right)_{r_1+r_2} \left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} x_1^{r_1} x_2^{r_2} x_3^{r_3} z_4^{r_4}, \quad (26)$$

$$= \sum_{r_1, r_2, r_3, r_4} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{3}{2}\right)_{r_1+r_2+r_3} \left(\frac{4}{2}\right)_{r_1+r_2+r_3+r_4}}{(1)_{r_1} \left(\frac{3}{2}\right)_{r_1+r_2} \left(\frac{4}{2}\right)_{r_1+r_2+r_3} \left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4}. \quad (27)$$

After the linear-fractional transformation, Eq. (15), the order of the differential equation for the hypergeometric function related to the box diagram is reduced from three to two, see Eqs. (21) and (22), [15]. The pentagon, Eq. (25), corresponds to a hypergeometric function satisfying a differential equation of order two.⁴ As follows from Eqs. (10) and (17), the massive hexagon is expressible in terms of hypergeometric functions of five variables satisfying a differential equations of order three. However, since the difference between parameters of hypergeometric functions, Eq. (10) or (17), are integer or half-integer, these functions possess extended symmetries with respect to non-linear transformations of their arguments [26] (multivariable generalizations of quadratic transformations related to Gauss hypergeometric functions).⁵ It is still an open question, whether or not it is possible, to reduce the order of differential equations with the help of non-linear transformations.

2.2. Off-shell massless case

Let us consider an off-shell massless one-loop N -point diagram, where for some i , we have $\{p_i^2\} \neq 0$, cf., Eq. (7). In these kinematics, the three-point diagram (vertex) is not algebraically reducible to a simpler diagram. The $I_2^{(d)}$ integral can be written (up to some irrelevant normalization) as

$$I_2^{(d)} = \frac{1}{\Gamma\left(\frac{d-1}{2}\right)}, \quad (28)$$

and the iterative solution of Eq. (8) is

$$\begin{aligned} H_{N \geq 3}^{(d)} \Big|_{\{p_i^2\} \neq 0} &= \sum_{r_1, r_2, r_3, \dots, r_{N-2}=0}^{\infty} \frac{\Gamma\left(\frac{d-2}{2} + r_1 + r_2 + \dots + r_{N-2}\right)}{\Gamma\left(\frac{d-1}{2} + r_1 + r_2 + \dots + r_{N-2}\right)} \\ &\times \frac{\Gamma\left(\frac{d-3}{2} + r_2 + r_3 + \dots + r_{N-2}\right)}{\Gamma\left(\frac{d-2}{2} + r_2 + r_3 + \dots + r_{N-2}\right)} \frac{\Gamma\left(\frac{d-4}{2} + r_3 + r_4 + \dots + r_{N-2}\right)}{\Gamma\left(\frac{d-3}{2} + r_3 + r_4 + \dots + r_{N-2}\right)} \\ &\times \frac{\Gamma\left(\frac{d-5}{2} + r_4 + \dots + r_{N-2}\right)}{\Gamma\left(\frac{d-4}{2} + r_4 + \dots + r_{N-2}\right)} \dots \frac{\Gamma\left(\frac{d-N+1}{2} + r_{N-2}\right)}{\Gamma\left(\frac{d-N+2}{2} + r_{N-2}\right)} z_1^{r_1} z_2^{r_2} \dots z_{N-2}^{r_{N-2}}. \end{aligned} \quad (29)$$

From Eqs. (10) and (29) we see, that the structure of hypergeometric functions related with off-shell massive and off-shell massless integrals is related as follows [17]:

$$H_{N+1}^{(d)} \Big|_{\{p_i^2\} \neq 0} \sim H_N^{(d-1)}, \quad (30)$$

⁴ At present, a full classification of Horn-type hypergeometric functions of four variables does not exist [4].

⁵ All hypergeometric functions, defined by Eqs. (11)–(27), belong to the class of multiple Gauss hypergeometric functions [4]: the series representation can be written as infinite sum(s) with respect to the index of summation over the parameters of ${}_2F_1$ hypergeometric functions: in Eqs. (20), (23), (27) the Gauss hypergeometric functions enter via the last index of summation; in Eq. (21) via summation over r_1 ; in Eq. (22) via summation over r_2 or r_3 ; in Eq. (24) via summation over r_1 or r_2 ; in Eq. (25) via summation over r_2 or r_3 ; in Eq. (26) via summation over r_3 . Exploring the transformation properties of Gauss hypergeometric functions, see Eq. (15) for an example, the transformation of hypergeometric functions, Eqs. (11)–(27), can be performed.

where d is the dimension of space–time and N denotes the number of external legs. The symbol $\sim$ in Eq. (30) indicates that this relation is valid for hypergeometric functions related with the corresponding Feynman diagram. Eq. (30) is also valid for hypergeometric functions, Eq. (17), after application of the linear-fractional transformation, Eq. (15).

3. Differential reduction of Horn-type hypergeometric functions of three variables

3.1. System of differential equations

Let us consider the system of linear differential operators of second order L_j for the hypergeometric functions $\omega(\vec{z})$:

$$L_i \omega(\vec{z}) : \quad \theta_i^2 \omega(\vec{z}) = \left[\theta_i \sum_{j=1; \{j \neq i\}}^3 P_{ij} \theta_j + \sum_{m=1}^3 R_{im} \theta_m + S_i \right] \omega(\vec{z}), \quad i = 1, 2, 3, \quad (31)$$

where $\vec{z} = (z_1, z_2, z_3)$ with z_1, z_2, z_3 being variables, $\{P_{ij}, R_{ab}, S_j\}$ are rational functions, $\theta_j = z_j \partial_{z_j}$ for $j = 1, 2, 3$, and $\theta_{i_1 \dots i_k} = \theta_{i_1} \dots \theta_{i_k}$. Taking the derivative, $\theta_k L_i \omega(\vec{z})$, we finally obtain from Eq. (31):

$$\begin{aligned} \theta_k L_i \omega(\vec{z}) : & \left[(1 - P_{ik} P_{ki}) \theta_k \theta_i^2 - \sum_{j=1; \{j \neq i \neq k\}}^3 (P_{ij} + P_{ik} P_{kj}) \theta_i \theta_k \theta_j \right] \omega(\vec{z}) \\ & = \left\{ [P_{ik} R_{ki} P_{ik} + R_{ik} P_{ki} + R_{ii} + P_{ik} R_{kk} + (\theta_k P_{ik})] \theta_{ik} + \sum_{j=1; \{j \neq i \neq k\}}^3 [P_{ik} R_{ki} P_{ij} + P_{ik} R_{kj} + (\theta_k P_{ij})] \theta_i \theta_j \right. \\ & \quad + \sum_{j=1; \{j \neq i \neq k\}}^3 [R_{ik} P_{kj} + R_{ij}] \theta_k \theta_j + \sum_{m=1}^3 [P_{ik} R_{ki} R_{im} + R_{ik} R_{km} + (\theta_k R_{im})] \theta_m \\ & \quad \left. + P_{ik} S_k \theta_i + S_i \theta_k + P_{ik} R_{ki} S_i + R_{ik} S_k + (\theta_k S_i) \right\} \omega(\vec{z}), \quad i, k = 1, 2, 3. \end{aligned} \quad (32)$$

For a function of three variables, the sum $\sum_{j=1; \{j \neq i \neq k\}}^3$ can be replaced by the index j , where $j \neq i \neq k$. The conditions of complete integrability are defined via the relations:

$$\theta_i [\theta_j L_k] \omega(\vec{z}) = \theta_j [\theta_i L_k] \omega(\vec{z}), \quad i, j, k = 1, 2, 3. \quad (33)$$

The number of independent solutions of the system of differential equations, Eq. (31), of three variables is defined by coefficients on the l.h.s. of Eq. (32) and the validity of Eq. (33). When the coefficients

$$(1 - P_{ik} P_{ki}), \quad i, k = 1, 2, 3, \quad (34)$$

and

$$\sum_{j=1; \{j \neq i \neq k\}}^3 (P_{ij} + P_{ik} P_{kj}), \quad i, k = 1, 2, 3, \quad (35)$$

are not equal to zero for all i, j, k , Eqs. (31) and (32) can be reduced to the Pfaff system of eight independent differential equations:

$$d\vec{f} = R\vec{f}, \quad (36)$$

where $\vec{f} = (\omega(\vec{z}), \theta_1 \omega(\vec{z}), \theta_2 \omega(\vec{z}), \theta_3 \omega(\vec{z}), \theta_{12} \omega(\vec{z}), \theta_{13} \omega(\vec{z}), \theta_{23} \omega(\vec{z}), \theta_{123} \omega(\vec{z}))$. When some of the coefficients in Eq. (34) are zero, the coefficients in front of the terms $\theta_{123} \omega(\vec{z})$, defined by Eq. (35), start to play a role. For non-zero values of Eq. (35), the terms $\theta_{123} \omega(\vec{z})$ can be excluded, and the rank of differential system is reduced to seven independent functions. When for some values of i and k both coefficients, defined by Eqs. (34) and (35) are zero, a further simplification can be performed, so that the rank of system is reduced to six or to an even smaller number.

The locus of singularities L_{ij} of the linear system of differential equations of second order of three variables defined by Eq. (31) follows from singularities of higher rank differential operators on the l.h.s. of Eqs. (31) and (32):

$$L_{ij} = \cup_{i=1}^3 \{z_i = 0\} \cup_{i,k=1}^3 \{P_{ik}^{-1} = 0\} \cup_{i,j,k=1}^3 \{(1 - P_{ik} P_{ki})^{-1} = 0\} \cup_{i,j,k=1}^3 \{(P_{ij} + P_{ik} P_{kj})^{-1} = 0\}. \quad (37)$$

This result is valid only for the full (non-degenerate) case of Pfaff system including eight elements.

3.2. Lauricella hypergeometric function F_D

3.2.1. General consideration

Let us consider the $F_D^{(r)}$ functions of r variables, defined around $z_i = 0$ as

$$F_D^{(r)}(a; b_1, \dots, b_k; c; z_1, \dots, z_r) = \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{|\vec{m}|}}{(c)_{|\vec{m}|}} \prod_{j=1}^r (b_j)_{m_j} \frac{z_1^{m_1}}{m_1!} \dots \frac{z_r^{m_r}}{m_r!}. \quad (38)$$

For $r = 1$ this function coincides with the Gauss hypergeometric function, for $r = 2$, it coincides with Appell function F_1 [27]. As follows from the definition, Eq. (38), this function is symmetric with respect to the transformation

$$b_i \leftrightarrow b_j, \quad z_i \leftrightarrow z_j.$$

Generally, F_D functions and their properties have been analyzed in detail in many references [6,28]. The differential operators for F_D hypergeometric function, Eq. (3), are given by

$$D_i F_D^{(r)} : \quad \partial_i \left(c - 1 + \sum_{j=1}^r \theta_j \right) F_D^{(r)} = \left(a + \sum_{j=1}^r \theta_j \right) (b_i + \theta_i) F_D^{(r)}, \quad i = 1, \dots, r. \quad (39)$$

where

$$F_D \equiv F_D^{(r)}(a; b_1, \dots, b_r; c; z_1, \dots, z_r). \quad (40)$$

They can be written in canonical form, cf. Eq. (31):

$$L_i F_D : \quad \theta_i^2 F_D = \left[-\theta_i \sum_{j \neq i} \theta_j + \frac{(a + b_i)z_i - (c - 1)}{1 - z_i} \theta_i + \frac{b_i z_i}{1 - z_i} \sum_{j \neq i} \theta_j + \frac{a b_i z_i}{1 - z_i} \right] F_D, \quad i = 1, \dots, r. \quad (41)$$

From these equations we have:

$$\begin{aligned} P_{ij} = P_{ji} = -1, \quad S_i &= \frac{a b_i z_i}{1 - z_i} \equiv a P_i, \\ R_{ii} &= \frac{(a + b_i)z_i - (c - 1)}{1 - z_i} \equiv R_i, \quad R_{im} = \frac{b_i z_i}{1 - z_i} \equiv P_i, \quad m \neq i. \end{aligned} \quad (42)$$

Upon substitution of these values for P_{ij} , R_{ab} , S_i into Eq. (32), we obtain

$$\left([P_k - P_i + R_i - R_k] \theta_i \theta_k - [R_{ki} R_{im} - R_{ik} R_{km}] \sum_{m=1}^r \theta_m - S_k \theta_i + S_i \theta_k - P_k S_i + P_i S_k \right) F_D = 0. \quad (43)$$

Eq. (43) can be simplified with the help of Eq. (42) by taking into account that the sum of the last two terms in Eq. (43), $P_i S_k - P_k S_i$, is equal to zero, and by splitting the sum over m into i , k and j , where $j \neq i \neq k$. In this way, we get

$$\left(R_k - P_k - R_i + P_i \right) \theta_i \theta_k F_D = \left(P_k [P_i - R_i - a] \theta_i - P_i [P_k - R_k - a] \theta_k \right) F_D, \quad (44)$$

where

$$R_i - P_i \equiv R_{ii} - R_{ik} = \frac{a z_i - (c - 1)}{1 - z_i}. \quad (45)$$

Eq. (44) can be rewritten in a more familiar form, see [28]:

$$[(z_i - z_j) \theta_i \theta_j - b_j z_j \theta_i + b_i z_i \theta_j] F_D = 0. \quad (46)$$

After factorization of z_i , z_j , Eq. (46) can be expressed as follows,

$$[(z_i - z_j) \partial_{ij} + b_i \partial_j - b_j \partial_i] F_D = 0. \quad (47)$$

In this way, all second derivatives of an F_D function are expressible in terms of the corresponding first derivatives and function, see Eqs. (41) and (46). As a consequence, there are only $r + 1$ linearly independent solutions of linear differential equations, Eq. (39). The locus of the singularities L_{ij} of an F_D function is defined from the singularities of the differential equations, Eqs. (41) and (46):

$$L_{ij} = \cup_{i=1}^r \{z_i = 0\} \cup_{1=i < j=r} \{z_i - z_j = 0\} \cup_{i=1}^r \{z_i = 1\}. \quad (48)$$

The Pfaff system for an F_D hypergeometric function has the following form:

$$d\omega(\vec{z}) = \left(\sum_{i < j} A_{ij} d \log(z_i - z_j) \right) \omega(\vec{z}),$$

where $\omega(\vec{z}) = \{F_D, \theta_j F_D\}$ and the matrices A_{ij} have been constructed explicitly in [28].

3.2.2. Differential reduction of F_D

The direct differential operators are the following:

$$\begin{aligned} aF_D^{(r)}(a+1; b_1, \dots, b_r; c; z_1, \dots, z_r) &= \left(a + \prod_{i=1}^r \theta_i\right) F_D, \\ b_i F_D^{(r)}(a; b_1, \dots, b_i+1, \dots, b_r; c; z_1, \dots, z_r) &= (b_i + \theta_i) F_D, \\ (c-1)F_D^{(r)}(a; b_1, \dots, b_r; c-1; z_1, \dots, z_r) &= \left(c-1 + \prod_{i=1}^r \theta_i\right) F_D, \end{aligned} \quad (49)$$

and F_D is defined by Eq. (40). The inverse differential operators have been constructed in [6]:

$$(c-a)F_D^{(r)}(a-1; b_1, \dots, b_r; c; z_1, \dots, z_r) = \left[\sum_{j=1}^r (1-z_j)\theta_j - \sum_{j=1}^r b_j z_j + c - a\right] F_D, \quad (50)$$

$$\left(c - \sum_{j=1}^r b_j\right) F_D^{(r)}(a; b_1, \dots, b_i-1, \dots, b_r; c; z_1, \dots, z_r) = \left[z_i \sum_{j=1}^r (1-z_j)\partial_j - a z_i + c - \sum_{j=1}^r b_j\right] F_D, \quad (51)$$

$$(c-a) \left(c - \sum_{j=1}^r b_j\right) F_D^{(r)}(a; b_1, \dots, b_r; c+1; z_1, \dots, z_r) = c \left[\sum_{j=1}^r (1-z_j)\partial_j + c - a - \sum_{j=1}^r b_j\right] F_D, \quad (52)$$

where F_D is defined by Eq. (40). In this case, the results of the differential reduction, Eq. (6), have the following form

$$S(\vec{z})F_D((a; \vec{b}; c) + \vec{m}; \vec{z}) = S_0(\vec{z})F_D(a; \vec{b}; c; \vec{z}) + \sum_{i=1}^r S_i(\vec{z}) \frac{\partial}{\partial z_i} F_D^{(r)}(a; \vec{b}; c; \vec{z}), \quad (53)$$

where $\vec{m}$ is a set of integers and S, S_i are polynomials.

3.3. Hypergeometric function F_S

3.3.1. General consideration

The Lauricella–Saran hypergeometric function of three variables F_S [29] (F_7 in the notation of [2]) is defined around the point $z_1 = z_2 = z_3 = 0$ as follows

$$F_S(a_1; a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \sum_{m_1, m_2, m_3=0}^{\infty} \frac{(a_1)_{m_1} (a_2)_{m_2+m_3}}{(c)_{m_1+m_2+m_3}} \prod_{j=1}^3 (b_j)_{m_j} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3}}{m_1! m_2! m_3!}. \quad (54)$$

It is one of the 14 functions of three variables of order two,⁶ introduced by Lauricella [2]. In this case, the differential operators, Eq. (3), are

$$D_1 F_S : \quad \partial_1 \left(c - 1 + \sum_{j=1}^3 \theta_j\right) F_S = (a_1 + \theta_1) (b_1 + \theta_1) F_S, \quad (55)$$

$$D_i F_S : \quad \partial_i \left(c - 1 + \sum_{j=1}^3 \theta_j\right) F_S = (a_i + \theta_i) (b_i + \theta_i) F_S, \quad i = 2, 3, \quad (56)$$

where

$$F_S = F_S(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3). \quad (57)$$

The canonical form of these differential equations are the following:

$$L_1 F_S : \quad \partial_1^2 F_S = \left[-\frac{1}{1-z_1} \theta_1 (\theta_2 + \theta_3) + \frac{(a_1 + b_1)z_1 - (c-1)}{1-z_1} \theta_1 + \frac{a_1 b_1 z_1}{1-z_1}\right] F_S, \quad (58)$$

$$L_2 F_S : \quad \partial_2^2 F_S = \left[-\theta_2 \theta_3 - \frac{1}{1-z_2} \theta_2 \theta_1 + \frac{(a_2 + b_2)z_2 - (c-1)}{1-z_2} \theta_2 + \frac{b_2 z_2}{1-z_2} \theta_3 + \frac{a_2 b_2 z_2}{1-z_2}\right] F_S, \quad (59)$$

$$L_3 F_S : \quad \partial_3^2 F_S = \left[-\theta_3 \theta_2 - \frac{1}{1-z_3} \theta_3 \theta_1 + \frac{(a_2 + b_3)z_3 - (c-1)}{1-z_3} \theta_3 + \frac{b_3 z_3}{1-z_3} \theta_2 + \frac{a_2 b_3 z_3}{1-z_3}\right] F_S. \quad (60)$$

⁶ The complete set of programs for the differential reduction for other functions from the Lauricella–Srivastava list [4] will be presented in a separate publication.

These equations define the values of functions P_{ij} , R_{ab} , S_i entering in Eq. (31):

$$\begin{aligned} R_{12} = R_{13} = R_{21} = R_{31} &= 0, & P_{23} = P_{32} &= -1, \\ P_{12} = P_{13} &= -\frac{1}{1-z_1}, & P_{21} &= -\frac{1}{1-z_2}, & P_{31} &= -\frac{1}{1-z_3}, \\ R_{11} &= \frac{(a_1+b_1)z_1-(c-1)}{1-z_1}, & R_{ii} &= \frac{(a_i+b_i)z_i-(c-1)}{1-z_i}, & i &= 2, 3, \\ R_{23} &= \frac{b_2z_2}{1-z_2}, & R_{32} &= \frac{b_3z_3}{1-z_3}, & S_1 &= \frac{a_1b_1z_1}{1-z_1}, & S_i &= a_2 \frac{b_i z_i}{1-z_i}, & i &= 2, 3. \end{aligned} \quad (61)$$

With the substitution of these values of P_{ij} into Eq. (32) and, since $1 - P_{23}P_{32} = 0$, we can express the third mixing derivatives of the hypergeometric function, $\theta_{123}\omega(\vec{z})$, via second derivatives of the hypergeometric function.

The series representation of the hypergeometric function F_5 can be rewritten in the following form:

$$F_5(a_1, a_2, a_3; b_1, b_2, b_3; c; z_1, z_2, z_3) = \sum_{m_1=0}^{\infty} \frac{(a_1)_{m_1} (b_1)_{m_1}}{(c)_{m_1}} \frac{z_1^{m_1}}{m_1!} F_1(a_2; b_2, b_3; c+m_1; z_2, z_3), \quad (62)$$

where $F_1(a; b_1, b_2; c; z_1, z_2)$ is the Appell function of two variables: $F_1 \equiv F_D^{(2)}$. From this representation it is easy to get the following relation:

$$[(z_2 - z_3)\theta_{23}]F_5 = (b_3z_3\theta_2 - b_2z_2\theta_3)F_5. \quad (63)$$

Eq. (32) allows us to express all higher derivatives of hypergeometric functions F_5 in terms of second derivatives only. In particular,

$$\begin{aligned} \theta_2 L_1 : \left(1 - \frac{1}{(1-z_1)(1-z_2)}\right) \theta_{112} F_5 &= \left\{ (P_{12}R_{22} + R_{11})\theta_{12} + P_{12}R_{23}\theta_{13} + P_{12}S_2\theta_1 + S_1\theta_2 \right\} F_5 \\ &= \left\{ \left(\frac{(a_1+b_1)z_1-(c-1)}{1-z_1} - \frac{(a_2+b_2)z_2-(c-1)}{(1-z_1)(1-z_2)} \right) \theta_{12} \right. \\ &\quad \left. - \frac{b_2z_2}{(1-z_1)(1-z_2)} \theta_{13} - \frac{a_2b_2z_2}{(1-z_1)(1-z_2)} \theta_1 + \frac{a_1b_1z_1}{1-z_1} \theta_2 \right\} F_5, \end{aligned} \quad (64)$$

$$\begin{aligned} \theta_3 L_1 : \left(1 - \frac{1}{(1-z_1)(1-z_3)}\right) \theta_{113} F_5 &= \left\{ (P_{13}R_{33} + R_{11})\theta_{13} + P_{13}R_{32}\theta_{12} + P_{13}S_3\theta_1 + S_1\theta_3 \right\} F_5 \\ &= \left\{ \left(\frac{(a_1+b_1)z_1-(c-1)}{1-z_1} - \frac{(a_2+b_3)z_3-(c-1)}{(1-z_1)(1-z_3)} \right) \theta_{13} \right. \\ &\quad \left. - \frac{b_3z_3}{(1-z_1)(1-z_3)} \theta_{12} - \frac{a_2b_3z_3}{(1-z_1)(1-z_3)} \theta_1 + \frac{a_1b_1z_1}{1-z_1} \theta_3 \right\} F_5, \end{aligned} \quad (65)$$

$$\begin{aligned} \theta_3 L_2 = -\theta_2 L_3 : \frac{(z_3 - z_2)}{(1-z_2)(1-z_3)} \theta_{123} F_5 &= \left\{ P_{21}R_{32}\theta_{12} - P_{31}R_{23}\theta_{13} - (R_{22} - R_{23} + R_{32} - R_{33}) \frac{1}{z_2 - z_3} (b_3z_3\theta_2 - b_2z_2\theta_3) \right. \\ &\quad \left. - R_{23}(a_2 + R_{33} - R_{32})\theta_3 + R_{32}(a_2 + R_{22} - R_{23})\theta_2 \right\} F_5 \\ &= \left\{ \frac{b_2z_2}{(1-z_2)(1-z_3)} \theta_{13} - \frac{b_3z_3}{(1-z_2)(1-z_3)} \theta_{12} \right\} F_5. \end{aligned} \quad (66)$$

The last equation, Eq. (66), coincides with the derivative of the Eq. (63) with respect to z_1 . The remaining two differential equations we can write in the following form:

$$\begin{aligned} \theta_1 L_2 : \left(1 - \frac{1}{(1-z_1)(1-z_2)}\right) \theta_{122} F_5 &= -\left(1 - \frac{1}{(1-z_1)(1-z_2)}\right) \theta_{123} F_5 \\ &\quad + \left\{ (P_{21}R_{11} + R_{22})\theta_{12} + R_{23}\theta_{13} + P_2S_1\theta_2 + S_2\theta_1 \right\} F_5 \\ &= -\left(1 - \frac{1}{(1-z_1)(1-z_2)}\right) \theta_{123} F_5 \\ &\quad + \left\{ \left(\frac{(a_2+b_2)z_2-(c-1)}{1-z_2} - \frac{(a_1+b_1)z_1-(c-1)}{(1-z_1)(1-z_2)} \right) \theta_{12} \right. \\ &\quad \left. + \frac{b_2z_2}{(1-z_2)} \theta_{13} - \frac{a_1b_1z_1}{(1-z_1)(1-z_2)} \theta_2 + \frac{a_2b_2z_2}{1-z_2} \theta_1 \right\} F_5, \end{aligned} \quad (67)$$

$$\begin{aligned}
\theta_{13} L_3 : \left(1 - \frac{1}{(1-z_1)(1-z_3)} \right) \theta_{133} F_S &= - \left(1 - \frac{1}{(1-z_1)(1-z_3)} \right) \theta_{123} F_S \\
&+ \{ (P_{31} R_{11} + R_{33}) \theta_{13} + R_{32} \theta_{12} + P_{31} S_1 \theta_3 + S_3 \theta_1 \} F_S \\
&= - \left(1 - \frac{1}{(1-z_1)(1-z_2)} \right) \theta_{123} F_S \\
&+ \left\{ \left(\frac{(a_2 + b_3)z_3 - (c-1)}{1-z_3} - \frac{(a_1 + b_1)z_1 - (c-1)}{(1-z_1)(1-z_3)} \right) \theta_{13} \right. \\
&\left. + \frac{b_3 z_3}{(1-z_3)} \theta_{12} - \frac{a_1 b_1 z_1}{(1-z_1)(1-z_3)} \theta_3 + \frac{a_2 b_3 z_3}{1-z_3} \theta_1 \right\} F_S,
\end{aligned} \tag{68}$$

where the mixed derivative $\theta_{123} F_S$ is defined by Eq. (66).

In this way, we have proven:

Theorem 1. The Lauricella–Saran hypergeometric function F_S of three variables, Eq. (54), has six linearly independent solutions around the points $z_1 = z_2 = z_3 = 0$.

The locus of singularities L_{ij} of the hypergeometric function F_S follows from the singularities of the differential operators, Eqs. (58)–(60), (66)–(68):

$$L_{ij} = \cup_{i=1}^3 \{z_i = 0\} \cup_{i=1}^3 \{z_i = 1\} \cup \{z_2 = z_3\} \cup_{i=2}^3 \{z_1 + z_i = z_1 z_i\}. \tag{69}$$

3.3.2. Differential reduction of F_S

The direct differential operators are the following:

$$\begin{aligned}
a_1 F_S(a_1 + 1, a_2; \vec{b}; c; \vec{x}) &= (a_1 + \theta_1) F_S, \\
a_2 F_S(a_1, a_2 + 1; \vec{b}; c; \vec{x}) &= (a_2 + \theta_2 + \theta_3) F_S, \\
b_i F_S(a_1, a_2; \dots, b_i + 1, \dots; c; \vec{x}) &= (b_i + \theta_i) F_S, \\
(c-1) F_S(\vec{a}; \vec{b}; c-1; \vec{x}) &= \left(c-1 + \prod_{j=1}^3 \theta_j \right) F_S.
\end{aligned} \tag{70}$$

and F_S is defined by Eq. (57). The corresponding inverse differential operators we define for the parameters $\mathbf{X} \in \{a_1, a_2, b_1, b_2, b_3\}$ as follows:

$$F_S(\mathbf{X}; c; \vec{z}) = [A_{\mathbf{X}, F_S} + B_{\mathbf{X}, F_S} \theta_1 + C_{\mathbf{X}, F_S} \theta_2 + D_{\mathbf{X}, F_S} \theta_3 + E_{\mathbf{X}, F_S} \theta_{12} + F_{\mathbf{X}, F_S} \theta_{13}] F_S(\mathbf{X} + 1; c; \vec{z}), \tag{71}$$

and for the parameter c :

$$\begin{aligned}
F_S(a_1, a_2; b_1, b_2, b_3; c; \vec{z}) \\
= [A_{c, F_S} + B_{c, F_S} \theta_1 + C_{c, F_S} \theta_2 + D_{c, F_S} \theta_3 + E_{c, F_S} \theta_{12} + F_{c, F_S} \theta_{13}] F_S(a_1, a_2; b_1, b_2, b_3; c-1; \vec{z}).
\end{aligned} \tag{72}$$

The full list of inverse differential operators are the following:

$$\begin{aligned}
A_{a_1, F_S} &= \frac{a_1^2 + a_1(b_1 z_1 + D_1 + D_3 - 2b_1) + a_2(b_1 z_1 + D_2 - a_1) + (b_1 z_1 - c + 1)(D_2 - a_1)}{D_0 D_2}, \\
B_{a_1, F_S} &= \frac{(z_1 - 1)(a_2 + D_2)}{D_0 D_2}, \quad C_{a_1, F_S} = \frac{b_1 z_1 (z_2 - 1)}{z_2 D_0 D_2}, \quad D_{a_1, F_S} = \frac{b_1 z_1 (z_3 - 1)}{z_3 D_0 D_2}, \\
E_{a_1, F_S} &= -\frac{z_1 + z_2 - z_1 z_2}{z_2 D_0 D_2}, \quad F_{a_1, F_S} = -\frac{z_1 + z_3 - z_1 z_3}{z_3 D_0 D_2},
\end{aligned} \tag{73}$$

$$\begin{aligned}
A_{a_2, F_S} &= \frac{(b_2 z_2 + b_3 z_3 + D_1)(a_1 + D_1) - b_1 D_1}{D_0 D_1}, \quad B_{a_2, F_S} = \frac{(z_1 - 1)(b_2 z_2 + b_3 z_3)}{z_1 D_0 D_1}, \\
C_{a_2, F_S} &= \frac{(z_2 - 1)(b_1 + D_0)}{D_0 D_1}, \quad D_{a_2, F_S} = \frac{(z_3 - 1)(b_1 + D_0)}{D_0 D_1}, \\
E_{a_2, F_S} &= -\frac{z_1 + z_2 - z_1 z_3}{z_1 D_0 D_1}, \quad F_{a_2, F_S} = -\frac{z_1 + z_3 - z_1 z_3}{z_1 D_0 D_1},
\end{aligned} \tag{74}$$

$$A_{c, F_S} = -\frac{(c-1)}{D_0 D_1 D_2 D_3} \left[a_1(a_2 + D_3)(D_1 + D_3) + D_1(D_2 + a_2 - a_1)D_3 \right], \tag{75}$$

$$B_{c, F_S} = -\frac{(c-1)(z_1 - 1)(a_2(D_1 + D_2) + D_2 D_3)}{z_1 D_0 D_1 D_2 D_3},$$

$$C_{c, F_S} = \frac{(c-1)(1 - z_2)(a_1(D_1 + D_2) + D_1 D_3)}{z_2 D_0 D_1 D_2 D_3},$$

Table 1Exceptional set of parameters for the hypergeometric functions $F_D^{(r)}$ and F_S .

$F_D^{(r)}$	$\{a, b_j, c - a, c - \sum_{j=1}^r b_j\} \in \mathbb{Z}$
F_S	$\{a_1, a_2, b_j, c - a_1 - a_2, c - b_1 - b_2 - b_3, a_1 + b_2 + b_3 - c, a_2 + b_1 - c\} \in \mathbb{Z}$

$$\begin{aligned}
D_{c,FS} &= \frac{(c-1)(1-z_3)(a_1(D_1+D_2)+D_1D_3)}{z_3D_0D_1D_2D_3}, \\
E_{c,FS} &= \frac{(c-1)(z_1+z_2-z_1z_2)(D_1+D_2)}{z_1z_2D_0D_1D_2D_3}, & F_{c,FS} &= \frac{(c-1)(z_1+z_3-z_1z_3)(D_1+D_2)}{z_1z_3D_0D_1D_2D_3}, \\
A_{b_1,FS} &= \frac{a_2(a_1z_1+D_3)+(a_1z_1+D_1-a_2)D_3}{D_1D_3}, & B_{b_1,FS} &= \frac{(z_1-1)(a_2+D_3)}{D_1D_3}, \\
C_{b_1,FS} &= \frac{a_1z_1(z_2-1)}{z_2D_1D_3}, & D_{b_1,FS} &= \frac{a_1z_1(z_3-1)}{z_3D_1D_3}, \\
E_{b_1,FS} &= -\frac{z_1+z_2-z_1z_2}{z_2D_1D_3}, & F_{b_1,FS} &= -\frac{z_1+z_3-z_1z_3}{z_3D_1D_3},
\end{aligned} \tag{76}$$

$$\begin{aligned}
A_{b_2,FS} &= \frac{a_1(a_2z_2+D_3)+D_3(a_2z_2+D_3-b_1)}{D_2D_3}, & B_{b_2,FS} &= \frac{a_2(z_1-1)z_2}{z_1D_2D_3}, \\
C_{b_2,FS} &= \frac{(z_2-1)(a_1+D_3)}{D_2D_3}, & D_{b_2,FS} &= \frac{z_2(z_3-1)(a_1+D_3)}{z_3D_2D_3}, \\
E_{b_2,FS} &= -\frac{z_1+z_2-z_1z_2}{z_1D_2D_3}, & F_{b_2,FS} &= -\frac{z_2(z_1+z_3-z_1z_3)}{z_1z_3D_2D_3}.
\end{aligned} \tag{77}$$

$$\begin{aligned}
A_{b_3,FS} &= \frac{a_1(a_2z_3+D_3)+D_3(a_2z_3+D_3-b_1)}{D_2D_3}, & B_{b_3,FS} &= \frac{a_2(z_1-1)z_3}{z_1D_2D_3}, \\
C_{b_3,FS} &= \frac{(z_2-1)z_3(a_1+D_3)}{z_2D_2D_3}, & D_{b_3,FS} &= \frac{(z_3-1)(a_1+D_3)}{D_2D_3}, \\
E_{b_3,FS} &= -\frac{z_3(z_1+z_2-z_1z_2)}{z_1z_2D_2D_3}, & F_{b_3,FS} &= -\frac{z_1+z_3-z_1z_3}{z_1D_2D_3},
\end{aligned} \tag{78}$$

where

$$D_0 = a_1 + a_2 - (c-1), \tag{79}$$

$$D_1 = a_2 + b_1 - (c-1), \tag{80}$$

$$D_2 = a_1 + b_2 + b_3 - (c-1), \tag{81}$$

$$D_3 = b_1 + b_2 + b_3 - (c-1), \tag{82}$$

and

$$D_1 + D_2 = D_0 + D_3. \tag{83}$$

The results of the differential reduction, Eq. (6), have the following form in this case:

$$S(\vec{z})F_S((\vec{a}; \vec{b}; c) + \vec{m}; \vec{z}) = S_0(\vec{z})F_S(\vec{a}; \vec{b}; c; \vec{z}) + \sum_{i=1}^3 S_i(\vec{z}) \frac{\partial}{\partial z_i} F_S(\vec{a}; \vec{b}; c; \vec{z}) + \sum_{j=2}^3 S_{1j}(\vec{z}) \frac{\partial^2}{\partial z_1 \partial z_j} F_S(\vec{a}; \vec{b}; c; \vec{z}), \tag{84}$$

where $\vec{m}$ is a set of integers, S , S_j and S_{ij} are polynomials.

3.4. Exceptional values of parameters: F_D and F_S

It was pointed out in [8], that the subset of parameters for which the results of the differential reduction, Eqs. (53) and (84), have simpler forms, can be defined from the conditions

- (i) that the hypergeometric function entering the l.h.s. of Eqs. (50)–(52), (73)–(78), is expressible in terms of simpler hypergeometric functions ($F_D^{(r-1)}$ for $F_D^{(r)}$ and ${}_2F_1$, F_1 or F_3 for F_S hypergeometric function);
- (ii) that some of the coefficients entering the inverse differential relations are equal to zero (infinity).

For the hypergeometric functions $F_D(\equiv F_D^{(r)})$ and F_S , the exceptional sets of parameters are listed in Table 1.

4. Mathematica based program for the differential reduction of F_D and F_S hypergeometric functions

In this section, we will present the Mathematica based programs **FdFunction** and **FsFunction** for the differential reduction of Horn-type hypergeometric functions F_D of r variables and F_S of three variables.⁷ In particular, in the application to Lauricella functions F_D , the reduction algorithm, Eq. (6), has the following form:

$$R(x, y)F_D^{(r)}(a + m_a; \vec{b} + \vec{m}_b; c + m_c; \vec{z}) = [P_0(\vec{z}) + P_1(\vec{z})\theta_{z_1} + \dots + P_r(\vec{z})\theta_{z_r}]F_D^{(r)}(a; \vec{b}; c; \vec{z}), \quad (85)$$

where $\vec{m}_b, m_a, m_c$ are sets of integers and $\vec{b}, a, c$ denote the set of parameters. R, P_i are some polynomial and $\theta_{z_i} = z_i \partial_{z_i}$. The differential reduction algorithm in application to the Lauricella–Saran function F_S is:

$$R(\vec{z})F_S(\vec{a} + \vec{m}_a; \vec{b} + \vec{m}_b; c + m_c; \vec{z}) = [P_0(\vec{z}) + P_1(\vec{z})\theta_{z_1} + P_2(\vec{z})\theta_{z_2} + P_3(\vec{z})\theta_{z_3} + P_{12}(\vec{z})\theta_{z_1}\theta_{z_2} + P_{13}(\vec{z})\theta_{z_1}\theta_{z_3}]F_S(\vec{a}; \vec{b}; c; \vec{z}), \quad (86)$$

where, again, $\vec{m}_a, \vec{m}_b, m_c$ denote sets of integers, $\vec{a}, \vec{b}, c$ sets of parameters, and $R, \{P_j\}, \{P_{ij}\}$ some polynomials.

The program is freely available from [30] subject to the license conditions specified. The current version, 1.0, deals with non-exceptional values of a parameters only.

4.1. Package FdFunction

The package can be loaded in the standard way:

```
<< "FdFunction.m"
```

and it includes the following basic routines:

```
FdIndexChange[changingVector, parameterVector], \quad (87)
```

and

```
FdDiffSeries[...], FdSeries[...] \quad (88)
```

The list “changingVector” in Eq. (87) provides the set of integers by which the values of parameters of the Lauricella function F_D are to be changed, i.e., the vector $m_a, \{\vec{m}_b\}, m_c$ in Eq. (85). The set of initial parameters of F_D function are defined in the list “parameterVector” corresponding to the vector $a + m_a; \vec{b} + \vec{m}_b, c + m_c$ and arguments $\vec{z}$ in the l.h.s. of Eq. (85).

The structure of the output of **FdIndexChange**[] is the following:

$$\{\{A_1, A_2, \dots, A_{r+1}\}, \{\text{parameterVectorNew}\}\}, \quad (89)$$

where

- (i) “parameterVectorNew” is the set of new parameters of $F_D^{(r)}$ hypergeometric function;
- (ii) $A_1, A_2, \dots, A_{r+1}$ are the rational functions corresponding to the ratios of $P_0/R, P_1/R, P_2/R \dots$ of functions entering in Eq. (85).

The functions **FdDiffSeries**[] and **FdSeries**[] are designed for the numerical evaluation of F_D hypergeometric functions. They return the Taylor series of F_D in its derivatives, respectively:

```
FdDiffSeries[numberOfvariable, vectorInit, numbSer], \quad (90)
```

```
FdSeries[vectorInit, numbSer], \quad (91)
```

where

- (i) “numberOfvariable” is the list of variable numbers for differentiation;
- (ii) “vectorInit” is the set of Fd parameters;
- (iii) “numbSer” is the number of terms in Taylor expansion.

Let us present a number of examples for the usage⁸

Example 1. Differential reduction of the hypergeometric function $F_D^{(2)}$ of two variables.⁹

FdIndexChange [{-1, {1, 0}, 1}, {a, {b₁, b₂}, c, {z₁, z₂}}]

$$\left\{ \left\{ \frac{c(-1+z_2) - b_2 z_2 + z_1(-1+a+b_1(-1+z_2) + z_2 - a z_2 + b_2 z_2)}{c(-1+z_2)}, \right. \right. \\ \frac{1 - z_2 - b_2 z_2 + z_1(-1+b_1(-1+z_2) + z_2 + b_2 z_2) + a(-1+z_1+z_2-z_1 z_2)}{(-1+a)c(-1+z_2)}, \\ \frac{1 - a(1+z_1(-2+z_2)) - b_2 z_2 + c z_2 + z_1(-2-c+b_1(-1+z_2) + z_2 + b_2 z_2)}{(-1+a)c(-1+z_2)}, \\ \left. \left. -1+a, \{1+b_1, b_2\}, 1+c, \{z_1, z_2\} \right\} \right\} \quad (92)$$

⁷ The programs have been tested for Mathematica version 8.0.

⁸ All functions in the package **HYPERDIRE** generate output without additional simplification. This is done for the maximum efficiency of the algorithm. To bring the output into a simpler form, we recommend to use in addition the command **Simplify**. All examples considered here have been treated with the command **Simplify**[...]. Subsequent to the call of **HYPERDIRE**.

⁹ When $r = 2$, the Lauricella function F_D coincides with the Appell function F_1 and the package **AppellF1F4** [8] can be used for the differential reduction.

In an explicit form:

$$F_D^{(2)}(a; b_1, b_2, c; z_1, z_2) = \left[\frac{c(-1+z_2) - b_2 z_2 + z_1(-1+a+b_1(-1+z_2) + z_2 - a z_2 + b_2 z_2)}{c(-1+z_2)} + \frac{1 - z_2 - b_2 z_2 + z_1(-1+b_1(-1+z_2) + z_2 + b_2 z_2) + a(-1+z_1+z_2 - z_1 z_2)}{(-1+a)c(-1+z_2)} \theta_1 + \frac{1 - a(1+z_1(-2+z_2)) - b_2 z_2 + c z_2 + z_1(-2-c+b_1(-1+z_2) + z_2 + b_2 z_2)}{(-1+a)c(-1+z_2)} \theta_2 \right] \times F_D^{(2)}(a-1; b_1+1, b_2, c+1; z_1, z_2). \quad (93)$$

Example 2. Reduction of the hypergeometric function $F_D^{(3)}$ of three variables.

FdIndexChange $[-1, \{1, -1, 0\}, 0], \{a, \{b_1, b_2, b_3\}, c, \{z_1, z_2, z_3\}\}$

$$\left\{ \left\{ \frac{z_1-1}{z_2-1}, \frac{z_1-1}{(a-1)(z_2-1)}, \frac{z_1(a-c+(b_2-1)z_2) - (a-c+b_2-1)z_2}{(a-1)(b_2-1)(z_2-1)z_2}, \frac{z_1-1}{(a-1)(z_2-1)} \right\}, \{a-1, \{b_1+1, b_2-1, b_3\}, c, \{z_1, z_2, z_3\}\} \right\}.$$

This has the explicit form:

$$F_D^{(3)}(a; b_1, b_2, b_3; c; z_1, z_2, z_3) = \left[\frac{z_1-1}{z_2-1} + \frac{z_1-1}{(a-1)(z_2-1)} \theta_1 + \frac{z_1(a-c+(b_2-1)z_2) - (a-c+b_2-1)z_2}{(a-1)(b_2-1)(z_2-1)z_2} \theta_2 + \frac{z_1-1}{(a-1)(z_2-1)} \theta_3 \right] F_D^{(3)}(a-1; b_1+1, b_2-1, b_3; c; z_1, z_2, z_3). \quad (94)$$

Example 3. Reduction of hypergeometric function $F_D^{(5)}$ of five variables.

FdIndexChange $[-1, \{0, 1, 0, 0, -1\}, 0], \{a, \{b_1, b_2, b_3, b_4, b_5\}, c, \{z_1, z_2, z_3, z_4, z_5\}\}$

$$\left\{ \left\{ \frac{z_2-1}{z_5-1}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2(a+(b_5-1)z_5-c) - z_5(a+b_5-c-1)}{(a-1)(b_5-1)(z_5-1)z_5} \right\}, \{a-1, \{b_1, b_2+1, b_3, b_4, b_5-1\}, c, \{z_1, z_2, z_3, z_4, z_5\}\} \right\}$$

This has the explicit form:

$$F_D^{(5)}(a; b_1, b_2, b_3, b_4, b_5; c; z_1, z_2, z_3, z_4, z_5) = \left[\frac{z_2-1}{z_5-1} + \frac{z_2-1}{(a-1)(z_5-1)} \theta_1 + \frac{z_2-1}{(a-1)(z_5-1)} \theta_2 + \frac{z_2-1}{(a-1)(z_5-1)} \theta_4 + \frac{z_2(a+(b_5-1)z_5-c) - z_5(a+b_5-c-1)}{(a-1)(b_5-1)(z_5-1)z_5} \theta_5 \right] F_D^{(5)}(a-1; b_1+1, b_2-1, b_3; c; z_1, z_2, z_3, z_4, z_5).$$

The hypergeometric function F_D is not built into the current version of Mathematica. The series representation of the hypergeometric function $F_D^{(r)}$, Eq. (38), is implemented in our package. The functions **FdDiffSeries**[] and **FdSeries**[] allow to make numerical cross-checks of the results of the differential reduction. The corresponding examples for using these functions are all collected in the file `example-FdFunction.m`, which is available in [30].

4.2. Package *FsFunction*

Again, the program can be loaded in a standard way:

```
<< "FsFunction.m"
```

and its structure and output are similar to the **FdFunction** package. The package **FsFunction** includes the following basic routines:

$$\mathbf{FsIndexChange}[\text{changingVector}, \text{parameterVector}], \quad (95)$$

Here, again, “changingVector” is the list of integers by which the values of the parameters of the function F_S are to be changed, i.e., the vectors $\vec{m}_a, \vec{m}_b, m_c$ in Eq. (86), while the set of initial parameters of the function F_S is defined in the list “parameterVector” corresponding to the vector $\vec{a} + \vec{m}_a; \vec{b} + \vec{m}_b, c + m_c$ and the arguments $\vec{z}$ in the l.h.s. of Eq. (86).

The structure of the output of **FsIndexChange**[] is the following:

$$\{A, B, C, D, E, F\}, \{\text{parameterVectorNew}\}, \quad (96)$$

where

- (i) “parameterVectorNew” is the set of new parameters of the function F_S ;
- (ii) A, B, C, D, E, F are the rational functions corresponding to the ratios of $P_0/R, P_1/R, P_2/R, P_3/R, P_{12}/R$ and P_{13}/R entering Eq. (86).

Example 4. Reduction of F_S .

FsIndexChange [[1, −1, 0, 0, 0, 0], { $a_1, a_2, b_1, b_2, b_3, c, z_1, z_2, z_3$ }]

$$\left\{ \left\{ \frac{a_1 + b_1 z_1 + b_2 + b_3 - c + 1}{a_1 + b_2 + b_3 - c + 1}, -\frac{1 - z_1}{a_1 + b_2 + b_3 - c + 1}, -\frac{z_2(-a_1 - b_2 - b_3 + c - 1) - b_1 z_1(z_2 - 1)}{(a_2 - 1)z_2(a_1 + b_2 + b_3 - c + 1)}, \right. \right. \\ \left. -\frac{z_3(-a_1 - b_2 - b_3 + c - 1) - b_1 z_1(z_3 - 1)}{(a_2 - 1)z_3(a_1 + b_2 + b_3 - c + 1)}, \right. \\ \left. -\frac{z_2 - z_1(z_2 - 1)}{(a_2 - 1)z_2(a_1 + b_2 + b_3 - c + 1)}, -\frac{z_3 - z_1(z_3 - 1)}{(a_2 - 1)z_3(a_1 + b_2 + b_3 - c + 1)} \right\}, \\ \left. \left\{ a_1 + 1, a_2 - 1, b_1, b_2, b_3, c, z_1, z_2, z_3 \right\} \right\} \quad (97)$$

This has the explicit form:

$$F_S(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \left[\frac{a_1 + b_1 z_1 + b_2 + b_3 - c + 1}{a_1 + b_2 + b_3 - c + 1} - \frac{1 - z_1}{a_1 + b_2 + b_3 - c + 1} \theta_1 \right. \\ \left. - \frac{z_2(-a_1 - b_2 - b_3 + c - 1) - b_1 z_1(z_2 - 1)}{(a_2 - 1)z_2(a_1 + b_2 + b_3 - c + 1)} \theta_2 \right. \\ \left. - \frac{z_3(-a_1 - b_2 - b_3 + c - 1) - b_1 z_1(z_3 - 1)}{(a_2 - 1)z_3(a_1 + b_2 + b_3 - c + 1)} \theta_3 \right. \\ \left. - \frac{z_2 - z_1(z_2 - 1)}{(a_2 - 1)z_2(a_1 + b_2 + b_3 - c + 1)} \theta_1 \theta_2 - \frac{z_3 - z_1(z_3 - 1)}{(a_2 - 1)z_3(a_1 + b_2 + b_3 - c + 1)} \theta_1 \theta_3 \right] \\ \times F_S(a_1 + 1, a_2 - 1; b_1, b_2, b_3; c; z_1, z_2, z_3). \quad (98)$$

Example 5. Reduction of F_S

FsIndexChange [[1, 0, 0, 0, 1, 2], { $a_1, a_2, b_1, b_2, b_3, c, z_1, z_2, z_3$ }]

$$\left\{ \left\{ -\frac{b_1 z_1(z_2 - 1)(-a_2 z_3 - a_1 + c) - z_2(a_2(z_3(b_3 - c) + b_2(z_3 - 1)) + c(c + 1)) + a_2 b_3 z_3 - a_2 c z_3 + c^2 + c}{c(c + 1)(z_2 - 1)}, \right. \right. \\ \left. \frac{z_1(a_2 z_3 + a_1 - c) - a_2 z_3 + c}{c(c + 1)}, \right. \\ \left. -\frac{z_2(a_2 - b_2 z_3 - b_3 z_3 + b_2 + c z_3 - 2c - 1) - a_2 z_3 - b_1 z_1(z_2 - 1)(z_3 - 1) + b_3 z_3 + c + z_3}{c(c + 1)(z_2 - 1)}, \right. \\ \left. \frac{z_2(z_3(b_3 - c) + b_2(z_3 - 1) + c) + b_1 z_1(z_2 - 1)(z_3 - 1) - b_3 z_3 + c z_3 - c}{c(c + 1)(z_2 - 1)}, \frac{z_1(z_2 - 1) - z_2(z_3 - 1)}{c(c + 1)(z_2 - 1)}, -\frac{-z_3 z_1 + z_1 + z_3}{c^2 + c} \right\}, \\ \left. \left\{ a_1 + 1, a_2, b_1, b_2, b_3 + 1, c + 2, z_1, z_2, z_3 \right\} \right\}.$$

This has the explicit form:

$$F_S(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) \\ = \left[-\frac{b_1 z_1(z_2 - 1)(-a_2 z_3 - a_1 + c) - z_2(a_2(z_3(b_3 - c) + b_2(z_3 - 1)) + c(c + 1))}{c(c + 1)(z_2 - 1)} \right. \\ \left. + \frac{a_2 b_3 z_3 - a_2 c z_3 + c^2 + c}{c(c + 1)(z_2 - 1)} + \frac{z_1(a_2 z_3 + a_1 - c) - a_2 z_3 + c}{c(c + 1)} \theta_1 \right]$$

$$\begin{aligned}
& - \frac{z_2 (a_2 - b_2 z_3 - b_3 z_3 + b_2 + c z_3 - 2c - 1) - a_2 z_3 - b_1 z_1 (z_2 - 1) (z_3 - 1)}{c(c+1)(z_2 - 1)} \theta_2 + \frac{b_3 z_3 + c + z_3}{c(c+1)(z_2 - 1)} \theta_2 \\
& + \frac{z_2 (z_3 (b_3 - c) + b_2 (z_3 - 1) + c) + b_1 z_1 (z_2 - 1) (z_3 - 1) - b_3 z_3 + c z_3 - c}{c(c+1)(z_2 - 1)} \theta_3 \\
& \times \left[\frac{(z_1 + (z_2 - 1) - z_2) (z_3 - 1)}{c(c+1)(z_2 - 1)} \theta_1 \theta_2 - \frac{-z_3 z_1 + z_1 + z_3}{c^2 + c} \theta_1 \theta_3 \right] \times F_5(a_1 + 1, a_2; b_1, b_2, b_3 + 1; c + 2; z_1, z_2, z_3). \quad (99)
\end{aligned}$$

Also the hypergeometric function F_5 is not built into the current version of `Mathematica`, whereas our package implements the series representation of the hypergeometric function F_5 , Eq. (54). Again, this series representation is suitable for numerical checks of the results of the differential reduction and the corresponding examples are gathered in the file `example-FsFunction.m` available from [30].

5. On the construction of coefficients of the ε -expansion of Horn-type hypergeometric functions

For physical applications, the construction of analytical coefficients of the Laurent expansion of hypergeometric functions around particular values of parameters (integer, half-integer, rational) is necessary [11,12,31]. Many efforts have been made in the past in attempts to write results of the Laurent expansion in terms of multiple polylogarithms [32]. There are a number of different though entirely equivalent ways to describe the hypergeometric functions [3]:

- (i) as a multiple series;
- (ii) as a solution of a system of differential equations;
- (iii) as an integral of the Euler type.

Each of these approaches can be used for the construction of the Laurent expansion of hypergeometric functions and each of them has some technical advantage or disadvantage in comparison with the other ones.

The most universal technique which does not depend on the order of differential equation is based on the algebra of multiple sums [11,12]. For the hypergeometric functions, where algorithms [11,12] are applicable, the result of the ε -expansion are automatically written in terms of multiple polylogarithms. The algorithms [11] have been implemented in a few packages [33]. However there are some technical problems with the extension of this approach to rational values of parameters and its application specific classes of hypergeometric functions.

The differential equation approach [18] allows to analyze arbitrary sets of parameters simultaneously and to construct the solution in terms of iterated integrals, but for any hypergeometric function the Pfaff system of differential equations should be constructed.

Purely numerical approaches [34] can be applied to arbitrary values of the parameters. However this technique typically does not produce stable numerical result around the region of singularities of hypergeometric function.

The integral representation is well suited for the construction of the ε -expansion for a limited number of types of hypergeometric functions [35].

For hypergeometric functions considered in this paper, the physically interesting cases correspond to the construction of the ε -expansion for the hypergeometric function F_D around integer values of parameters [36], and for F_5 around half-integer values of parameters [15], respectively. For the Appell functions F_D , the following theorem about structure of coefficients of all-order ε -expansion is valid:

Theorem 2. *The ε -expansions of the Appell hypergeometric function F_D of r -variables and its first derivative have the following structure:*

$$F_D(a\varepsilon; \{b_i\varepsilon\}; 1 + c\varepsilon; \vec{z}) = 1 + a \sum_{k=2}^{\infty} r_k \Phi_k \left(\left\{ 1, \vec{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (100a)$$

$$z_i \frac{d}{dz_i} F_D(a\varepsilon; \{b_j\varepsilon\}; 1 + c\varepsilon; \vec{z}) = a \sum_{k=2}^{\infty} d_k \rho_{k-1} \left(\left\{ 1, \vec{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (100b)$$

where $a, \{b_j\}, c$ are arbitrary numbers, ε is an arbitrary small parameter, r_k and d_k are the polynomial in $a, \{b_j\}, c$ and Φ_k and ρ_k include only multiple polylogarithms of weight k .

The proof of this theorem is based on the explicit construction of the iterative solution of the system of differential equations.

Up to functions of weight 3 the explicit value of the coefficients are the following

$$\begin{aligned}
F_D^{(r)}(a\varepsilon; \{b_i\varepsilon\}; 1 + c\varepsilon; \vec{z}) &= 1 + \sum_{k=2}^{\infty} \omega_0^{(k)}(\vec{z}) \varepsilon^k, \\
z_i \frac{d}{dz_i} F_D^{(r)}(a\varepsilon; \{b_k\varepsilon\}; 1 + c\varepsilon; \vec{z}) &\equiv z_i \frac{ab_i \varepsilon^2}{1 + c\varepsilon} F_D^{(r)}(1 + a\varepsilon; 1 + b_i\varepsilon, \{b_j\varepsilon\}; 2 + c\varepsilon; \vec{z}) \\
&= \sum_{k=2}^{\infty} \omega_i^{(k)}(\vec{z}) \varepsilon^k, \quad (101)
\end{aligned}$$

where

$$\frac{\omega_0^{(2)}(\vec{z})}{a} = \sum_{j=1}^r b_j \text{Li}_2(z_j) \quad (102)$$

$$\frac{\omega_0^{(3)}(\vec{z})}{a} = \sum_{j=1}^r (a - c + b_j) b_j S_{1,2}(z_j) - c \sum_{j=1}^r b_j \text{Li}_3(z_j) + \sum_{i < j=1; i \neq j}^r b_i b_j \left[\text{Li}_{1,2} \left(\frac{z_i}{z_j}, z_j \right) + \text{Li}_{1,2} \left(\frac{z_j}{z_i}, z_i \right) \right], \quad (103)$$

and

$$\frac{\omega_j^{(2)}(\vec{z})}{a} = -b_j \ln(1 - z_j), \quad (104)$$

$$\frac{\omega_j^{(3)}(\vec{z})}{a} = b_j \left[\frac{1}{2} (a + b_j - c) \ln^2(1 - z_j) - c \text{Li}_2(z_j) + \sum_{k=1; k \neq j}^r b_k \text{Li}_{1,1} \left(\frac{z_k}{z_j}, z_j \right) \right], \quad (105)$$

where $j = 1, \dots, r$ and

$$\text{Li}_{k_1, k_2, \dots, k_n}(x_1, x_2, \dots, x_n) = \sum_{0 < m_1 < m_2 < \dots < m_{n-1} < m_n} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \times \dots \times \frac{x_n^{m_n}}{m_n^{k_n}}, \quad (106)$$

denotes a multiple polylogarithm [32]. $\text{Li}_k(x)$ is a classical and $S_{1,2}$ a Nielsen polylogarithm, respectively.

Starting from $\omega_j^{(4)}(\vec{z})$, the result of the ε -expansion has more a complicated form. Since the F_D hypergeometric function is symmetric with respect to the exchange of the variables $(b_i, z_i) \leftrightarrow (b_j, z_j)$ we present here result only for $\omega_1^{(4)}(\vec{z})$. For the case of two variables, we have:

$$\begin{aligned} \frac{\omega_1^{(4)}(x, y)}{ab_1} &= \Delta_{b_1}^2 \text{Li}_{1,1,1}(1, 1, x) + [ab_1 - c \Delta_{b_1}] \text{Li}_{2,1}(1, x) - c \Delta_{b_1} \text{Li}_{1,2}(1, x) + c^2 \text{Li}_3(x) \\ &\quad + b_2 \Delta_{b_1} \text{Li}_{1,1,1} \left(\frac{y}{x}, 1, x \right) + b_2 \Delta_{b_2} \text{Li}_{1,1,1} \left(1, \frac{y}{x}, x \right) + b_1 b_2 \text{Li}_{1,1,1} \left(\frac{x}{y}, \frac{y}{x}, x \right) \\ &\quad - c b_2 \text{Li}_{1,2} \left(\frac{y}{x}, x \right) + (a - c) b_2 \text{Li}_{2,1} \left(\frac{y}{x}, x \right), \end{aligned} \quad (107)$$

where

$$\Delta_b = a - c + b.$$

Eq. (107) coincides with results of [33].

For r -variable case ($r \geq 3$) the iterative solution is:

$$\begin{aligned} &\frac{1}{ab_1 b_r} \left[\omega_1^{(4)}(z_1, \dots, z_{r-1}, z_r) - \omega_1^{(4)}(z_1, \dots, z_{r-1}, z_r = 0) \right] \\ &= (a - c) \left[\text{Li}_1 \left(\frac{z_r}{z_1} \right) \text{Li}_{1,1}(1, z_1) - \text{Li}_{1,1,1} \left(1, z_1, \frac{z_r}{z_1} \right) - \text{Li}_{1,2}(1, z_r) \right] \\ &\quad + b_r \text{Li}_{1,1,1} \left(1, \frac{z_r}{z_1}, z_1 \right) - c \text{Li}_{1,2} \left(\frac{z_r}{z_1}, z_1 \right) \\ &\quad + b_1 \left[\text{Li}_{1,1,1} \left(\frac{z_1}{z_r}, \frac{z_r}{z_1}, z_1 \right) + \text{Li}_{1,1,1} \left(\frac{z_r}{z_1}, 1, z_1 \right) \right] \\ &\quad + \sum_{j=1; j \neq 1; j \neq r}^r b_j \left[\text{Li}_{1,1,1} \left(\frac{z_j}{z_r}, \frac{z_r}{z_1}, z_1 \right) + \text{Li}_{1,1,1} \left(\frac{z_r}{z_j}, \frac{z_j}{z_1}, z_1 \right) \right], \end{aligned} \quad (108)$$

and further simplifications can be done by the help of the stuffle relation:

$$\begin{aligned} \text{Li}_1 \left(\frac{z_r}{z_1} \right) \text{Li}_{1,1}(1, z_1) &= \text{Li}_{1,1,1} \left(1, z_1, \frac{z_r}{z_1} \right) + \text{Li}_{1,1,1} \left(1, \frac{z_r}{z_1}, z_1 \right) + \text{Li}_{1,1,1} \left(\frac{z_r}{z_1}, 1, z_1 \right) \\ &\quad + \text{Li}_{1,2}(1, z_r) + \text{Li}_{2,1} \left(\frac{z_r}{z_1}, z_1 \right). \end{aligned} \quad (109)$$

As an example for the application of these coefficients, let us evaluate the function $F_D^{(r)}(1; 1, \dots, 1; 2; z_1, \dots, z_r)$. Using the differential reduction algorithm, we get the following expression:

$$F_D^{(r)}(1 + a; \{1 + b_k\}; 1 + c; z_1, \dots, z_r) = \frac{c}{a} \sum_{i=1}^r \frac{z_i^{r-2}}{\prod_{j=1; j \neq i}^r (z_i - z_j)} \frac{\theta_i}{b_i} F_D^{(r)}(a; \{b_k\}; c; z_1, \dots, z_r). \quad (110)$$

After replacing, $a \rightarrow a\varepsilon$, $b_k \rightarrow b_k\varepsilon$, $c \rightarrow 1 + c\varepsilon$ and taking limit $\varepsilon \rightarrow 0$, we get (see [37]):

$$\begin{aligned} F_D^{(r)}(1; \{1\}; 2; z_1, \dots, z_r) &= \sum_{i=1}^r \frac{z_i^{r-2}}{\prod_{j=1; j \neq i}^r (z_i - z_j)} \frac{\omega_i^{(2)}(\vec{z})}{ab_i} \\ &\equiv - \sum_{i=1}^r \frac{z_i^{r-2}}{\prod_{j=1; j \neq i}^r (z_i - z_j)} \ln(1 - z_i). \end{aligned} \quad (111)$$

In the end, we also present the theorem¹⁰ about structure of the ε -expansion for two other Appell hypergeometric functions of two variables:

Theorem 3. *The ε -expansions of the Appell hypergeometric functions F_2 and F_3 of 2-variables and their derivatives have the following structure:*

$$F_2(a\varepsilon, \{b_i\varepsilon\}; \{1 + c_i\varepsilon\}; z_1, z_2) = 1 + \sum_{k=2}^{\infty} r_k \Phi_k \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (112a)$$

$$z_i \frac{d}{dz_i} F_2(a\varepsilon, \{b_i\varepsilon\}; \{1 + c_i\varepsilon\}; z_1, z_2) = \sum_{k=2}^{\infty} d_k \rho_{k-1} \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (112b)$$

$$z_1 z_2 \frac{d}{dz_1} \frac{d}{dz_2} F_2(a\varepsilon, \{b_i\varepsilon\}; \{1 + c_i\varepsilon\}; z_1, z_2) = \sum_{k=3}^{\infty} q_k \sigma_{k-2} \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (112c)$$

$$F_3(a_1\varepsilon, a_2\varepsilon; b_1\varepsilon, b_2\varepsilon; 1 + c\varepsilon; z_1, z_2) = 1 + \sum_{k=2}^{\infty} \tilde{r}_k \tilde{\Phi}_k \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (113a)$$

$$z_i \frac{d}{dz_i} F_3(a_1\varepsilon, a_2\varepsilon; b_1\varepsilon, b_2\varepsilon; 1 + c\varepsilon; z_1, z_2) = \sum_{k=2}^{\infty} \tilde{d}_k \tilde{\rho}_{k-1} \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (113b)$$

$$z_1 z_2 \frac{d}{dz_1} \frac{d}{dz_2} F_3(a_1\varepsilon, a_2\varepsilon; b_1\varepsilon, b_2\varepsilon; 1 + c\varepsilon; z_1, z_2) = \sum_{k=4}^{\infty} \tilde{q}_k \tilde{\sigma}_{k-2} \left(\left\{ 1, \bar{z}, \frac{z_i}{z_j} \right\} \right) \varepsilon^k, \quad (113c)$$

where $\{a_i, b_j\}$, c are arbitrary numbers, ε is an arbitrary small parameter, r_k, d_k, q_k and $\tilde{r}_k, \tilde{d}_k, \tilde{q}_k$ are polynomials in $\{a_i, b_j\}$, c , Φ_k, ρ_k, σ_k and $\tilde{\Phi}_k, \tilde{\rho}_k, \tilde{\sigma}_k$ include only multiple polylogarithms of weight $\mathbf{k}$.

Example.

$$F_3(a_1\varepsilon, a_2\varepsilon; b_1\varepsilon, b_2\varepsilon; 1 + c\varepsilon; z_1, z_2) = 1 + \varepsilon^2 \sum_{j=1}^2 a_j b_j \text{Li}_2(z_j) + \varepsilon^3 \sum_{j=1}^2 a_j b_j \{ (a_j + b_j - c) S_{1,2}(z_j) - c \text{Li}_3(z_j) \} + O(\varepsilon^4). \quad (114)$$

The ε -expansion for F_5 around half-integer values of parameters will be considered in another paper.

6. Conclusion

The differential-reduction algorithm for Horn-type hypergeometric functions allows one to compare different functions in this class whose values for the parameters differ by integers. This proceeds in an entirely algorithmic manner suitable for automation in a computer algebra system. In this paper, we have presented the Mathematica-based programs **FdFunction** and **FsFunction** for the differential reduction of the generalized hypergeometric function F_D of r variables and the Lauricella–Saran hypergeometric function F_5 of three variables. Both functions are related with one-loop massive Feynman diagrams and both belong to the class of Horn-type hypergeometric function of order two.

For hypergeometric function F_5 , Eq. (54), we have presented a detailed analysis of the dimension of the solution space (see Theorem 1 in Section 3.3). Regarding the expansion of the hypergeometric functions in a small coefficient ε around specific values of their parameters, we have proven two theorems (Theorems 2 and 3). These concern the structure of the all-order ε -expansion of the hypergeometric functions, $F_D^{(r)}, F_2, F_3$ around integer values of parameters. As illustration, the first three coefficients (up to functions of weight 3) have been constructed explicitly.

Acknowledgments

We are grateful to T. Riemann, O. Tarasov, and O. Veretin for useful discussions. This work was supported in part by the Deutsche Forschungsgemeinschaft in SFB/TR 9, and by the Heisenberg–Landau Program. V.V.B. was supported in part by the Russian Foundation for Basic Research RFFI through Grant No. 12-02-31703.

Appendix

For completeness, we present also the integral representation of functions F_D and F_5 :

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(a)} F_D(a; b_1, \dots, b_k; c_1, \dots, c_k; z_1, \dots, z_k) = \int_0^1 du u^{a-1} (1-u)^{c-a-1} (1-uz_1)^{-b_1} \dots (1-uz_n)^{-b_n}, \quad (115)$$

¹⁰ The proof is straightforward and similar to the technique described in [18,38].

$$\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)}{\Gamma(c)}F_S(a_1; a_2; b_1, b_2, b_3; c; z_1, z_2, z_3)$$

$$= \int_{u,v \geq 0} \int_{u+v \leq 1} du dv u^{a_1-1} v^{a_2-1} (1-u-v)^{c-a_1-a_2-1} (1-uz_1)^{-b_1} (1-vz_2)^{-b_2} (1-vz_3)^{-b_3}. \quad (116)$$

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