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Amplitude Dependent Betatron Tuneshifts at Injection in the HERA Proton Ring

P. N. Chirkov *

* IHEP Protvino, Branch of Institute of Nuclear Physics, 142 284 Protvino, Russia

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1 The formalism

The equation of betatron oscillations in a circular accelerator in the presence of the magnetic field imperfections $\vec{h} = (h_x, h_y)$ has the form:

$$\frac{d^2\zeta_1}{d\theta^2} + g_\zeta(\theta)\zeta_1 = -\Delta g_\zeta(\theta)\zeta_1 + \frac{R_o^2}{R} F_\zeta \tag{1}$$

where ζ_1 is an equation symbol for either x_1 or y_1, the later being the particle transverse displacements from the reference orbit;

$$F_x = \Delta p/p_o - h_y(x_1, y_1, \theta)/H_o\ , \qquad F_y = h_x(x_1, y_1, \theta)/H_o;$$

R_0 and R are the average and curvature radii, respectively, of the reference orbit in field H_o; θ is a generalized azimuth which may be expressed by the longitudinal coordinate s as $\theta = s/R_0$; $\Delta p/p_o$ is relative momentum deviation from the reference value p_o.

The periodic functions $g_\zeta(\theta)$ determine the focusing properties of the unperturbed magnetic structure of an accelerator at the reference momentum p_o. The variation of these functions due to particle momentum deviations in such large machines as HERA may be written as follows:

$$\Delta g_\zeta(\theta) \simeq \mp \frac{R_o^2}{R} \frac{G(\theta)}{H_o} \frac{\Delta p}{p_o} \tag{2}$$

where signs "$-,+$" apply to x- and y-direction of the betatron motion, respectively; $G(\theta)$ is the field gradient in accelerator quadrupoles.

The components $h_{x,y}$ of the magnetic field imperfections are expressed in terms of the longitudinal vector potential $A(s)$:

$$h_x = -\frac{\partial A}{\partial y}, \qquad h_y = \frac{\partial A}{\partial x}, \qquad A(s) = \sum_{n=1}^{\infty} A_n (x_1 + iy_1)^n + c.c., \qquad A_n = H_o \frac{b_n + ia_n}{2r^{(n-1)}n} \tag{3}$$

where b_n, a_n are relative additions to the field introduced by the (n-1)-th power normal and skew nonlinearities, respectively, taken at $(x_1 = r, y_1 = 0)$; r is the reference radius, taken as 25 mm for the HERA p-ring. The results of HERA dipole magnet measurements at the field level corresponding to injection energy 40 GeV are summarized in table 1, where $\bar{b}_n, \bar{a}_n$ are systematic values and $< b_n, a_n >$ are values of r.m.s. spread of quantities b_n, a_n Ref.[1].

Let $\zeta_o(\theta)$ be a periodic solution of Eq.(1) when its right-hand side contains only components that do not depend on x_1 and y_1. Then, according to Eqs.(1)-(3) variation of quantity $\zeta = \zeta_1 - \zeta_o$

Table 1: SC-dipole field nonlinearities in units of 10^{-4} at injection

n	2	3	4	5	6	7	8	9	10
$\bar{b}_n$	-0.02	-32.43	0.10	12.33	-0.13	-2.16	0.15	0.45	-0.13
$< b_n >$	0.62	3.19	0.37	1.12	0.35	0.53	0.41	0.47	0.52
$\bar{a}_n$	-0.21	-0.35	0.83	0.22	-0.71	-0.2	0.24	0.10	-0.25
$< a_n >$	2.54	0.74	1.25	0.66	0.56	0.54	0.39	0.67	0.50

characterizing the betatron oscillation with respect to the closed orbit ζ_o is described by the following equation:

$$\begin{aligned} \frac{d^2\zeta}{d\theta^2} + g_\zeta(\theta)\zeta &= -\frac{R_o^2}{RH_o} \cdot \frac{\partial}{\partial \zeta} A^{ef}(\zeta_o + \zeta, \theta) \\ \text{where} \qquad A^{ef} &= A - \frac{G}{2}\frac{\Delta p}{p_o}(x_1^2 - y_1^2) \end{aligned} \tag{4}$$

On applying the averaging technique up to the i-th order ($i = 1, 2$) in perturbation, Refs.[2], [3], one can derive from Eq.(4) the following canonical equations for slowly varying amplitudes squared, $\vec{I} = (I_x, I_y)$, and phases $\vec{\eta} = (\eta_x, \eta_y)$:

$$\frac{dI_\zeta}{d\theta} = -\frac{\partial D^{(i)}}{\partial \eta_\zeta}, \qquad \frac{d\eta_\zeta}{d\theta} = \frac{\partial D^{(i)}}{\partial I_\zeta},$$
$$\zeta = r\sqrt{\frac{\beta_\zeta(\theta)}{\beta_{max}}}\sqrt{I_\zeta}\cos(\mu_\zeta(\theta) + \eta_\zeta)$$

where $\beta_\zeta(\theta)$ is beta–function and $\mu_\zeta = Q_\zeta\theta + \chi_\zeta$ is unperturbed phase advance with a periodic part $\chi_\zeta(\theta)$ and tune Q_ζ. Thus, $\sqrt{I_\zeta}$ is the ζ-oscillation amplitude normalized to r and taken at azimuth where $\beta_\zeta = \beta_{max}$. It should be noted that the quantity I_ζ coincides with an action variable of the unperturbed ζ-oscillation up to a constant factor.

2 Amplitude Dependent Tune Shifts Obtained by First Order Perturbation Theory

In the first approximation of averaging technique the Hamiltonian $D^{(1)}$ has the form Ref.[3]

$$D^{(1)} = \langle D \rangle, \qquad \text{where} \quad D = \frac{2\beta_{max}R_0}{r^2 HR} A^{ef}\{\zeta(I_\zeta, \eta_\zeta, \theta), \theta\}$$

The operator $\langle \ldots \rangle$ denoting the averaging over θ removes fast harmonics from the expansion of D. By taking into account the periodic dependence of D on $\vec{\mu} = (\mu_x, \mu_y)$ and azimuth θ one can put down

$$D = \sum_{k=-\infty}^{\infty} \sum_{\vec{n}} D_{\vec{n},k}(I_x, I_y) \exp\{i(\vec{n}\vec{Q} - k)\theta + i\vec{n}\vec{\eta}\}$$

$$D_{\vec{n},k} = \frac{1}{(2\pi)^3} \iiint_0^{2\pi} D\{r\sqrt{\frac{\beta_\zeta(\theta)}{\beta_{max}}}\sqrt{I_\zeta}\cos\alpha_\zeta, \theta\} \exp\{i[k\theta + \vec{n}\vec{\chi}(\theta) - \vec{n}\vec{\alpha}]\} \; d\alpha_x d\alpha_y d\theta$$

where $\vec{n} = (n_x, n_y)$, $\quad n_{x,y}, k$ are integers.

Systematic variation of quantities I_ζ, η_ζ occurs when the Hamiltonian $D^{(1)}$ does not vanish. Nonvanishing contributions are provided by the slow oscillating harmonics with $|\vec{n}\vec{Q} - k| \ll 1$. Such harmonics can be grouped into two sets: nonresonant and resonant ones. The nonresonant set confines a single "harmonic" $\{\vec{n} = (0,0), k = 0\}$ which determines the betatron tune shift

$$\Delta\vec{Q} = \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{\partial D_{\vec{o},o}}{\partial \vec{I}}.$$

According to the previous treatment one obtains the following expression for $D_{\vec{o},o}$:

$$\begin{aligned}
D_{\vec{o},o} &= \vec{\xi}^{(o)}\vec{I}\,\frac{\Delta p}{p_o} + \sum_{n\geq 2}\sum_{j=0}^{n_o}\sum_{k=0}^{j} \alpha_n(j,k)\; I_y^k I_x^{j-k}, \\
&\qquad \vec{\xi}^{(o)} = \frac{1}{4\pi}\int_0^{L_{ring}} \begin{pmatrix} -\beta_x \\ +\beta_y \end{pmatrix} \frac{G}{H_o R} ds, \\
\alpha_n(j,k) &= \frac{\beta_{max}}{\pi r}\frac{(-1)^k (n-1)!}{(k!(j-k)!)^2} \sum_{m=0}^{n-2j} (m!(n-2j-m)!)^{-1} \\
&\times \int_0^{L_{ring}} \frac{\beta_y^k \beta_x^{(j-k)}}{(4\beta_{max})^j} \left(\frac{y_o}{r}\right)^m \left(\frac{x_o}{r}\right)^{n-2j-m} \left(b_n \cos\left(\frac{\pi}{2}m\right) - a_n \sin\left(\frac{\pi}{2}m\right)\right)\frac{ds}{R}.
\end{aligned} \tag{5}$$

where $n_o = [n/2]$ is the integer; $\vec{\xi}^{(o)}$ is the natural chromaticity. Thus, tune shift caused by magnetic field nonlinearities in the first approximation may be written in the form:

$$\Delta\vec{Q} = \vec{\xi}^{(o)}\frac{\Delta p}{p_o} + \sum_{n\geq 2}\sum_{j=1}^{n_o} \begin{pmatrix} \sum_{k=0}^{j-1} & (j-k) & \alpha_n(j,k) I_y^k I_x^{j-k-1} \\ \sum_{k=1}^{j} & k & \alpha_n(j,k) I_y^{k-1} I_x^{j-k} \end{pmatrix}$$

2.1 Tune shift in case of the closed orbit corrected perfectly

Consider a case when the effects of field multipoles $a_1(s)$ and $b_1(s)$ are eliminated perfectly by means of an appropriate correction system. Then the course of the closed orbit is determined by particle momentum only and has the form:

$$x_o(s) = \psi(s)\frac{\Delta p}{p_o} \quad \text{and} \quad y_o(s) = 0,$$

where $\psi(s)$ is dispersion function of x-direction. It is assumed that the corresponding dispersion function in y-direction is small enough. Then according to Eq.(5) the tune shift is caused entirely by normal nonlinearities and the expression for the coefficient $\alpha_n(j,k)$ becomes

$$\alpha_n(j,k) = \frac{\beta_{max}}{\pi r}\frac{(-1)^k (n-1)!}{(k!(j-k)!)^2 (n-2j)!} \int_0^{L_{ring}} b_n(s) \frac{\beta_y^k \beta_x^{j-k}}{(4\beta_{max})^j} \left(\frac{\psi}{r}\frac{\Delta p}{p_o}\right)^{n-2j} \frac{ds}{R}. \tag{6}$$

It is useful to note that $\Delta\vec{Q}^{(n)}$ caused by the multipole b_n in the case of the perfectly corrected orbit varies with $\Delta p/p_o$ and $\vec{I}$ in accordance with the relationship:

$$\Delta\vec{Q}^{(n)}\left(\gamma\frac{\Delta p}{p_o}, \vec{I}\right) = \gamma^{n-2}\Delta\vec{Q}^{(n)}\left(\frac{\Delta p}{p_o}, \frac{\vec{I}}{\gamma^2}\right),$$

where γ is an arbitrary factor.

By taking into account the periodic dependence of β_ζ and ψ on s in regular cells of the magnetic structure, where practically all superconducting dipoles are located, according to Eq.(6) one has to expect that $\Delta\vec{Q}^{(n)}$ will be mainly determined by the systematic values $\bar{b}_n$. Then, the coefficient $\alpha_n(j,k)$ can be written as follows:

$$\begin{aligned}
\alpha_n(j,k) &= \bar{b}_n \left(\frac{\Delta p}{p_o} 10^3\right)^{n-2j} \Lambda_n(j,k) \\
\Lambda_n(j,l) &= N \frac{\beta_{max}}{\pi r} \frac{(-1)^k (n-1)!}{(k!(j-k)!)^2 (n-2j)!} \int_0^{L_{cell}} \frac{\beta_y^k \beta_x^{j-k}}{(4\beta_{max})^j} \left(\frac{\psi}{r} 10^{-3}\right)^{n-2j} \frac{ds}{R},
\end{aligned} \tag{7}$$

where $N = 104$ is the total number of cells in arcs Ref.[4]. Table 2 shows numerical values of coefficients $\Lambda_n(j,k)$ for the HERA p-ring, the $\beta_{x,y}(s)$ and $\psi(s)$ functions being calculated under the following conditions: $\mu_{x,o} = \mu_{y,o} = 90^o$ is phase advance of the betatron oscillation per a cell; $L_{cell} = 47.017$ m is the length of a cell; $L_{dip} = 8.824$ m is the length of a superconducting dipole; $L_{qu} = 1.861$ m is the length of a superconducting quadrupole. The value of β_{max} was taken as large as 79.5076 m.

Table 2: Values of the $\Lambda_n(j,k)$ coefficients for the HERA p-ring

n	j	k=0	k=1	k=2	k=3	k=4	k=5
2	1	764.752	-764.752				
3	1	88.788	-76.316				
4	1	7.394	-5.890				
	2	162.749	-456.311	162.749			
5	1	0.6446	-0.4173				
	2	39.9539	-98.3748	30.4981			
6	1	0.0501	-0.0286				
	2	6.2419	-13.6752	3.6581			
	3	57.7496	-273.7438	273.7438	-57.7496		
7	1	0.0038	-0.0019				
	2	0.7922	-1.5664	0.3606			
	3	22.0776	-95.1110	82.0910	-15.5374		
8	1	0.0003	-0.0001				
	2	0.0891	-0.1613	0.0320			
	3	4.9833	-19.7722	14.7968	-2.4786		
	4	26.1479	-180.1552	286.7942	-180.1552	26.1479	
9	1	0.00002	-0.00001				
	2	0.00927	-0.01555	0.00268			
	3	0.86584	-3.20257	2.09637	-0.30707		
	4	13.66114	-88.16419	123.59486	-67.72154	9.11442	
10	1	0.000002	-0.000001				
	2	0.000912	-0.001436	0.000218			
	3	0.128051	-0.446233	0.258542	-0.032820		
	4	4.048180	-24.710072	30.882272	-14.659025	1.806552	
	5	13.657135	-128.510121	274.713414	-274.713414	128.510121	-13.657135

Thus, taking into account only the systematic values $\bar{b}_n$ of superconducting dipoles located in arcs the expression for tune shift can be put down in the form:

$$\Delta\vec{Q}^{(n)} = \bar{b}_n \sum_{j=1}^{n_o} \left(\frac{\Delta p}{p_o}10^3\right)^{n-2j} \begin{pmatrix} \sum\limits_{k=0}^{j-1} & (j-k) & \Lambda_n(j,k) I_y^k I_x^{j-k-1} \\ \sum\limits_{k=1}^{j} & k & \Lambda_n(j,k) I_y^{k-1} I_x^{j-k} \end{pmatrix} \tag{8}$$

On using this expression and values indicated in table 2, one determines $\Delta\vec{Q}$ for particles with various $\Delta p/p_o$ and $\vec{I}$ at any set of systematic multipole errors. Consequently, it is possible to plot beam particle distribution in the $\{Q_x, Q_y\}$-plane or represent tune spread in the form of the relevant tables.

Now, let us consider only a simple case of the 1-D x-oscillation which has taken place during dynamic aperture measurement in November 1994. In such a case (i.e. when $I_y \to 0$) formulae (8) is simplified to

$$\Delta\vec{Q}^{(n)} = \bar{b}_n \sum_{j=1}^{n_o} \left(\frac{\Delta p}{p_o}10^3\right)^{n-2j} \begin{pmatrix} j\Lambda_n(j,0) \\ \Lambda_n(j,1) \end{pmatrix} I_x^{j-1}$$

The table 3 shows the ΔQ_x and ΔQ_y versus the 1-D x-oscillation amplitude $r\sqrt{I_x}$ at $(\Delta p/p_o)10^3 = \pm 0.2$ and 0.0 with allowance for all systematic normal nonlinearities indicated in table 1. It is assumed that the tune shifts introduced by the systematic normal $\bar{b}_2$ and $\bar{b}_3$ multipoles and the natural chromaticity $\xi^{(o)}$ are compensated perfectly by means of the relevant correction systems.

Naturally, in this case the tune shift at the sufficiently large amplitude is mainly determined by effect of $\bar{b}_5$. The question how perfectly the $\vec{\Delta Q}^{(5)}$ tune shift can be eliminated with a special lumped correction system will be considered in the next section.

2.2 The $\Delta\vec{Q}^{(5)}$ tune shift compensation

The previous treatment shows that at injection the main contribution to the amplitude dependent tune shift is coming from $\bar{b}_5$. Consider the reduction of $\Delta\vec{Q}^{(5)}$ by means of decapole correction coils mounted on the beam pipe in half the number of all dipoles located in arcs Ref.[4].

In order to distinguish the harmful b_5 from controlling decapole of these coils the latter is designated as c_5, that is

$$c_5 = \frac{1}{H_o}\frac{\partial^4 h_y}{\partial x_1^4}\frac{r^4}{4!}$$

Set c_5 equal in all decapole coils and let $\lambda_5(j,k)$, similar to $\Lambda_5(j,k)$, be coefficients determining the tune shift produced by c_5. Values of $\lambda_5(j,k)$ are shown in table 4 where for comparison the $\Lambda_5(j,k)$ coefficients from table 2 are indicated too. At calculation of $\lambda_5(j,k)$ the coil length was taken as 2.824 m.

It is easy to check that the ratio $\Lambda_5(j,k)/\lambda_5(j,k)$ is slightly different for various pairs of (j,k). Therefore, such a decapole correction system can not eliminate $\Delta\vec{Q}^{(5)}$ absolutely. Consider this question in more details.

According to data in table 4, tune shifts of particles with quite large amplitudes are mainly determined by terms proportional to $(\Delta p/p_o)^1$. In this case one can represent approximately

$$\Delta\vec{Q}^{(5)}_{tot} = \Delta\vec{Q}^{(5)}_{dip} + \Delta\vec{Q}^{(5)}_{cor} \simeq \frac{\Delta p}{p_o}10^3 \begin{pmatrix} t_{xx} & t_{xy} \\ t_{xy} & t_{yy} \end{pmatrix} \vec{I}$$

where:

$$\begin{pmatrix} t_{xx}/2 \\ t_{xy} \\ t_{yy}/2 \end{pmatrix} = \begin{pmatrix} \Lambda_5(2,0) \\ \Lambda_5(2,1) \\ \Lambda_5(2,2) \end{pmatrix} \bar{b}_5 + \begin{pmatrix} \lambda_5(2,0) \\ \lambda_5(2,1) \\ \lambda_5(2,2) \end{pmatrix} c_5$$

We choose value of c_5 such to minimize quantity $|\Delta\vec{Q}^{(5)}_{tot}|$ on the triangular amplitude range $0 \leq I_x + I_y \leq I_o$ in the $\{I_x, I_y\}$-plane. Under this condition and taking into account values of $\Lambda_5(2,k)$ and $\lambda_5(2,k)$, is easy to derive the compensation relation:

$$c_5 \simeq -6.8156\,\bar{b}_5 \tag{9}$$

Figs. 1a and 1b show tune shifts that are introduced by all $\bar{b}_{n\geq 4}$ for beam particles with different $I_x + I_y = I_o$ and $\Delta p/p_o = \pm 0.7\cdot 10^{-3}$. Fig. 1a is the case when decapole correction is switched off, $c_5 = 0$. Fig. 1b is the case under condition (9).

Thus, the decapole correction system existing in the p-ring is able to compensate quite well the tune shift produced by one of the most dangerous nonlinearities at injection.

Acknowledgements

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References

[1] "Dr. F. Willeke gave me these data in October 1991".

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Table 3: Tune shift for the 1-D x-oscillation

$(\Delta p/p_o)\cdot 10^3$	-0.2		0.0		$+0.2$	
$rI_x^{1/2}$, mm	ΔQ_x	ΔQ_y	ΔQ_x	ΔQ_y	ΔQ_x	ΔQ_y
1	-0.00003	0.00003	0.00001	-0.00001	0.00005	-0.00005
2	-0.00011	0.00013	0.00002	-0.00003	0.00016	-0.00019
3	-0.00024	0.00029	0.00005	-0.00006	0.00034	-0.00042
4	-0.00042	0.00051	0.00008	-0.00011	0.00059	-0.00074
5	-0.00066	0.00079	0.00013	-0.00018	0.00092	-0.00115
6	-0.00095	0.00113	0.00018	-0.00025	0.00131	-0.00164
7	-0.00129	0.00154	0.00024	-0.00034	0.00178	-0.00222
8	-0.00168	0.00201	0.00031	-0.00043	0.00231	-0.00288
9	-0.00212	0.00254	0.00039	-0.00054	0.00290	-0.00362
10	-0.00262	0.00313	0.00047	-0.00065	0.00356	-0.00443
11	-0.00316	0.00378	0.00056	-0.00077	0.00428	-0.00532
12	-0.00375	0.00449	0.00065	-0.00089	0.00505	-0.00628
13	-0.00439	0.00526	0.00074	-0.00102	0.00588	-0.00730
14	-0.00508	0.00608	0.00084	-0.00115	0.00676	-0.00838
15	-0.00581	0.00698	0.00094	-0.00128	0.00789	-0.00952
16	-0.00659	0.00789	0.00104	-0.00141	0.00867	-0.01072
17	-0.00741	0.00888	0.00114	-0.00154	0.00969	-0.01196
18	-0.00828	0.00992	0.00124	-0.00166	0.01076	-0.01325
19	-0.00919	0.01101	0.00133	-0.00178	0.01188	-0.01458
20	-0.01015	0.01216	0.00142	-0.00189	0.01300	-0.01594
21	-0.01115	0.01337	0.00151	-0.00198	0.01416	-0.01734
22	-0.01220	0.01464	0.00158	-0.00205	0.01536	-0.01875
23	-0.01330	0.01598	0.00164	-0.00209	0.01657	-0.02018
24	-0.01445	0.01740	0.00167	-0.00209	0.01779	-0.02158
25	-0.01566	0.01892	0.00168	-0.00204	0.01902	-0.02298

Table 4: Comparison $\lambda_5(j,k)$ and $\Lambda_5(j,k)$ coefficients

	$\lambda_5(j,k)$			$\Lambda_5(j,k)$		
j	$k=0$	$k=1$	$k=2$	$k=0$	$k=1$	$k=2$
1	0.0706	-0.0677		0.6446	-0.4173	
2	4.2348	-16.1416	4.1190	39.9539	-98.3748	30.4981

Figure 1a. **Tune shifts at c_5=0:**
"+" for the 1-D x-oscillations,
"•" for the 1-D y-oscillations;
$(\Delta p/p_o)10^3 = +0.7$ **for solid lines,**
$(\Delta p/p_o)10^3 = -0.7$ **for dashed lines;**
$r\sqrt{I_o} = 10$ **mm(1), 15 mm(2), 20 mm(3), 25 mm(4).**

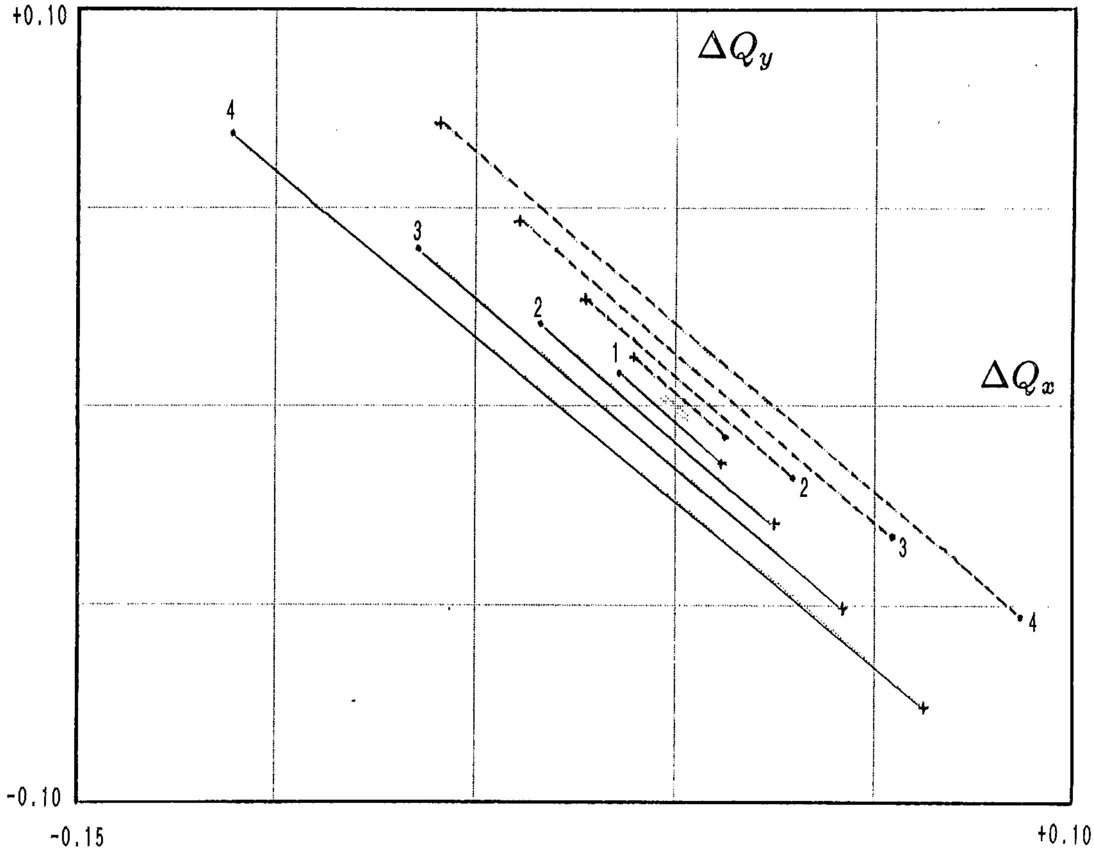

Figure 1b. Tune shifts with decapole correction:
"+" for the 1-D x-oscillations,
"•" for the 1-D y-oscillations;
$(\Delta p/p_o)10^3 = +0.7$ for solid lines,
$(\Delta p/p_o)10^3 = -0.7$ for dashed lines;
$r\sqrt{I_o} = 10$ mm(1), 15 mm(2), 20 mm(3), 25 mm(4).