PROGRESS IN ANALYTICAL CALCULATIONS FOR THE ANOMALOUS MAGNETIC MOMENT OF THE MUON*

P.A. Baikov

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University
1(2), Leninskie gory, Moscow 119991, Russian Federation

A. Maier

Physik Department T31, Technische Universität München
James-Franck-Straße, 85748 Garching, Germany

P. Marquard

Deutsches Elektronen Synchrotron DESY
Platanenallee 6, 15738 Zeuthen, Germany

(Received October 22, 2013)

We present results for certain classes of diagrams contributing to the anomalous magnetic moment of the muon at five-loop order. Our method is based on first constructing an approximating function for the vacuum polarization function of the photon at four-loop order which later can be numerically integrated to obtain the anomalous magnetic moment of the muon.

DOI:10.5506/APhysPolB.44.2267
PACS numbers: 12.20.–m, 14.60.Ef

1. Introduction

The anomalous magnetic moments of electron and muon are some of the best measured and theoretically predicted quantities. The QED corrections have recently been calculated numerically up to five loops in [1, 2]. Up to next-to-next-to-leading order complete analytical results are available [3, 4].

---

* Presented at the XXXVII International Conference of Theoretical Physics "Matter to the Deepest" Ustroń, Poland, September 1–6, 2013.

At the four-loop order only partial results exist, contributions from corrections to the vacuum polarization function of the photon have been calculated in [5], contributions due to light lepton loops in [6] and due to heavy leptons in [7]. Recently, in Ref. [8] some five-loop corrections have been calculated by using the leading term in the high-energy expansion of the vacuum polarization function of the photon. Since this approach leads to a surprisingly large deviation from the numerical result, the method has been improved in Ref. [9]. In the following, we will review the main ingredients of the analysis in Ref. [9] and discuss the results.

2. Setup of the calculation and results

The QED corrections to the anomalous magnetic moment a_μ can be calculated in a perturbation theory and can thus be written in the form of power series in the fine structure constant α

$$a_\mu = \sum_{k=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^k a_\mu^{(2k)} , \tag{1}$$

where $a_\mu^{(2k)}$ can be further decomposed — following the conventions in Ref. [1] — as

$$a_\mu^{(2k)} = A_1^{(2k)} + A_2^{(2k)}(m_e/m_\mu) + A_2^{(2k)}(m_\tau/m_\mu) + A_3^{(2k)}(m_e/m_\mu, m_\tau/m_\mu) . \tag{2}$$

$A_1^{(2k)}$ contains the universal contributions, which in the case of the muon anomalous magnetic moment only contain muon loops. The diagrams contributing to $A_2^{(2k)}(m_e/m_\mu)$ and $A_2^{(2k)}(m_\tau/m_\mu)$ have at least one electron or tau loop, respectively. In $A_3^{(2k)}(m_e/m_\mu, m_\tau/m_\mu)$, contributions from diagrams with both electron and tau loops are collected. In this paper, we are mainly interested in contributions to $A_2^{(2k)}(m_e/m_\mu)$ without any muon loops.

The contributions to the anomalous magnetic moment of the muon due to photon polarization effects can be calculated (*cf.* Fig. 1) by using [10]

$$a_\mu = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \frac{1}{1+\Pi(s_x)} , \qquad s_x = -\frac{x^2}{1-x} m_\mu^2 , \tag{3}$$

where Π denotes the vacuum polarization function as defined in Eq. (4). This formula can be obtained by considering the one-loop result for $g-2$ for the case of a heavy photon in combination with the dispersion relation for $\Pi(q^2)$. The classes of diagrams accessible by this method are shown in Fig. 2. Thus, we have to find a suitable approximation for $\Pi(q^2)$ which, in turn, can be integrated to obtain a_μ.

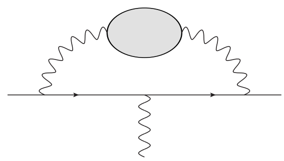

Fig. 1. Prototype diagram.

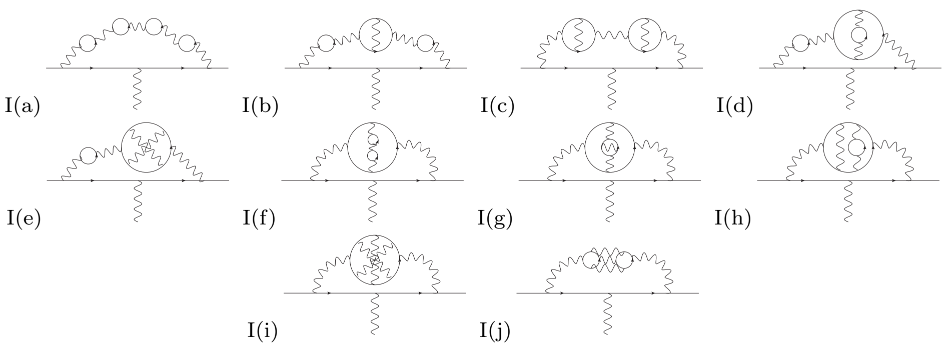

Fig. 2. Classes of diagrams accessible by the used method.

We define the vacuum polarization function $\Pi(q^2)$ as usual by

$$\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \Pi\left(q^2\right) = ie^2 \int dx \, \langle 0|e^{iqx} T j^\mu(x) j^\nu(0)|0\rangle \,, \tag{4}$$

with the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and write it as an expansion in the fine-structure constant α

$$\begin{aligned} \Pi\left(q^2\right) &= \frac{\alpha}{\pi}\Pi^{(1)}\left(q^2\right) + \left(\frac{\alpha}{\pi}\right)^2 \Pi^{(2)}\left(q^2\right) + \left(\frac{\alpha}{\pi}\right)^3 \Pi^{(3)}\left(q^2\right) \\ &\quad + \left(\frac{\alpha}{\pi}\right)^4 \Pi^{(4)}\left(q^2\right) + \mathcal{O}\left(\alpha^5\right) . \end{aligned} \tag{5}$$

In the following, we will collect the available results for the low- and high-energy region and the threshold region, which will later be used to construct an approximating function. For details on the calculation of the listed results, please refer to Ref. [9]. In the following, n_h and **si** label contributions from lepton loops and singlet diagrams.

In the low-energy limit, the polarization function can be expanded in a power series in $z = q^2/(4m_q^2) < 1$

$$\Pi^{(n)}_{\rm le} = \sum_{k=1}^{\infty} \Pi^{(n)}_{{\rm le},k} z^k \,. \tag{6}$$

For $\Pi^{(4)}_{\rm le}$, one obtains

$$\begin{aligned}\Pi^{(4)}_{\rm le} &= z\left(n_h^2(0.066\,\mathbf{si} + 0.571) + 0.112n_h^3 + 0.834n_h\right)\\ &\quad +z^2\left(0.025n_h^3 + n_h^2(0.140\,\mathbf{si} + 0.366) + 2.230n_h\right)\\ &\quad +z^3\left(0.012n_h^3 + n_h^2(0.126\,\mathbf{si} + 0.277) + 3.396n_h\right)\,.\end{aligned}$$

In the high-energy region, we write the result in the form

$$\Pi^{(n)}_{\rm he} = \sum_{k=0}^{\infty} \Pi^{(n)}_{{\rm he},k} z^{-k} \tag{7}$$

with

$$\begin{aligned}\Pi^{(4)}_{\rm he} &= n_h^3\left(-0.009\log^3(-4z)+0.019\log^2(-4z)-0.076-0.086\log(-4z)\right)\\ &+n_h^2\big(-0.021\log^2(-4z) + \log(-4z)(0.496\,\mathbf{si} - 0.258)\\ &+(0.638 - 1.619\,\mathbf{si})\big) + n_h(0.180\log(-4z) - 1.972)\\ &+\Big[n_h^2\big((-2.546\,\mathbf{si} - 0.015) + 0.188\log^3(-4z) - 0.938\log^2(-4z)\\ &+2.414\log(-4z)\big) + n_h^3\big(-0.028\log^3(-4z) + 0.181\log^2(-4z)\\ &-0.666\log(-4z) + 0.684\big) + n_h\big(0.141\log^2(-4z) - 0.281\log^3(-4z)\\ &+1.265 - 2.048\log(-4z)\big)\Big]/z\,.\end{aligned}$$

The polarization function in the threshold region can be written as

$$\Pi^{(n)}_{\rm thr} = 16\pi^2 \sum_{k=2-n}^{\infty} \Pi^{(n)}_{{\rm thr},k} \left(\sqrt{1-z}\,\right)^k \tag{8}$$

with the four-loop contribution

$$\begin{aligned}\Pi^{(4)}_{\rm thr} &= \frac{14.640n_h}{1-z} + \frac{-184.800n_h - 70.130n_h\log(1-z)}{\sqrt{1-z}} + 8.278n_h\log^2(1-z)\\ &\quad + \log(1-z)\left(-185.400n_h - 3.553n_h^2\right) - 6.220n_h\log^3(1-z) + C\,,\end{aligned}$$

where C denotes an unknown constant.

To obtain an interpolation between these regions, a Padé approximation has been used. We show the result for the approximating function in Fig. 3, where the envelope together with the relative error is displayed. Inserting the approximation for $\Pi^{(4)}$ into Eq. (3), the results shown in Table I are obtained.

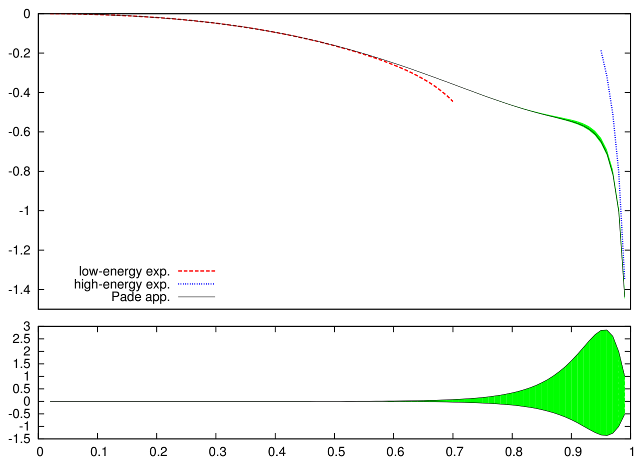

Fig. 3. Padé approximation for $\Pi^{(4)}\left(-\frac{x^2}{1-x}m_\mu^2\right)$. We show in the top part the approximants and in the bottom part the relative error with respect to the local mean of all approximants obtained.

TABLE I

Results for $A_2^{(10)}(m_e/m_\mu)$ with pure electronic insertions. The errors listed in the second column are estimated from the spread between different Padé approximants, which is negligible for classes I(a)–I(e). Please note that the authors of Ref. [8] only used the asymptotic form of $\Pi(s)$ and did not provide any error estimate.

	This work	Ref. [8]	Refs. [11–14]	
I(a)	20.142 813	20.183 2	20.142 93(23)	[11]
I(b)	27.690 061	27.718 8	27.690 38(30)	[11]
I(c)	4.742 149	4.817 59	4.742 12(14)	[11]
I(d)+I(e)	6.241 470	6.117 77	6.243 32(101)(70)	[11]
I(e)	−1.211 249	−1.331 41	−1.208 41(70)	[11]
I(f)+I(g)+I(h)	$4.446\ 8^{+6}_{-4}$	4.391 31	4.446 68(9)(23)(59)	[11, 12]
I(i)	$0.074\ 6^{+8}_{-19}$	0.252 37	0.087 1(59)	[13]
I(j)	$-1.246\ 9^{+4}_{-3}$	−1.214 29	−1.247 26(12)	[14]

The results listed for classes I(a)–I(c) are exact since we numerically integrated the known one- and two-loop results for the vacuum polarization function. Classes I(d) and I(e) are calculated using the highly constrained Padé approximants, which have been constructed using 30 terms in the low- and high-energy expansion. Due to the vast amount of information, the results for $g-2$ using different approximants have very little spread and the final result is thus very precise. The situation is quite different for classes I(f)–I(j) which require the knowledge of $\Pi(q^2)$ at four-loop order. Since there is only a limited number of terms in the relevant expansions, the Padé approximation is less precise and the precision of our result for $g-2$ is limited. In general, we find good agreement with the results from Refs. [11–14], but for some classes a certain tension remains.

3. Conclusions

We calculated the contribution to the anomalous magnetic moment of the muon arising from corrections to the vacuum polarization function of the photon at five-loop order. To this end we constructed an approximation of the vacuum polarization function of the photon at four-loop order based on expansion in the low- and high-energy and the threshold region. We find good agreement with the results presented in Refs. [11–14].

We like to thank K.G. Chetyrkin and J.H. Kühn for initiating the project, and fruitful discussions. In addition, we thank K.G. Chetyrkin for valuable advice on computing the vacuum polarization function in high-energy limit. The work of P.B. was supported by RFBR grant 11-02-01196. P.M. has been supported in part by DFG Sonderforschungsbereich Transregio 9, Computergestützte Theoretische Teilchenphysik, and by the EU Network LHCPHENOnet PITN-GA-2010-264564.

REFERENCES

[1] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, *Phys. Rev. Lett.* **109**, 111808 (2012) [arXiv:1205.5370 [hep-ph]].

[2] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, *Phys. Rev. Lett.* **109**, 111807 (2012) [arXiv:1205.5368 [hep-ph]].

[3] S. Laporta, E. Remiddi, *Phys. Lett.* **B379**, 283 (1996) [arXiv:hep-ph/9602417].

[4] S. Laporta, *Phys. Lett.* **B312**, 495 (1993) [arXiv:hep-ph/9306324].

[5] P. Baikov, D.J. Broadhurst, In "Pisa 1995, New Computing Techniques in Physics Research", pp. 167–172 [arXiv:hep-ph/9504398].

[6] R. Lee *et al.*, *J. High Energy Phys.* **1303**, 162 (2013) [arXiv:1301.6481 [hep-ph]].

[7] A. Kurz, T. Liu, P. Marquard, M. Steinhauser, "Anomalous Magnetic Moment with Heavy Virtual Leptons", in preparation.

[8] P. Baikov, K. Chetyrkin, J. Kühn, C. Sturm, *Nucl. Phys.* **B867**, 182 (2013) [arXiv:1207.2199 [hep-ph]].

[9] P. Baikov, A. Maier, P. Marquard, arXiv:1307.6105 [hep-ph], submitted to *Nucl. Phys.* **B** (2013).

[10] B. Lautrup, E. de Rafael, *Nucl. Phys.* **B70**, 317 (1974).

[11] T. Kinoshita, M. Nio, *Phys. Rev.* **D73**, 053007 (2006) [arXiv:hep-ph/0512330].

[12] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, *Phys. Rev.* **D78**, 113006 (2008) [arXiv:0810.5208 [hep-ph]].

[13] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, *Phys. Rev.* **D83**, 053003 (2011) [arXiv:1012.5569 [hep-ph]].

[14] T. Aoyama *et al.*, *Phys. Rev.* **D78**, 053005 (2008) [arXiv:0806.3390 [hep-ph]].