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Beam characterization of FLASH from beam profile measurement by intensity transport equation and reconstruction of the Wigner distribution function

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ABSTRACT

Beam parameters of the free-electron laser FLASH @13.5 nm in two different operation modes were determined from beam profile measurements and subsequent reconstruction of the Wigner distribution function behind the ellipsoidal focusing mirror at beamline BL2. 40 two-dimensional single pulse intensity distributions were recorded at each of 65 axial positions around the waist of the FEL beam with a magnifying EUV sensitized CCD camera. From these beam profile data the Wigner distribution function based on different levels of averaging could be reconstructed by an inverse Radon transform. For separable beams this yields the complete Wigner distribution, and for beams with zero twist the information is still sufficient for wavefront determination and beam propagation through stigmatic systems. The obtained results are compared to wavefront reconstructions based on the transport of intensity equation. A future setup for Wigner distribution measurements of general beams is discussed.

Keywords: wavefront, Wigner distribution, spatial coherence

1. INTRODUCTION

A couple of important applications of free electron lasers (FEL) require more comprehensive beam characterization than can be gained from standard techniques as beam profiling¹ or caustic measurements². In some situations even wavefront sensing techniques³⁻⁸ are not fully satisfying, in particular if the intensity and phase distributions are required at arbitrary planes where measurement is not possible and a numerical propagation of the measured field amplitude is necessary. The latter approach requires highly accurate wavefront data and a complete coherent wave field, which may be correct for seeded FEL's but surely becomes insufficient when considering self-amplified spontaneous emission (SASE)-type FEL's with inherent stochastic features¹⁵⁻¹⁷. On the other hand, phase space tomography i.e. the reconstruction of the Wigner distribution function²⁴ from several profiles along the beam caustic which has already been demonstrated for X-ray synchrotron sources²¹⁻²³ and recently been applied to FLASH^{9,10}, delivers in principle a complete beam characterization including wavefront, propagation and spatial coherence^{18,19,20}. However, unlike Hartmann and Hartmann-Shack measurements as well as coherence experiments based on Young's double slit experiment¹¹⁻¹⁴, the technique gives only average results and no single pulse information. Furthermore, the Wigner-reconstructions of the FLASH beam carried out so far are based on a very limited data set obtained from standard caustic measurements, which maps only a 3D-sub-manifold of the 4D-phase space.

Following a brief introduction of the Wigner distribution we will identify those beam properties that can be estimated from standard caustic measurements and other ones which require a more complex setup. In a second section wavefront reconstructions from the Wigner function are compared to an approach using the transport of intensity equation, and finally we propose a setup for mapping almost the complete 4D-phase space.

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2. THEORY

2.1 The Wigner distribution

The Wigner distribution h of a quasi-monochromatic paraxial beam is defined in terms of the mutual intensity J as a two-dimensional Fourier transform of the latter according to²⁰:

$$h(\mathbf{x}, \mathbf{u}) = \left(\frac{k}{2\pi}\right)^2 \iint J\left(\mathbf{x} - \frac{\mathbf{s}}{2}, \mathbf{x} + \frac{\mathbf{s}}{2}\right) e^{ik\mathbf{u}\mathbf{s}} d^2s \quad (1),$$

where $\mathbf{x}=(x, y)$, $\mathbf{s}=(s_x, s_y)$ are two-dimensional spatial and $\mathbf{u}=(u, v)$ angular coordinates in a plane perpendicular to the direction of beam propagation and k the mean wave number of light, respectively. As J is Hermitian, h is real, although it may become negative in some regions. However, its marginal distributions with respect to $\mathbf{x}$ and $\mathbf{u}$ are always non negative and yield the irradiance (near field) $I(\mathbf{x})$ and the radiant intensity (far field) $I_{FF}(\mathbf{u})$ respectively²⁶.

The propagation of the Wigner distribution through static and lossless paraxial systems, signified by a 4x4 optical ray propagation ABCD matrix S from an input (i) to an output (o) plane writes^{20, 25}:

$$h_i(D\mathbf{x} - B\mathbf{u}, -C\mathbf{x} + A\mathbf{u}) = h_o(\mathbf{x}, \mathbf{u}) \quad (2).$$

Likewise, the four dimensional Fourier transform $\tilde{h}$ of h obeys a similar transformation law under propagation²⁰:

$$\tilde{h}_i(A^T \mathbf{w} + C^T \mathbf{t}, B^T \mathbf{w} + D^T \mathbf{t}) = \tilde{h}_o(\mathbf{w}, \mathbf{t}), \quad (3)$$

where $(\mathbf{w}, \mathbf{t})$ are the Fourier-space coordinates corresponding to $(\mathbf{x}, \mathbf{u})$.

Considering a set $\{p\}$ of parameters and a set of irradiance profiles $I_{\{p\}}(x, y)$ recorded at positions which are connected to an arbitrary reference plane via the corresponding ray transformation matrices $S_{\{p\}}$, one obtains, according to the marginal property of h and well known Fourier relations:

$$\iint h_{\{p\}}(x, y, u, v) d^2uv = I_{\{p\}}(x, y) \xleftrightarrow{FT} \tilde{I}_{\{p\}}(w_x, w_y) = \tilde{h}_{\{p\}}(w_x, w_y, 0, 0) \quad (4)$$

and from (3) and (4)²⁵:

$$\tilde{h}_{ref}(A_{\{p\}}^T \mathbf{w}, B_{\{p\}}^T \mathbf{w}) = \tilde{I}_{\{p\}}(\mathbf{w}). \quad (5)$$

2.2. Reconstruction and evaluation of the Wigner distribution from standard caustic measurements

Considering an experimental arrangement as shown in fig. 2, the parameter set $\{p\}$ is simply the camera z -position and eq. (5) rewrites:

$$\tilde{h}_{ref}(\mathbf{w}, z \cdot \mathbf{w}) = \tilde{I}_{\{z\}}(\mathbf{w}) \quad (6).$$

The corresponding map $(\mathbf{w}') \rightarrow (\mathbf{w}, \mathbf{t})$:

$$\begin{pmatrix} w_x \\ w_y \\ t_x \\ t_y \end{pmatrix} = w_x' \begin{pmatrix} 1 \\ 0 \\ z \\ 0 \end{pmatrix} + w_y' \begin{pmatrix} 0 \\ 1 \\ 0 \\ z \end{pmatrix}$$

can be looked upon as a 1D set of planes with set parameter z . It obeys the relation $w_x t_y - w_y t_x = 0$ which is the functional equation of the 3D-submanifold M :

$$M: \{(\mathbf{w}, \mathbf{t}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid w_x t_y - w_y t_x = 0\}$$

of the reciprocal phase space covered by the standard caustic measurement¹. In the following we will make use of the notation:

$$\tilde{h}|_M = \begin{cases} \tilde{h}(\mathbf{w}, \mathbf{t}) & \text{if } (\mathbf{w}, \mathbf{t}) \in M \\ 0 & \text{else} \end{cases}.$$

¹ Here we assume, the complete range of z -positions from $(-\infty, \infty)$ has either been measured or remaining gaps are filled in by interpolation.

We can now ask the question which amount of information can be extracted from a Wigner distribution whose 4D Fourier transform is known merely on M. Regarding the marginal distributions it is straightforward that they can be reconstructed:

$$I(\mathbf{x}) = \iint h(\mathbf{x}, \mathbf{u}) d\mathbf{u} d\mathbf{v} \xleftrightarrow{FT} \tilde{h}\left(\frac{\mathbf{w}, \mathbf{0}}{\in M}\right) \quad \text{and} \quad I_{FF}(\mathbf{u}) = \iint h(\mathbf{x}, \mathbf{u}) d\mathbf{x} d\mathbf{y} \xleftrightarrow{FT} \tilde{h}\left(\frac{\mathbf{0}, \mathbf{t}}{\in M}\right) \quad (7).$$

From (7) it follows immediately, that the pure angular and pure spatial 2nd moments of the Wigner distribution can be calculated as well. Considering the mixed moments, we have first:

$$\langle ux \rangle = P^{-1} \iint ux h(\mathbf{x}, \mathbf{u}) d^2 \mathbf{x} d^2 \mathbf{u} = P^{-1} \frac{i}{k} \partial_{w_x} \partial_{t_x} \tilde{h}(\mathbf{w} = \mathbf{0}, \mathbf{t} = \mathbf{0}) \quad (8),$$

with P the total beam power. The evaluation of the right hand side in the second equality of (8) requires the knowledge of $\tilde{h}$ in the (w_x, t_x) -plane near the origin. Therefore, as $(w_x, 0, t_x, 0) \in M$, $\langle ux \rangle$ can be determined and the same holds for the $\langle yv \rangle$ term. Obviously, the anti-diagonal parts of the mixed-moment matrix $\langle uy \rangle$ and $\langle vx \rangle$ cannot in general be calculated separately using only reciprocal phase space values on M. However, using the propagation law eq. (3) one obtains:

$$\partial_{w_y} \partial_{t_x} \tilde{h}_{z=0}|_{(0,0)} + \partial_{w_x} \partial_{t_y} \tilde{h}_{z=0}|_{(0,0)} = \partial_{w_x} \frac{\partial_{w_y} \tilde{h}_{z=dz} - \partial_{w_y} \tilde{h}_{z=0}}{dz} |_{(0,0)} \quad (9).$$

As $\partial_{w_x} \partial_{w_y} \tilde{h}$ can be calculated whenever $(\mathbf{w}, \mathbf{0}) \in M$ and $\tilde{h}|_M$ can be reconstructed at arbitrary z-planes, $\langle uy \rangle$ and $\langle vx \rangle$ may be determined, provided the condition $\langle uy \rangle = -\langle vx \rangle$ or equivalently $\langle uy \rangle - \langle vx \rangle = 0$ holds. The latter equation is known as the familiar definition of the twisted phase parameter. In conclusion we may state that the second order moments of the complete 4D Wigner distribution can be determined from the 3D function $\tilde{h}|_M$ provided that the beam under test has vanishing twist.

In a similar fashion the local moments of the beam may be treated, however, here we concentrate on the most important case of the first angular moments determining the wavefront w and the transverse part $\mathbf{S}_\perp = (\mathbf{S}_x, \mathbf{S}_y)$ of the Poynting-vector $\mathbf{S}$:

$$I(\mathbf{x})^{-1} \iint \begin{pmatrix} u \\ v \end{pmatrix} h(\mathbf{x}, \mathbf{u}) d\mathbf{u} d\mathbf{v} = I(\mathbf{x})^{-1} \begin{pmatrix} S_x \\ S_y \end{pmatrix} = \begin{pmatrix} \partial_x w \\ \partial_y w \end{pmatrix} + \mathbf{v} = \nabla_\perp w + \mathbf{v} \quad (10),$$

where the source-free part $\mathbf{v}$ satisfies $\text{div } \mathbf{v} = 0$. By taking the Fourier transform, eq. (10) writes:

$$\mathbf{w} \cdot \tilde{\mathbf{S}}_\perp = \mathbf{w} \cdot \nabla_t \tilde{h}(\mathbf{w}, \mathbf{t})|_{\mathbf{t}=\mathbf{0}} = \frac{\tilde{h}_{z=dz} - \tilde{h}_{z=0}}{dz} |_{\mathbf{t}=\mathbf{0}} \xleftrightarrow{FT} \nabla_\perp \cdot \mathbf{S}_\perp = \frac{I_{z=dz} - I_{z=0}}{dz} \quad (11)$$

thus, from $(\mathbf{w}, \mathbf{0}) \in M$ it follows, that $\mathbf{w} \cdot \tilde{\mathbf{S}}$ is attainable from $\tilde{h}|_M$ and so is $\nabla_\perp w$. Of course, according to the divergence operation in (11), only the curl-free part of $\mathbf{S}_\perp / I$ can be determined.

In order to figure out the possible ray transformation ABCD-matrices, that leave the functional equation defining M invariant, we write the latter in symplectic form:

$$(w_x, t_x) \cdot \mathbf{J} \cdot \begin{pmatrix} w_y \\ t_y \end{pmatrix} = (w_x, t_x) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_y \\ t_y \end{pmatrix} = 0 \quad (12),$$

from which it is obvious, that x and y terms are not permitted to mix up by allowable transformations. We then get:

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} A_y & C_y \\ B_y & D_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (13),$$

and therefore $A_x = A_y$, $B_x = B_y$, $C_x = C_y$ and $D_x = D_y$. 4x4 ray matrices whose 2x2 submatrices are proportional to the 2x2 unit matrix are called stigmatic and we may state: The information contained in $\tilde{h}|_M$ is sufficient for the propagation of arbitrary beams through stigmatic optical systems.

In the remainder of this section we will discuss briefly the effect of the reduced information contained in $\tilde{h}|_M$ on the determination of spatial coherence. For that purpose we express the global degree of coherence K^{27} in terms of $\tilde{h}$:

$$K = \lambda^2 \frac{\iint h(\mathbf{x}, \mathbf{u})^2 d^2 \mathbf{x} d^2 \mathbf{u}}{[\iint h(\mathbf{x}, \mathbf{u}) d^2 \mathbf{x} d^2 \mathbf{u}]^2} = \lambda^2 \frac{\iint |\tilde{h}(\mathbf{w}, \mathbf{t})|^2 d^2 \mathbf{w} d^2 \mathbf{t}}{|\tilde{h}(\mathbf{0}, \mathbf{0})|^2} \quad (14),$$

which follows from Parseval's theorem.

$$\text{Obviously, as } M \text{ is a null set in } \mathbb{R}^4 : K|_M = \lambda^2 \frac{\iint |\tilde{h}|_M(\mathbf{w}, \mathbf{t})|^2 d^2 \mathbf{w} d^2 \mathbf{t}}{|\tilde{h}(\mathbf{0}, \mathbf{0})|^2} = 0 \quad (15).$$

However, eq. (15), although true analytically, does not hold, if the discrete space of numerical calculation with voxel-size δ is considered. In that case, points with distance $< \delta/2$ from M in each dimension will contribute to the finite sum replacing (15) and one may write:

$$K|_{M, \delta} = \sum_{\substack{\text{voxels } i \\ \text{supp}_i \cap M \neq \emptyset}} \langle |\tilde{h}|_{M \cap \text{supp}_i} \rangle_i^2 \mathbf{Vol}_{4D}(i) \quad (16),$$

where we have normalized $\tilde{h}|_M$ to $\tilde{h}|_M(\mathbf{0}, \mathbf{0}) = 1$. Furthermore, if we introduce dimensionless variables $w_{x,y} = w_{x,y}/w_{\max}$, $t_{x,y} = t_{x,y}/t_{\max}$, the 4D volume and the support of i are then given by:

$$\mathbf{Vol}_{4D}(i) = \delta^4 = \frac{16}{N^4}$$

$$\text{supp}_i = \left\{ (\mathbf{w}, \mathbf{t}) : |w_x - w_{xi}| < \frac{\delta}{2} \wedge |w_y - w_{yi}| < \frac{\delta}{2} \wedge |t_x - t_{xi}| < \frac{\delta}{2} \wedge |t_y - t_{yi}| < \frac{\delta}{2} \right\}$$

with N the number of voxels per dimension and $(\mathbf{w}_i, \mathbf{t}_i)$ the center coordinates of voxel i .

The amount eq. (16) underestimates the true global degree of coherence depends on the mapping strategy, the amount of interpolation and on the shape of $\tilde{h}$ itself. Fig. 1 shows the base-10 logarithm of the fraction

$$\frac{K|_{M, \delta}}{K|_{\delta}} = \frac{\sum_{\substack{\text{voxels } i \\ \text{supp}_i \cap M \neq \emptyset}} \langle |\tilde{h}|_{M \cap \text{supp}_i} \rangle_i^2 \mathbf{Vol}_{4D}(i)}{\sum_{\text{voxels } i} \langle |\tilde{h}| \rangle_i^2 \mathbf{Vol}_{4D}(i)} \quad (17)$$

for an isotropic Gaussian Wigner distribution h , using nearest neighbor mapping and no interpolation as a function of the width of $\tilde{h}$ in Fourier space. As expected, $K|_{M, \delta}/K|_{\delta}$ decreases with higher resolutions approaching to the continuum limit. On the other hand, for N fixed the fraction increases with decreasing size of the Wigner distribution in Fourier space, as the 4D space is more densely sampled near the origin of the coordinate system than in the outer parts. In cases where the Wigner function is known to be quite isotropic, at least after an appropriate coordinate scaling, K may be corrected by means of eq. (17). Table 1 show K values with and without isotropic correction for several caustic sequences recorded at FLASH beamline BL2. The values calculated according to eq. (16) are almost a factor of ten smaller than results reported earlier which were based on recordings using a 3mm aperture stop for beam size reduction¹⁰.

Table 1: Global degree of coherence K calculated from Wigner reconstruction based on standard caustic measurements for several sequences with varying number of profiles or bunch charge q , respectively. The last column shows corrected values based on the assumption of an isotropic Gaussian shaped Wigner function of 5 voxel $1/e^2$ -diameter.

	No. of voxels	No. of Profiles in reconstruction	K [eq. (16)]	K corrected [eq. (17)]
Sequence 1; $q=0.1$ nC	64^4	65 @ 65 z-Pos.	$1.56 \cdot 10^{-2}$	0.04
Sequence 2; $q=0.1$ nC	64^4	1300 @ 65 z-Pos.	$9.8 \cdot 10^{-3}$	0.025
Sequence 3; $q=0.5$ nC	64^4	65 @ 65 z-Pos.	$5.5 \cdot 10^{-3}$	0.014
Sequence 4; $q=0.5$ nC	64^4	1300 @ 65 z-Pos.	$4.9 \cdot 10^{-3}$	0.013

However, the decrease of spatial coherence with increasing bunch charge due to the higher saturation of the FEL is quite reasonable although the corrected K values shown in the last column of table 1 are still considerably smaller than results obtained from Young's double pinhole experiment¹². The distinguished amount of averaging as well as the incompleteness of phase space mapping may be a reason for the observed difference.

Figure 1: : Base-10 logarithm of $K_{M,\delta}/K_\delta$ calculated according to eq. (17) for an isotropic Gaussian $\tilde{h}$ -distribution as function of 1/e-distribution size in Fourier space (cf text).

2.3 Wavefront reconstruction from the transport of intensity equation

The conservation of energy flux in beam propagation through lossless, homogeneous and isotropic media requires:

$$\nabla \cdot \mathbf{S} = 0 \quad (18),$$

where the Poynting vector $\mathbf{S}$ in a scalar theory can be written as:

$$\mathbf{S} = \frac{1}{2ik} (u^* \nabla u - u \nabla u^*) \quad (19).$$

By writing the field amplitude u in terms of intensity and wavefront : $u = \sqrt{I} e^{ikw}$ it follows from (18):

$$\nabla \cdot (I \nabla w) = 0 \quad (20),$$

and, since $(\nabla w)_z \approx 1$ (paraxial approximation) one obtains the transport equation²⁹:

$$\frac{\partial I}{\partial z} = -\nabla_\perp I \cdot \nabla_\perp w - I \Delta_\perp w \quad (21),$$

$\nabla_\perp$ denoting the 2D Nabla operator. Eq. (21) states, that the wavefront (up to a constant term) can be determined from the irradiance distribution in adjacent planes z_0 and z_0+dz perpendicular to the direction of beam propagation.

For a numerical treatment w is written as a series expansion in terms of the Zernike polynomials $Z_l(x,y)$ ²⁸:

$$w(x, y) = \sum_{l=0}^L C_l Z_l(x, y) \quad (22).$$

With (22) it follows from (21):

$$\frac{\partial I}{\partial z} = -\sum_l C_l \left\{ \frac{\partial I}{\partial x} \frac{\partial Z_l}{\partial x} + \frac{\partial I}{\partial y} \frac{\partial Z_l}{\partial y} + I \left(\frac{\partial^2 Z_l}{\partial x^2} + \frac{\partial^2 Z_l}{\partial y^2} \right) \right\} \quad (23).$$

The discrete version of (23) at sampling points (x_k, y_k) writes:

$$\partial_z I_k + \sum_l C_l [\partial_x I_k \partial_x Z_{k,l} + \partial_y I_k \partial_y Z_{k,l} + I_k (\partial_x^2 Z_{k,l} + \partial_y^2 Z_{k,l})] = 0 \quad ; \quad k = 1, \dots, N \quad (24)$$

or
$$B_k + A_{kl} C_l = 0 \quad k=1, \dots, N.$$

After approximation of the derivatives by the familiar central differences (p: detector pixel size)

$$\partial_z I_k = \frac{I(x_k, y_k, z_0 + dz) - I(x_k, y_k, z_0)}{dz}$$

$$\partial_x I_k = \frac{I(x_k + p, y_k, z_0 + dz) - I(x_k - p, y_k, z_0 + dz)}{4p} + \frac{I(x_k, y_k + p, z_0) - I(x_k, y_k - p, z_0)}{4p} \quad \partial_y I_k = \dots ,$$

Eq. (24) becomes a linear $N \times L$ system with the total number of sampling points N and the degree of series expansion L , that can numerically be solved by standard methods (e.g. singular value decomposition) of linear theory.

3. EXPERIMENTAL

Figure 2: Experimental setup for determination of the Wigner distribution of the FLASH beam at BL2 from standard caustic measurements (cf. text).

Figure 2 shows the setup for the FEL beam caustic measurement conducted at FLASH BL2 at $\lambda=13.5\text{nm}$ fundamental wavelength. A carbon-coated ellipsoidal mirror with 2m focal length focuses the beam down to approximately $50\mu\text{m}$ FWHM. An Al filter was used to attenuate the beam for the caustic measurement. The caustic sensor, a phosphorous screen imaged onto a 12bit 2/3" CCD chip with 1280×1024 pixels resolution by a 10x magnifying microscope on a translation stage, senses the FEL beam near the focal plane of the ellipsoidal mirror. Beam profile measurements were taken at $N_z = 65$ z positions within an interval of 240mm distributed around the beam waist in order to fill the phase space as uniform as possible and covering 8 Rayleigh lengths z_R in both axial directions. At each z-position 20-40 single pulses were recorded.

Figure 3: Single pulse beam profiles behind the elliptical mirror recorded at different z-position along the caustic of FLASH BL2. The selected area of the detector corresponds to $0.85 \times 0.66 \text{ mm}^2$.

4. RESULTS

Figure 4: Wavefront at the waist of the FLASH beam at BL2 behind the elliptical mirror reconstructed from the Wigner distribution (upper row) and from the transport equation (lower row) for values of the bunch charge of 0.1nC and 0.5nC, respectively. The profiles for wavefront reconstruction from the transport equation were taken at z-positions of 132mm and 133mm with respect to the camera reference frame.

Figure 4 shows reconstructed wavefront distributions at the waist of the FEL beam obtained from the Wigner distribution based on 1 and 20 sequences á 65 profiles recorded ± 8 Rayleigh lengths around waist as well as from the transport equation using 2 single profiles near the waist position. For both methods a Zernike expansion degree of 28 polynomials has been used. The agreement is quite satisfactory, revealing the fact, that the wavefront determination requires only information localized to the vicinity of the reconstruction plane. Despite the average concordance, however, the pulse-to-pulse variation is considerable, showing 30% fluctuation of the peak-valley wavefront irregularity, as shown in fig. 5. On the other hand, the average wavefront shape near waist happens to be rather insensitive to the charge of the electron gun as can be discovered from figures 4 and 5 showing distributions for 0.1nC and 0.5nC bunch charge respectively.

Figure 5: Pulse-to-pulse fluctuation of the wavefront near the waist of the FEL beam (BL2) @ 0.5nC bunch charge, reconstructed from the transport equation for different pulses of a sequence. Profiles from $z=132\text{mm}$ and 133mm (left and middle figure) as well as from 131.5mm and 133.5mm (right figure) are used for reconstruction.

Table 2: Wavefront-RMS and peak valley values as well as number (ISO 10110-5) of the three largest selected Zernike coefficients from different evaluation methods for the FEL beam at waist position.

value [nm]	w_{RMS}	w_{pv}	Zernike Largest	coefficient 2 nd largest	3 rd largest
Wigner 65 Profiles	8.4	31.5	C16 Fourfold	C4 Astigmatism	C8 Spherical
Wigner 20 Seq. á 65 Profiles	8.7	30	C16 Fourfold	C4 Astigmatism	C8 Spherical
Intensity Transport ◊ over 8 single reconstructions	4.3	25	C16 Fourfold	C4 Astigmatism	C3 Defocus

Despite the broader database, the Wigner-based wavefront reconstruction delivers more distinct distributions than the transport-of-intensity approach when compared in terms of the wavefront RMS-deformation and, less pronounced, in the peak-valley data. The pulse-to-pulse fluctuations give rise to a considerable randomness in data taken from only two profiles. In turn, as the random part spreads evenly over the whole Zernike-spectrum and happens to be uncorrelated to the wavefront signal, the amplitude of the latter, located at lower spatial frequencies is reduced to some extent. Table 2 gives an overview for the wavefront RMS- and peak-valley-deformation as well as the number of the three largest Zernike coefficients according to ISO 10110-5 @ 0.1nC bunch charge.

5. SETUP FOR SAMPLING THE COMPLETE 4D PHASE SPACE

Figure 6: Experimental setup for measuring a 4D Wigner function.

Figure 6 shows a principal setup, that enables the measurement of a 4D section of phase space. Compared to the caustic setup shown in fig. 2, the grazing incidence ellipsoidal mirror is replaced by a toroidal mirror, whose surface normal is tilted by a small angle α with respect to the incoming beam. Furthermore, in addition to the camera z-movement the toroid can be rotated around its normal by an angle $\varphi \in [0, 2\pi]$. Both parameters z and φ together with the detector plane

coordinates x and y , respectively, constitute a parameter space, being sufficient for 4D phase space mapping. Without loss of generality let the reference plane be a fixed distance z' in front of the toroid mirror. Then the 4x4 transformation matrix $S_{\{p\}}=S_{\{\varphi, z\}}$ from the reference plane to the camera position writes:

$$S_{\{\varphi, z\}} = S_z S_{\text{tilt}}^{-1} R_\varphi^{-1} S_{\text{Toroid}} R_\varphi S_{\text{tilt}} S_z' \quad (25)$$

The matrices S_z , S_z' and S_{Toroid} denote free space propagation and the focussing action of the toroid according to :

$$S_z = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_{\text{Toroid}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/R_x & 0 & 1 & 0 \\ 0 & -2/R_y & 0 & 1 \end{pmatrix},$$

whereas R_φ and S_{tilt} symbolize rotation and tilt of the toroid mirror. They are given by:

$$S_{\text{tilt}} = \begin{pmatrix} \sqrt{\cos\alpha} & 0 & 0 & 0 \\ 0 & \sqrt{\cos\alpha}^{-1} & 0 & 0 \\ 0 & 0 & \sqrt{\cos\alpha}^{-1} & 0 \\ 0 & 0 & 0 & \sqrt{\cos\alpha} \end{pmatrix} \quad \text{and} \quad R_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix}, \text{ respectively.}$$

The resulting 4D-map explicitly writes:

$$\begin{pmatrix} \mathbf{w} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^T \mathbf{w}' \\ \mathbf{B}^T \mathbf{w}' \end{pmatrix} = \begin{pmatrix} 1 - \frac{2z\cos^2\varphi}{R_x\cos\alpha} - \frac{2z\sin^2\varphi}{R_y\cos\alpha} & z\frac{\sin 2\varphi}{R_y} - z\frac{\sin 2\varphi}{R_x} & 1 - \frac{2z\cos^2\varphi\cos\alpha}{R_y} - \frac{2z\sin^2\varphi\cos\alpha}{R_x} & z\frac{\sin 2\varphi}{R_y} - z\frac{\sin 2\varphi}{R_x} \\ z + z' - \frac{2zz'\cos^2\varphi}{R_x\cos\alpha} - \frac{2zz'\sin^2\varphi}{R_y\cos\alpha} & zz'\frac{\sin 2\varphi}{R_y} - zz'\frac{\sin 2\varphi}{R_x} & z + z' - \frac{2zz'\cos^2\varphi\cos\alpha}{R_y} - \frac{2zz'\sin^2\varphi\cos\alpha}{R_x} & zz'\frac{\sin 2\varphi}{R_y} - zz'\frac{\sin 2\varphi}{R_x} \end{pmatrix} \mathbf{w}' \quad (26).$$

The proof, that eq. (26) covers the complete phase space, provided $\{w'_x, w'_y, \varphi, z\}$ passes through its whole data range, is straightforward, however, the calculations are quite lengthy, and we omit it here for reasons of space. The interested reader is referred to²⁵, where the analysis is performed for a similar setup containing a spherical and a rotating cylindrical lens.

6. CONCLUSION

In the preceding it has been shown, that the Wigner reconstruction based on a standard caustic measurement gives already valuable information of beam parameters and wavefront. Moreover, if one is interested only in stigmatic optical systems, the propagation of such an incompletely sampled beam can thoroughly be performed. In the case of complete spatial coherence, or if solely the wavefront distribution is of concern, however, it has been shown in sects. 3 and 4, that an analysis based on an even further reduced data set of only two neighbouring irradiance profiles can successfully be done by using the transport of intensity equation. Wavefronts in the waist of FLASH @ beamline BL2 have consistently been reconstructed from both techniques.

On the other hand, the results presented in section 3 clearly show, that the measurement of spatial coherence, the complete characterization of beam carrying a twisted phase or beam propagation through arbitrary systems cannot be performed based on such a reduced data set. Only in the very special case of a separable beam complete beam information is available from the incomplete sampled Wigner distribution. Although early experimental results supported the separability of FLASH BL2, this does obviously no longer hold in the absence of limiting apertures. In those cases a proper characterisation of partially coherent beams requires a full 4D extension of the standard caustic approach which is proposed and briefly discussed in section 5. In particular, when coherence properties of the beam are of major importance, the wide variety of information gained may outweigh the higher experimental effort.

7. ACKNOWLEDGEMENT

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