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COSMOLOGICAL CONSTANT in $F(\mathcal{R})$ SUPERGRAVITY ¹

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Abstract

A cosmological constant in the regime of low spacetime curvature is calculated in the recently proposed version of $F(\mathcal{R})$ supergravity with a generic cubic function F . The $F(\mathcal{R})$ supergravity is the $N = 1$ supersymmetric extension of $f(R)$ gravity. The cubic model is known to successfully describe a chaotic (slow-roll) inflation in the regime of high spacetime curvature. We find that a simple extension of the same model allows a *positive* cosmological constant in the regime of low spacetime curvature. The inflaton superfield in $F(R)$ supergravity (like inflaton in $f(R)$ gravity) violates the Strong Energy Condition and thus breaks the restriction of the standard supergravity (with usual matter) that can only have either a negative or vanishing cosmological constant.

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1 Introduction

The Standard (Λ -CDM) Model in cosmology gives a phenomenological description of the observed *Dark Energy* (DE) and *Dark Matter* (DM). It is based on the use of a small positive cosmological constant Λ and a *Cold Dark Matter* (CDM), and is consistent with all observations coming from the existing cosmological, Solar system and ground-based laboratory data. However, the Λ -CDM Model cannot be the ultimate answer to DE, since it implies its time-independence. For example, the ‘primordial’ DE responsible for inflation in the early universe was different from Λ and unstable. The *dynamical* (ie. time-dependent) models of DE can be easily constructed by using the $f(R)$ gravity theories, defined via replacing the scalar curvature R by a function $f(R)$ in the gravitational action. The $f(R)$ gravity provides the self-consistent non-trivial alternative to the Λ -CDM Model — see eg., refs. [1, 2, 3] for a review. The use of $f(R)$ gravity in the inflationary cosmology was pioneered by Starobinsky [4]. Viable $f(R)$ -gravity-based models of the current DE are also known [5, 6, 7], and the combined inflationary-DE models are possible too [1].

Despite of the apparent presence of the higher derivatives, an $f(R)$ gravity theory can be free of ghosts and tachyons. The corresponding stability conditions are well known – see Sec. 2 below. Under those conditions, it is always possible to prove the classical equivalence of an $f(R)$ gravity theory to the certain scalar-tensor theory of gravity [8, 9, 10]. Dynamics of the spin-2 part of metric in $f(R)$ gravity (compared to Einstein gravity) is not modified, but there is the extra propagating scalar field (called scalaron) given by the spin-0 part of metric. By the classical equivalence above we mean that both theories lead to the same inflaton scalar potential and, therefore, the same inflationary dynamics. However, the physical nature of inflaton in each theory is different. In the $f(R)$ gravity and $F(\mathcal{R})$ supergravity inflaton field is part of metric, whereas in the scalar-tensor gravity and supergravity inflaton is a matter particle. Therefore, the inflaton interactions with other matter fields are different in both theories. It gives rise to different inflaton decay rates and different reheating in the post-inflationary universe.

In our recent papers [11, 12, 13, 14, 15, 16, 17, 18] we proposed the new supergravity theory (we call it $F(\mathcal{R})$ -supergravity), and studied some of its physical applications (see also refs. [19, 20] for our earlier related work). The $F(\mathcal{R})$ -supergravity can be considered as the $N = 1$ locally supersymmetric extension of $f(R)$ gravity in four space-time dimensions.² Supergravity is well-motivated in High-Energy Physics theory beyond the Standard Model of elementary particles. Supergravity is also the low-energy effective action of Superstrings. As was demonstrated in ref. [11], an $F(\mathcal{R})$ supergravity is classically equivalent to the $N = 1$ Poincaré supergravity coupled to a dynamical (quintessence) chiral

²Another (unimodular) $F(R)$ supergravity theory was proposed in ref. [21].

superfield, whose (non-trivial) Kähler potential and superpotential are dictated by the chiral (holomorphic) function F . The classical equivalence is achieved via a non-trivial field redefinition [11] that gives rise to a non-trivial Jacobian in the path integral formulation of those quantum field theories (below their unitarity bounds). Hence, their classical equivalence is expected to be broken in quantum theory.³

The natural embedding of the Starobinsky $(R + R^2)$ -inflationary model into $F(\mathcal{R})$ supergravity was found in ref. [17]. It provides the very economical realization of chaotic inflation (at early times) in supergravity, which is consistent with observations [23] and gives a simple solution to the η -problem in supergravity [24].

The natural question arises, whether $F(\mathcal{R})$ supergravity is also capable to describe the present DE or have a positive cosmological constant. It is non-trivial because the standard supergravity with usual matter can only have a negative or vanishing cosmological constant [25]. It takes place since the usual (known) matter does not violate the *Strong Energy Condition* (SEC) [26]. A violation of SEC is required for an accelerating universe, and it is easily achieved in $f(R)$ gravity due to the fact that the quintessence field in $f(R)$ gravity is part of metric (ie. the unusual matter). Similarly, the quintessence scalar superfield in $F(\mathcal{R})$ supergravity is part of super-vielbein, and it also gives rise to a violation of SEC. In this Letter we further extend the Ansatz used in ref. [17] for F -function, and apply it to get a positive cosmological constant in the regime of low spacetime curvature (at late times).

Our paper is organized as follows. In sec. 2 we briefly recall the superspace construction of $F(\mathcal{R})$ supergravity, its relation to $f(R)$ gravity and the stability conditions. In sec. 3 we define our model of $F(\mathcal{R})$ supergravity, and compute its cosmological constant. Sec. 4 is our conclusion.

Throughout the paper we use the units $c = \hbar = M_{\text{Pl}} = 1$ in terms of the (reduced) Planck mass M_{Pl} , with the spacetime signature $(+, -, -, -)$. Our basic notation of General Relativity coincides with that of ref. [27]. An AdS-spacetime has a positive scalar curvature, and a dS-spacetime has a negative scalar curvature in our notation.

2 $F(\mathcal{R})$ supergravity and $f(R)$ gravity

A concise and manifestly supersymmetric description of supergravity is given by superspace. We refer the reader to the textbooks [28, 29, 30] for details of the superspace formulation of supergravity. A construction of the $F(\mathcal{R})$ supergravity action goes beyond the supergravity textbooks.

The most succinct formulation of $F(\mathcal{R})$ supergravity exist in a chiral 4D,

³See ref. [22] for the first steps in quantizing $f(R)$ gravity theories.

$N = 1$ superspace where it is defined by the action [11]

$$S_F = \int d^4x d^2\theta \mathcal{E} F(\mathcal{R}) + \text{H.c.} \quad (1)$$

in terms of a holomorphic function $F(\mathcal{R})$ of the covariantly-chiral scalar curvature superfield $\mathcal{R}$, and the chiral superspace density $\mathcal{E}$. The chiral $N = 1$ superfield $\mathcal{R}$ has the scalar curvature R as the field coefficient at its θ^2 -term. The chiral superspace density $\mathcal{E}$ (in a WZ gauge) reads

$$\mathcal{E} = e (1 - 2i\theta\sigma_a\bar{\psi}^a + 3\theta^2 X) \quad (2)$$

where $e = \sqrt{-g}$, ψ^a is gravitino, and $X = S - iP$ is the complex scalar auxiliary field (it does not propagate in the theory (1) despite of the apparent presence of the higher derivatives [11]).

A bosonic $f(R)$ gravity action is given by [1, 2, 3]

$$S_f = \int d^4x \sqrt{-g} f(R) \quad (3)$$

in terms of the real function $f(R)$ of the scalar curvature R . The relation between the master chiral superfield function $F(\mathcal{R})$ in eq. (1) and the corresponding bosonic function $f(R)$ in eq. (3) can be established by applying the standard formulae of superspace [28, 29, 30] and ignoring the fermionic contributions. For simplicity, we also ignore the complex nature of F and X in what follows.

The embedding of $f(R)$ gravity into $F(\mathcal{R})$ supergravity is given by [11, 12, 13]

$$f(R) = f(R, X(R)) \quad (4)$$

where the function $f(R, X)$ (or the gravity Lagrangian $\mathcal{L}$) is defined by

$$\mathcal{L} = f(R, X) = 2F'(X) \left[\frac{1}{3}R + 4X^2 \right] + 6XF(X) \quad (5)$$

and the function $X = X(R)$ is determined by solving an algebraic equation,

$$\frac{\partial f(R, X)}{\partial X} = 0 \quad (6)$$

The primes denote the derivatives with respect to the given argument. Equation (6) arises by varying the action (1) with respect to the auxiliary field X . It cannot be explicitly solved for X in a generic $F(\mathcal{R})$ supergravity theory.

The cosmological constant in $F(\mathcal{R})$ supergravity, in the regime of low space-time curvature, is thus given by

$$\Lambda = -f(0, X_0) \quad (7)$$

where $X_0 = X(0)$. It should be mentioned that X_0 represents the vacuum expectation value of the auxiliary field X that determines the scale of the supersymmetry breaking. Both inflation and DE imply $X_0 \neq 0$.

The $f(R)$ -gravity stability conditions in our notation are given by [1, 16]

$$f'(R) < 0 \quad (8)$$

and

$$f''(R) > 0 \quad (9)$$

The first (classical stability) condition (8) is related to the sign factor in front of the Einstein-Hilbert term (linear in R) in the $f(R)$ -gravity action, and it ensures that graviton is not a ghost. The second (quantum stability) condition (9) ensures that scalaron is not a tachyon. In $F(R)$ supergravity eq. (8) is replaced by a stronger condition [16],

$$F'(X) < 0 \quad (10)$$

Equation (10) guarantees the classical stability of the $f(R)$ -gravity embedding into the full $F(\mathcal{R})$ supergravity against small fluctuations of the axion field P [16].

To describe the early universe inflation (ie. in the regime of *high* spacetime curvature $R \rightarrow -\infty$), the function $f(R)$ should have the profile

$$f(R) = -\frac{1}{2}R + R^2 A(R) \equiv f_{\text{EH}}(R) + R^2 A(R) \quad (11)$$

with the slowly varying function $A(R)$ in the sense

$$|A'(R)| \ll \left| \frac{A(R)}{R} \right| \quad \text{and} \quad |A''(R)| \ll \left| \frac{A(R)}{R^2} \right| \quad (12)$$

The simplest choice $A = \text{const.} > 0$ gives rise to the Starobinsky model [4] with

$$f_S(R) = -\frac{1}{2}R + \frac{R^2}{12M_{\text{inf}}^2} \quad (13)$$

where the inflaton (scalaron) mass M_{inf} has been introduced.

To describe DE in the present universe, ie. in the regime with *low* spacetime curvature R , the function $f(R)$ should be close to the Einstein-Hilbert (linear) function $f_{\text{EH}}(R)$ with a small positive Λ ,

$$|f(R) - f_{\text{EH}}(R)| \ll |f_{\text{EH}}(R)|, \quad |f'(R) - f'_{\text{EH}}| \ll 1, \quad |Rf''(R)| \ll 1 \quad (14)$$

ie. $f(R) \approx -\frac{1}{2}R - \Lambda$ for small R with the very small and positive $\Lambda \approx 10^{-118}(M_{\text{Pl}}^4)$.

3 Cosmological constant

Equations (5) and (7) imply

$$\Lambda = -8F'(X_0)X_0^2 - 6X_0F(X_0) \quad (15)$$

where X_0 is a solution to the algebraic equation

$$4X_0^2F''(X_0) + 11X_0F'(X_0) + 3F(X_0) = 0 \quad (16)$$

As is clear from eq. (15), to have $\Lambda \neq 0$, one must have $X_0 \neq 0$, ie. a (spontaneous) supersymmetry breaking. However, in order to proceed further, one needs a reasonable Ansatz for the F -function in eq. (1).

The simplest opportunity is given by expanding the function $F(\mathcal{R})$ in Taylor series with respect to $\mathcal{R}$. Since the $N = 1$ chiral superfield $\mathcal{R}$ has X as its leading field component (in θ -expansion), one may expect that the Taylor expansion is a good approximation as long as $|X_0| \ll 1(M_{\text{Pl}})$. As was demonstrated in ref. [17], a viable (successful) description of inflation is possible in $F(\mathcal{R})$ supergravity, when keeping the *cubic* term $\mathcal{R}^3$ in the Taylor expansion of the $F(\mathcal{R})$ function. It is, therefore, natural to expand the function F up to the cubic term with respect to $\mathcal{R}$, and use it as our Ansatz here,

$$F(\mathcal{R}) = f_0 - \frac{1}{2}f_1\mathcal{R} + \frac{1}{2}f_2\mathcal{R}^2 - \frac{1}{6}f_3\mathcal{R}^3 \quad (17)$$

with some real coefficients f_0, f_1, f_2, f_3 . The Ansatz (17) differs from the one used in ref. [17] by the presence of the new parameter f_0 only. It is worth emphasizing here that f_0 is *not* a cosmological constant because one still has to eliminate the auxiliary field X . The stability conditions in the case (17) require

$$f_1 > 0 \quad , \quad f_2 > 0 \quad , \quad f_3 > 0 \quad (18)$$

and

$$f_2^2 < f_1f_3 \quad (19)$$

Inflation requires $f_3 \gg 1$ and $f_2^2 \gg f_1$ [17].⁴ As was shown in ref. [17], in the high-curvature regime the effective $f(R)$ -gravity action (originating from the $F(\mathcal{R})$ supergravity defined by eqs. (1) and (17) with $f_0 = 0$) takes the form of eq. (13) with $f_3 = 15M_{\text{inf}}^2$. To meet the WMAP observations [23], the parameter f_3 should be approximately $6.5 \cdot 10^{10}(N_e/50)^2$, where N_e is the number of e-foldings [17]. The cosmological constant in the high-curvature regime does not play a significant role and may be ignored there.

⁴The stronger condition $f_2^2 \ll f_1f_3$ was used in ref. [17] for simplicity.

In the low curvature regime, in order to recover the Einstein-Hilbert term, one has to fix $f_1 = 3/2$ [17]. Then the Ansatz (17) leads to the gravitational Lagrangian

$$f(R, X) = -5f_3X^4 + 11f_2X^3 - \frac{1}{3}f_3 \left(R + \frac{63}{2f_3} \right) X^2 + \left(6f_0 + \frac{2}{3}f_2R \right) X - \frac{1}{2}R \quad (20)$$

and the auxiliary field equation

$$X^3 - \frac{33f_2}{20f_3}X^2 + \frac{1}{30} \left(R + \frac{63}{2f_3} \right) X - \frac{1}{30f_3} (f_2R + 9f_0) = 0 \quad (21)$$

whose formal solution is available via the standard Cardano (Viète) formulae [31].

In the low-curvature regime we find a cubic equation for X_0 in the form

$$X_0^3 - \left(\frac{33f_2}{20f_3} \right) X_0^2 + \left(\frac{21}{20f_3} \right) X_0 - \left(\frac{3f_0}{10f_3} \right) = 0 \quad (22)$$

‘Linearizing’ eq. (22) with respect to X_0 brings the solution $X_0 = 2f_0/7$ whose substitution into the action (20) gives rise to a *negative* cosmological constant, $\Lambda_0 = -6f_0^2/7$. This way we recover the standard supergravity case.

Equations (20) and (22) allow us to write down the exact eq. (7) for the cosmological constant in the factorized form

$$\Lambda(X_0) = -\frac{11f_2}{4}X_0(X_0 - X_-)(X_0 - X_+) \quad (23)$$

where $X_{\pm}$ are the roots of the quadratic equation $x^2 - \frac{21}{11f_2}x + \frac{18f_0}{11f_2} = 0$, ie.

$$X_{\pm} = \frac{21}{22f_2} \left[1 \pm \sqrt{1 - \frac{2^3 \cdot 11}{7^2} f_0 f_2} \right] \quad (24)$$

Since $f_0 f_2$ is supposed to be very small, both roots $X_{\pm}$ are real and positive.

Equation (23) implies that $\Lambda > 0$ when either (I) $X_0 < 0$, or (II) X_0 is inside the interval (X_-, X_+) .

By using *Matematica* we were able to numerically confirm the existence of solutions to eq. (22) in the region (I) when $f_0 < 0$, but not in the region (II). So, to this end, we continue with the region (I) only. All real roots of eq. (22) are given by

$$\begin{aligned} (X_0)_1 &= 2\sqrt{-Q} \cos\left(\frac{\vartheta}{3}\right) + \frac{11f_2}{20f_3} , \\ (X_0)_2 &= 2\sqrt{-Q} \cos\left(\frac{\vartheta + 2\pi}{3}\right) + \frac{11f_2}{20f_3} , \\ (X_0)_3 &= 2\sqrt{-Q} \cos\left(\frac{\vartheta + 4\pi}{3}\right) + \frac{11f_2}{20f_3} , \end{aligned} \quad (25)$$

in terms of the Cardano-Viète parameters

$$\begin{aligned} Q &= -\frac{11f_2}{2^2 \cdot 5f_3} - \frac{7^2}{2^4 \cdot 5^2 f_3^2} \approx -\frac{11f_2}{20f_3} \ , \\ R &= -\frac{3 \cdot 7 \cdot 11f_2}{2^5 \cdot 5^2 f_3^2} + \frac{3f_0}{2^2 \cdot 5f_3} + \frac{11^3 f_2^3}{2^6 \cdot 5^3 f_3^3} \approx -\frac{1}{20f_3} \left(-\frac{21}{2}Q + 3f_0 \right) \end{aligned} \quad (26)$$

and the angle ϑ defined by

$$\cos \vartheta = \frac{R}{\sqrt{-Q^3}} \quad (27)$$

The Cardano discriminant reads $D = R^2 + Q^3$. All three roots are real provided that $D < 0$. It is known to be the case in the high-curvature regime [17], and it is also the case when f_0 is extremely small. Under our requirements on the parameters the angle ϑ is very close to zero, so the relevant solutions $X_0 < 0$ are given by the 2nd and 3rd lines of eq. (25), with $X_0 \approx f_0/10$.

4 Conclusion

We demonstrated that it is possible to have a *positive* cosmological constant (at low spacetime curvature or late times) in the particular $F(\mathcal{R})$ supergravity (without its coupling to super-matter) described by the Ansatz (17). The same Ansatz is applicable for describing a viable chaotic inflation in supergravity (at high spacetime curvature or early times). The positive cosmological constant was technically achieved as the *non-linear* effect with respect to the superspace curvature and spacetime curvature in the relatively narrow part of the parameter space (it is, therefore, highly constrained).

In the particular $F(\mathcal{R})$ supergravity model we considered, the effective $f(R)$ gravity function is essentially given by the Starobinsky function $(-\frac{1}{2}R + \frac{1}{12M_{\text{inf}}^2}R^2)$ in the high curvature regime, and by the DE-like function $(-\frac{1}{2}R - \Lambda)$ in the low curvature regime. Therefore, our model has a cosmological solution which describes an inflationary universe of the quasi-dS type with the Hubble function $H(t) \approx \frac{M_{\text{inf}}^2}{6}(t_{\text{end}} - t)$ at early times $t < t_{\text{end}}$ and an accelerating universe of the dS-type with $H = \Lambda$ at late times. It is similar to the known cosmological solutions unifying inflation and DE in $f(R)$ gravity [1].

Of course, describing the DE in the present universe requires an enormous fine-tuning of our parameters in the F -function. However, it is the common feature of all known approaches to the DE. This paper does not contribute to ‘explaining’ the smallness of the cosmological constant.

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