Solution to Bethe-Salpeter equation via Mellin-Barnes transform

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Abstract

We consider Mellin-Barnes transform of triangle ladder-like scalar diagram in $d = 4$ dimensions. It is shown how the multi-fold MB transform of the momentum integral corresponding to an arbitrary number of rungs is reduced to the two-fold MB transform. For this purpose we use Belokurov-Usyukina reduction method for four-dimensional scalar integrals in the position space. The result is represented in terms of Euler ψ -function and its derivatives. We derive new formulas for the MB two-fold integration in complex planes of two complex variables. We demonstrate that these formulas solve Bethe-Salpeter equation. We comment on further applications of the solution to the Bethe-Salpeter equation for the vertices in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We show that the recursive property of the MB transforms observed in the present work for that kind of diagrams has nothing to do with quantum field theory, theory of integral transforms, or with theory of polylogarithms in general, but has an origin in a simple recursive property for smooth functions which can be shown by using basic methods of mathematical analysis.

Keywords: Mellin-Barnes transform, UD functions, continuous functions

1 Introduction

In this paper we study Mellin-Barnes (MB) transform of three-point scalar ladder-like integrals in $d = 4$ space-time dimensions. These integrals contribute to Green's functions of four-dimensional massless scalar theories and can be represented in terms of UD functions [1, 2]. As has been proven in Refs. [4, 3, 5] at the level of graphs, the UD functions transform to themselves under Fourier transform. Also, this property can be proven by making a use of the MB transform [6].

As has been shown in Ref. [5], any other scalar three-point Green's function even in the non-integer number $d = 4 - 2\varepsilon$ of dimensions possesses such a property of invariance with respect to Fourier transform too. This happens due to the fact that any three-point function can diagrammatically be considered as a net of consequent three-point or four-point integrations in the position space since any of those integrations in the position space can be represented in terms of MB integral transforms with powers of spacetime intervals in denominators depending on the MB transform parameters. Doing consequently momentum integration via the MB transform, we come to an expression in which there is no momentum integral but there are integrations in complex planes of the MB transform parameters. We can apply arguments of Ref. [5, 6] to the expressions of such a type in order to demonstrate their invariance with respect to the Fourier transformation.

In general, technique of the MB transform is a powerful method to perform complicate multiloop calculations in quantum field theory [7, 8, 9, 10]. For example, the three-point one-loop function in the position space in the massless scalar theory is a combination of Appell's hypergeometric functions [11], which have appeared as a result of residue calculation via the MB transform. In Ref. [6] it has been shown that for the three-point integrations in the position space with the powers of space-time intervals in the denominators equal to the integer numbers shifted by some multipliers of ε , where ε is a parameter of dimensional regularization, $d = 4 - 2\varepsilon$, these Appell's hypergeometric functions can be reduced to a combination of the UD functions. On the other side, the four-point function in the position space in the scalar theory in $d = 4$ dimensions can be represented in terms of the UD functions too [12]. The invariance of those four-point ladder integrals under the Fourier transform can be proved by means of the MB transform too [12]. All these properties make UD functions very attractive for further investigation by making a use of the MB technique. The main advantage of this technique that we use in the present paper is that the three-point d -dimensional integration is transformed to a power-like form, which can be integrated in further loops as a three-point integration with the powers of space-time intervals in denominators which depend on the complex variables of the MB transform.

The first aim of this paper is to investigate further the properties of the UD functions via the MB transform. As a by-product, we derive new formulas for the MB two-fold integration in the complex planes of two variables of integration. The third aim is to demonstrate compatibility between these formulas for the MB transform and the Bethe-Salpeter equation for the infinite sum of triangle-like and box-like scalar ladders.

The fourth motivation for the investigation developed in this paper is to apply the experience obtained with studying the Bethe-Salpeter equation for the scalar ladders to calculation of an auxiliary vertex or Green function, which in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory does not depend on any scale, ultraviolet or infrared, in the limit of removing dimensional regularization, $\varepsilon = 0$, in $d = 4$. That vertex should satisfy the Bethe-Salpeter equation, particular for this Green's function. This is Lcc vertex in which an auxiliary field L couples to two (self-adjoint) Faddeev-Popov ghost fields c . To investigate that vertex by the Bethe-Salpeter equation has a sense since the ghost fields are the only fields of this vertex which stand in the measure of the path integral,

and therefore this equation restricts the vertex strongly.

As has been shown in Refs. [13, 14, 15], in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory starting with this vertex all the other correlators can be found in the four-dimensional limit (when $d = 4$) by using Slavnov-Taylor identity [16, 17, 18, 19, 20, 21, 22, 23]. The explicit two-loop result for that vertex has been found in Refs. [24, 25, 26]. All the poles in ε disappear in all number of loops for this vertex, however it cannot be analysed by the methods of conformal field theory since the auxiliary field L does not propagate. Nevertheless, the three-point function of dressed mean gluons of Refs. [13, 14, 15] can be fixed by conformal symmetry in analogy to [27, 28, 29, 30, 31], however, we cannot find higher-point correlators of the dressed mean fields from this Green's function by using ST identity.

As it can be seen from Refs. [13, 14, 15], the explicit structure of Lcc correlator includes logarithms and the UD functions of ratios of spacetime distances between arguments of the Green's functions. As it follows from Ref.[4], the ladder-like diagrams with horizontal gluon lines, that are important subclass of all the diagrams contributing to that vertex can be represented in terms of the UD functions too for an arbitrary number of the horizontal gluon lines. In the next paper we will demonstrate that any of the contributions to the correlator can be represented in terms of the UD functions [32].

In this paper we consider a particular solution to the Bethe-Salpeter equation, that represents an infinite sum of the triangle scalar ladder diagrams in four space-time dimensions in the MB representation. This particular solution has already been studied in the case of the box-ladder diagrams by Broadhurst and Davydychev in Ref. [34]. The authors of [34] calculated a sum of those box-ladder diagrams by making a use of the integral representation for the UD functions of Ref. [2]. As has been shown in Refs.[1, 2], the triangle-like and the box-like ladders in four space-time dimensions are related by conformal transformation in dual space and their MB transforms coincide. The difference of the present paper with the approach of Ref. [34] is that we do the calculation via the MB transformation. We derive recursive relations for the MB transforms of the momentum integrals corresponding to the UD triangle diagrams by using Belokurov-Usyukina trick [33] for the ladder diagrams in the position space, reducing the multi-fold MB integrals to the two-fold MB integrals.

2 Loop reduction in $d = 4$ dimensions

In this section we describe a trick with a help of which the triangle ladder diagram of n loops is reduced to the triangle ladder diagram of $n - 1$ loops. This trick has been published for the first time in paper [33]. The effect of the loop reduction has earlier been discovered in Ref. [35] for a propagator-type diagram of a particular topology in a special limit for the indices of diagram. Note, that in Ref. [33] the derivation has been performed without that special limit. Later, the key intermediate point of the trick has been published in Ref. [36]. Since all of that together has never been published in one article in detail, we collect all the results and intermediate steps we need for deriving our formulas for the MB transform of the triangle-ladder diagram.

The basic formulas we used to derive these diagrammatic relations are the chain integration for an any α_1 and α_2 , except for $d/2 + n$, $n \in N$,

$$\int Dx \frac{1}{[(x_1 - x)^2]^{\alpha_1} [(x_2 - x)^2]^{\alpha_2}} = \frac{A(\alpha_1, \alpha_2, d - \alpha_1 - \alpha_2)}{[(x_1 - x_2)^2]^{\alpha_1 + \alpha_2 - d/2}}, \quad (1)$$

$$\begin{aligned}
& \text{Diagram 1} = -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[\text{Diagram 2} \frac{J}{\varepsilon_2\varepsilon_3} \right. \\
& \quad \left. + \text{Diagram 3} \frac{1}{\varepsilon_1\varepsilon_2} + \text{Diagram 4} \frac{J}{\varepsilon_1\varepsilon_3} \right] \\
& \text{Diagram 5} = -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[\text{Diagram 6} \frac{J}{\varepsilon_2\varepsilon_3} \right. \\
& \quad \left. + \text{Diagram 7} \frac{J^{-1}}{\varepsilon_1\varepsilon_2} + \text{Diagram 8} \frac{J}{\varepsilon_1\varepsilon_3} \right] \\
& \text{Diagram 9} = \left[\text{Diagram 10} \frac{J}{\varepsilon_2\varepsilon_3} \right. \\
& \quad \left. + \text{Diagram 11} \frac{1}{\varepsilon_1\varepsilon_2} + \text{Diagram 12} \frac{J}{\varepsilon_1\varepsilon_3} \right]
\end{aligned}$$

The diagrams are Feynman-like diagrams with vertices and external lines labeled with expressions involving $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Diagram 1 is a triangle with a vertical line. Diagram 2 is a triangle with a vertical line. Diagram 3 is a triangle with a vertical line. Diagram 4 is a triangle with a vertical line. Diagram 5 is a triangle with a vertical line. Diagram 6 is a triangle with a vertical line. Diagram 7 is a triangle with a vertical line. Diagram 8 is a triangle with a vertical line. Diagram 9 is a triangle with a vertical line. Diagram 10 is a triangle with a vertical line. Diagram 11 is a triangle with a vertical line. Diagram 12 is a triangle with a vertical line.

Figure 1: Basic relations

where

$$A(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(d/2 - \alpha_1)\Gamma(d/2 - \alpha_2)\Gamma(d/2 - \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)},$$

$$\begin{aligned}
& \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} = -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3)\Gamma(1+\varepsilon_1)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} \\
& -\frac{\Gamma(\varepsilon_2)\Gamma(1+\varepsilon_3)\Gamma(2+\varepsilon_1)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} + \frac{\Gamma(1-\varepsilon_2)\Gamma(-\varepsilon_1)\Gamma(-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} \\
& = -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3)\Gamma(1+\varepsilon_1)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \frac{\Gamma(-\varepsilon_3)\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)}{\Gamma(2+\varepsilon_3)\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} \\
& -\frac{\Gamma(\varepsilon_2)\Gamma(1+\varepsilon_3)\Gamma(2+\varepsilon_1)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_3)\Gamma(2-\varepsilon_2)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_2)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} \\
& + \frac{\Gamma(1-\varepsilon_2)\Gamma(-\varepsilon_1)\Gamma(-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} = J \frac{1}{\varepsilon_2 \varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} \\
& -J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} + J \frac{1}{\varepsilon_1 \varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \nearrow \\ 2+\varepsilon_3 \quad 1+\varepsilon_1 \quad 1+\varepsilon_2 \\ \searrow \\ 1+\varepsilon_2 \end{array} =
\end{aligned}$$

Figure 2: Derivation of the basic relation. Part I. Lines without any index are the lines with index 1

and the star-triangle relation for $\alpha_1 + \alpha_2 + \alpha_3 = d$

$$\begin{aligned}
& \int Dx \frac{1}{[(x_1 - x)^2]^{\alpha_1} [(x_2 - x)^2]^{\alpha_2} [(x_3 - x)^2]^{\alpha_3}} = \\
& \frac{A(\alpha_1, \alpha_2, \alpha_3)}{[(x_1 - x_2)^2]^{d/2-\alpha_3} [(x_2 - x_3)^2]^{d/2-\alpha_1} [(x_1 - x_3)^2]^{d/2-\alpha_2}}.
\end{aligned}$$

$$\begin{aligned}
&= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{2+\varepsilon_3}{1-\varepsilon_1} \frac{1+\varepsilon_1}{1+\varepsilon_2} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{2+\varepsilon_3}{- \varepsilon_3} \frac{1+\varepsilon_2}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{2+\varepsilon_3}{1+\varepsilon_1} \frac{1+\varepsilon_2}{1+\varepsilon_3} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{2+\varepsilon_3}{1+\varepsilon_3} \frac{1+\varepsilon_2}{- \varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3}{1-\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3}{1-\varepsilon_3} \frac{1+\varepsilon_2}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= -J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} = J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)}{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= -J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{2+\varepsilon_3}{- \varepsilon_3} \frac{1+\varepsilon_2}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1)\Gamma(2+\varepsilon_1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(-\varepsilon_1-\varepsilon_3)\Gamma(1+\varepsilon_3)} \frac{2+\varepsilon_3}{1-\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&+ J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2)\Gamma(2-\varepsilon_1)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)\Gamma(\varepsilon_1)\Gamma(1+\varepsilon_3)} \frac{2+\varepsilon_3}{1-\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&= -J^2 \frac{1}{\varepsilon_2 \varepsilon_3^2} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1-\varepsilon_1} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1+\varepsilon_2} - J^2 \frac{1}{\varepsilon_1 \varepsilon_3^2} \frac{1}{1+\varepsilon_3} \frac{1+\varepsilon_1}{1-\varepsilon_2} \\
&= -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[\frac{J}{\varepsilon_2 \varepsilon_3} \frac{1+\varepsilon_1}{1-\varepsilon_1} + \frac{1}{\varepsilon_1 \varepsilon_2} \frac{1+\varepsilon_1}{1+\varepsilon_3} + \frac{J}{\varepsilon_1 \varepsilon_3} \frac{1+\varepsilon_1}{1-\varepsilon_2} \right]
\end{aligned}$$

Figure 3: Derivation of the basic relation. Part II. Lines without any index are the lines with index 1

Figure 4: Transformation of the first diagram in Fig. (1). Lines without any index are the lines with index 1

A new d -dimensional measure, introduced in ref. [24],

$$Dx \equiv \pi^{-\frac{d}{2}} d^d x$$

is used to avoid powers of π on the r.h.s. of the diagrams.

The bold dots correspond to the internal vertices of the diagrams, and this measure corresponds to the bold dots. We make an integration over coordinates of the internal vertices. In the first line of fig. (1) we present the formula which we want to derive. This formula is present in paper [36], however with a bit different indices of the diagrams. Later we will see how the r.h.s. of the first line of fig. (1) can be put in equivalence to fig. (4) of [36]. The second and the third lines of fig. (1) are consequences of the first line. How exactly the second and the third lines can be derived from the first line we explain further in the present paper. Usually, it can be done by inserting points of integration into the lines of the graphs. The third line is published in Ref. [33]. The second line has never been published. The condition for the ε -terms in the indices is

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.$$

In fig.(2) and in fig.(3), the latter is a continuation of fig.(2), we have shown how the first line of fig.(1) can be derived. Actually, the only mathematics that has been used here is the uniqueness

Figure 5: Transformation of the second diagram in Fig. (1)

$$\begin{aligned}
F \left[\text{Diagram 1} \right] &= -\frac{\pi^{-8}}{\varepsilon_3(1+\varepsilon_3)} \frac{1}{(p_3^2)^{-\varepsilon_3}} \frac{1}{(p_1^2)^{1-\varepsilon_1}} \frac{1}{(p_2^2)^{1-\varepsilon_2}} J^3 \left[\text{Diagram 2} \right] \\
F \left[\text{Diagram 3} \right] &= \pi^{-6} \frac{1}{p_2^2} \frac{1}{p_3^2} \frac{1}{(p_1^2)^{1-\varepsilon_1}} \frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1)}{\Gamma(1-\varepsilon_1)} \left[\text{Diagram 4} \right] \\
F \left[\text{Diagram 5} \right] &= \pi^{-6} \frac{1}{p_1^2} \frac{1}{p_3^2} \frac{1}{(p_2^2)^{1-\varepsilon_2}} \frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2)}{\Gamma(1-\varepsilon_2)} \left[\text{Diagram 6} \right] \\
F \left[\text{Diagram 7} \right] &= \pi^{-6} \frac{1}{p_3^2} \frac{1}{(p_1^2)^{1-\varepsilon_1}} \frac{1}{(p_2^2)^{1-\varepsilon_2}} J \left[\text{Diagram 8} \right]
\end{aligned}$$

Figure 6: Fourier transform of the diagrams in Fig. (1). The r.h.s. corresponds to the momentum integrals with indicated indices.

star-triangle relation and the integration by parts, both the relations are well-known in the scalar field theories [37, 38, 39] (for a short review, Ref. [40]). In all the figures from (1) till (9) we use for factor J the definition

$$J = \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)},$$

taken from Ref. [33].

In fig. (4) we show how the first diagram on the r.h.s. of the first line of fig. (1) transforms to the first diagram on the r.h.s. of the second line of figure (1). Again, we converted non-unique vertices to unique vertices by inserting the point of integration into one of the propagators attached to the vertex. Then, the star-triangle relation has been applied in a direct way or in a triangle-star way, until the diagram is converted to a diagram with desirable values of indices. The line without any index means that the index of the line is 1. We put the exact value of this index on the corresponding propagators of fig. (1) only. In the rest of figures we omit the index 1 just not to overload the diagram with indices.

In fig. (5) the second diagram of the first line of fig. (1) is transformed to the second diagram of the second line of fig. (1). Again, nothing else but the creating of unique vertices and triangles by putting new points of integration into the propagators has been used to obtain the desired indices. Thus, we have proven the first line and the second line of fig. (1).

The third line of fig. (1) can be obtained from the second line of fig. (1) by simple integration. Indeed, we eliminate the leftmost propagator on the l.h.s. of the second line by convoluting its

$$\begin{aligned}
F \left[\text{Diagram 1} \right] &= \frac{\pi^{-8}}{\varepsilon_3(1-\varepsilon_3)} \frac{1}{(p_3^2)^{\varepsilon_3}} \frac{1}{(p_1^2)^{1+\varepsilon_1}} \frac{1}{(p_2^2)^{1+\varepsilon_2}} J^{-3} \left[\text{Diagram 2} \right] \\
F \left[\text{Diagram 3} \right] &= \pi^{-6} \frac{1}{p_2^2} \frac{1}{p_3^2} \frac{1}{(p_1^2)^{1+\varepsilon_1}} \frac{\Gamma(1+\varepsilon_1)}{\Gamma(1-\varepsilon_1)} \frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)} \left[\text{Diagram 4} \right] \\
F \left[\text{Diagram 5} \right] &= \pi^{-6} \frac{1}{p_1^2} \frac{1}{p_3^2} \frac{1}{(p_2^2)^{1+\varepsilon_2}} \frac{\Gamma(1+\varepsilon_2)}{\Gamma(1-\varepsilon_2)} \frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)} \left[\text{Diagram 6} \right] \\
F \left[\text{Diagram 7} \right] &= \pi^{-6} \frac{1}{p_3^2} \frac{1}{(p_1^2)^{1+\varepsilon_1}} \frac{1}{(p_2^2)^{1+\varepsilon_2}} J^{-1} \left[\text{Diagram 8} \right]
\end{aligned}$$

Figure 7: Fourier transform of the diagrams in Fig. (1)

$$\begin{aligned}
& \left[\text{Diagram 9} \right] = J \frac{\pi^2}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_1 \varepsilon_2} \left[\text{Diagram 10} \right] 1 + \varepsilon_3 + \right. \\
& \left. + (p_2^2)^{\varepsilon_2} \frac{1}{\varepsilon_2 \varepsilon_3} \left[\text{Diagram 11} \right] 1 - \varepsilon_1 + (p_1^2)^{\varepsilon_1} \frac{1}{\varepsilon_1 \varepsilon_3} \left[\text{Diagram 12} \right] 1 - \varepsilon_2 \right]
\end{aligned}$$

Figure 8: Formula (25) of Ref. [1]

leftmost point with the line whose index is $2 - \varepsilon_3$,

$$\int d^4x \frac{1}{[(x_1 - x)^2]^{2-\varepsilon_3} [(x_2 - x)^2]^{2+\varepsilon_3}} = \pi^d \frac{\Gamma(\varepsilon_3)}{\Gamma(2-\varepsilon_3)} \frac{\Gamma(-\varepsilon_3)}{\Gamma(2+\varepsilon_3)} \delta^{(4)}(x_1 - x_2). \quad (2)$$

To derive formula (2) we need to replace in Eq. (1) each factor in the integrand with its integral Fourier transform. The formulas for the Fourier transform of the factor like those can be found in Ref. [40]. Indeed, since

$$\int d^d p e^{ipx} \frac{1}{(p^2)^\alpha} = \pi^{d/2} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \left(\frac{4}{x^2} \right)^{d/2 - \alpha} \Rightarrow$$

$$\frac{1}{(x^2)^\alpha} = \pi^{-d/2} 4^{-\alpha} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \int d^d p e^{ipx} \frac{1}{(p^2)^{d/2 - \alpha}},$$

we can write under the condition $\alpha_1 + \alpha_2 = d$

$$\begin{aligned} & \int d^d x \frac{1}{[(x_1 - x)^2]^{\alpha_1} [(x_2 - x)^2]^{\alpha_2}} = \\ & \pi^{-d} 4^{-\alpha_1 - \alpha_2} \frac{\Gamma(d/2 - \alpha_1)}{\Gamma(\alpha_1)} \frac{\Gamma(d/2 - \alpha_2)}{\Gamma(\alpha_2)} \int d^d x d^d p d^d q \frac{e^{ip(x-x_1)} e^{iq(x-x_2)}}{(p^2)^{d/2 - \alpha_1} (q^2)^{d/2 - \alpha_2}} = \\ & \pi^{-d} 4^{-\alpha_1 - \alpha_2} \frac{\Gamma(d/2 - \alpha_1)}{\Gamma(\alpha_1)} \frac{\Gamma(d/2 - \alpha_2)}{\Gamma(\alpha_2)} (2\pi)^d \int d^d p \frac{e^{ip(x_1 - x_2)}}{(p^2)^{d - \alpha_1 - \alpha_2}} = \\ & \pi^{-d} 4^{-\alpha_1 - \alpha_2} \frac{\Gamma(d/2 - \alpha_1)}{\Gamma(\alpha_1)} \frac{\Gamma(d/2 - \alpha_2)}{\Gamma(\alpha_2)} (2\pi)^{2d} \delta^{(d)}(x_1 - x_2) = \\ & \pi^d \frac{\Gamma(d/2 - \alpha_1)}{\Gamma(\alpha_1)} \frac{\Gamma(d/2 - \alpha_2)}{\Gamma(\alpha_2)} \delta^{(d)}(x_1 - x_2). \end{aligned}$$

As a consequence, we obtain in case of $\alpha_1 + \alpha_2 = d$

$$\int D x \frac{1}{[(x_1 - x)^2]^{\alpha_1} [(x_2 - x)^2]^{\alpha_2}} = \pi^{d/2} \frac{\Gamma(d/2 - \alpha_1)}{\Gamma(\alpha_1)} \frac{\Gamma(d/2 - \alpha_2)}{\Gamma(\alpha_2)} \delta^{(d)}(x_1 - x_2).$$

2.1 Description of Fourier Transform for the diagrams

In fig. (6) the Fourier transform is made for all the diagrams that appear in the first line of fig. (1). The procedure of Fourier transformation is done as follows. We replace each factor in the integrand of the position space representation with the integral Fourier transform of the corresponding factor in the momentum space representation. Space-time coordinates appear in the exponentials of the Fourier transforms. Integrating over the coordinates of internal vertices we create Dirac δ -functions, corresponding to the momentum conservation in each vertex of integration in the position space (internal vertex). The momentum integrals over loop momenta will be the Fourier transforms of the integrals in the position space. For example, the described procedure in a particular case of the one-rung integral in the position space would result in

$$\begin{aligned} \frac{1}{[31]} \int d^4 y d^4 z \frac{1}{[2y][1y][3z][yz][2z]} &= \frac{1}{(2\pi)^4} \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1 + p_2 + p_3) \times \\ &\times e^{ip_2 x_2} e^{ip_1 x_1} e^{ip_3 x_3} C^{(2)}(p_1^2, p_2^2, p_3^2). \end{aligned}$$

We assume the notation of Ref. [24], where $[Ny] = (x_N - y)^2$ and analogously for $[Nz]$ and $[yz]$, that is, $N = 1, 2, 3$ stands for $x_N = x_1, x_2, x_3$, respectively, which are the external points of

the triangle-like ladder diagram. In Refs. [1, 2] $C^{(n)}(p_1^2, p_2^2, p_3^2)$ is the definition for the result of momentum integrals for n -rung triangle ladder diagram in the momentum space representation. Just to make clear the definition for the operation of Fourier transformation used in fig. (6), we provide an example of the relation

$$F \left[\frac{1}{[31]} \int d^4 y d^4 z \frac{1}{[2y][1y][3z][yz][2z]} \right] = \frac{1}{(2\pi)^4} C^{(2)}(p_1^2, p_2^2, p_3^2).$$

The transformation from the position space to the momentum space is necessary to normalize on the results for the MB transform of the UD functions done in [1, 2] for the momentum space integrals.

Fig. (7) is fig. (6) with signs changed for all ε -terms. After this change of signs, the Fourier transform of the first line of fig. (1) takes the form depicted in fig. (8). In that diagrammatic relation we recognize Eq. (25) of Ref. [1].

3 Recursive formula for MB transform of the UD functions

We start this section with writing the definition of the momentum integral for the diagram in the r.h.s. of Fig. (8). To define the integral measure in the momentum space we use the notation

$$Dk \equiv \pi^{-\frac{d}{2}} d^d k. \quad (3)$$

This is done to avoid powers of π in formulas for the momentum integrals. For example, fig. (8) takes the form shown in fig. (9).

$$\begin{aligned} & \begin{array}{c} 1 + \varepsilon_2 \quad \leftarrow p_1 \\ 1 + \varepsilon_1 \quad \nearrow \\ p_3 \rightarrow \quad \triangleleft \quad 1 + \varepsilon_3 \quad 1 + \varepsilon_3 \\ 1 + \varepsilon_2 \quad \searrow \\ 1 + \varepsilon_1 \quad \leftarrow p_2 \end{array} = J \frac{1}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_1 \varepsilon_2} \triangleleft 1 + \varepsilon_3 \quad + \right. \\ & \left. + (p_2^2)^{\varepsilon_2} \frac{1}{\varepsilon_2 \varepsilon_3} \triangleleft 1 - \varepsilon_1 \quad + (p_1^2)^{\varepsilon_1} \frac{1}{\varepsilon_1 \varepsilon_3} \triangleleft 1 - \varepsilon_2 \right] \end{aligned}$$

Figure 9: Fig. (8) after measure redefinition (3)

We take a formula from paper [1] for the representation of the three-point momentum integral

$$J(\nu_1, \nu_2, \nu_3) = \int Dk \frac{1}{[(k+q_1)^2]^{\nu_1} [(k+q_2)^2]^{\nu_2} [(k+q_3)^2]^{\nu_3}}.$$

in terms of the MB transform

$$\begin{aligned}
J(\nu_1, \nu_2, \nu_3) &= \frac{1}{\Pi_i \Gamma(\nu_i) \Gamma(d - \Sigma_i \nu_i)} \frac{1}{(p_3^2)^{\Sigma \nu_i - d/2}} \oint_C dz_2 dz_3 x^{z_2} y^{z_3} \{ \Gamma(-z_2) \Gamma(-z_3) \\
&\Gamma(-z_2 - \nu_2 - \nu_3 + d/2) \Gamma(-z_3 - \nu_1 - \nu_3 + d/2) \Gamma(z_2 + z_3 + \nu_3) \Gamma(\Sigma \nu_i - d/2 + z_3 + z_2) \} \equiv \quad (4) \\
&\equiv \frac{1}{(p_3^2)^{\Sigma \nu_i - d/2}} \oint_C dz_2 dz_3 x^{z_2} y^{z_3} D^{(z_2, z_3)}[\nu_1, \nu_2, \nu_3],
\end{aligned}$$

where we have introduced the definition of function D ,

$$\begin{aligned}
D^{(z_2, z_3)}[\nu_1, \nu_2, \nu_3] &= \frac{\Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_2 - \nu_2 - \nu_3 + d/2) \Gamma(-z_3 - \nu_1 - \nu_3 + d/2)}{\Pi_i \Gamma(\nu_i)} \\
&\times \frac{\Gamma(z_2 + z_3 + \nu_3) \Gamma(\Sigma \nu_i - d/2 + z_3 + z_2)}{\Gamma(d - \Sigma_i \nu_i)}. \quad (5)
\end{aligned}$$

Here and in the rest of the paper we use the notation

$$x \equiv \frac{p_1^2}{p_3^2}, \quad y \equiv \frac{p_2^2}{p_3^2},$$

where the d -dimensional momenta p_1, p_2, p_3 satisfy the conservation law $p_1 + p_2 + p_3 = 0$ and are related to d -dimensional momenta q_1, q_2, q_3 by a parametrization

$$\begin{aligned}
p_1 &= q_3 - q_2, \\
p_2 &= q_1 - q_3, \\
p_3 &= q_2 - q_1.
\end{aligned}$$

As a consequence of this definition, p_1 appears to be a momentum that enters the one-loop triangle diagram in the vertex of the triangle which is opposite to the line with index ν_1 .

3.1 Description of notation in the MB representation

We absorb into the definition of the MB transform $D^{(z_2, z_3)}[\nu_1, \nu_2, \nu_3]$ of this three-point integral all the factors except for a power of the square of the external momentum p_3^2 . For $d = 4$ in the denominator we have a sum of the indices minus two for the power of p_3^2 .

We do not write the powers of i , supposing that we work in Euclidean space and the corresponding power of i can be recovered back after Wick rotation.

The contour of integration C passes a bit on the left of the imaginary axis, separates left and right poles and should be closed to the left infinity or to the right infinity. We choose to close the contour of integration in the complex plane to the right infinity. It could be closed to the left infinity too but it makes more complicate to take the residues into account since the residues in variables z_2 and z_3 are mixed in that case. Whether we have to close the contour to the right infinity or to the left infinity, the result should be the same function. We omit the factor $1/2\pi i$ that accompanies each integration over MB transform parameter. The inverse factor is generated in front of the residues.

It is known that this representation of the three-point integral can be derived by applying the two-fold MB transform to the integral over Feynman parameters, with producing the Euler beta functions [7] after integrating over these parameters. There is a difference with the representation used in Ref.[6]. The form of Ref.[6] can be recovered from (4) by a cyclic redefinition. In the rest of the paper, we use representation (4).

3.2 Multi-fold MB transforms of the UD functions

To develop the recursive equations for the MB transforms of the UD functions, we recover the definition of the MB transforms for the triangle ladder diagrams in $d = 4$ spacetime dimensions

$$C^{(n)}(p_1^2, p_2^2, p_3^2) = \frac{1}{(p_3^2)^n} \Phi^{(n)}(x, y) = \frac{1}{(p_3^2)^n} \oint_C dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2, z_3),$$

where we used the notation $\mathcal{M}^{(n)}(z_2, z_3)$ for the MB transform of the UD function $\Phi^{(n)}(x, y)$ of two arguments. The definition of incoming momenta for the triangle ladder diagrams is like in Ref.[1, 2, 5]. We include in the definition of $\mathcal{M}^{(n)}(z_2, z_3)$ all the factors that can appear in front of the MB transformation except for the power of the momentum p_3^2 .

The two-fold MB transform is known for the first UD function only. For a UD function of a higher order, the result is given in terms of the multi-fold MB transform [1, 2]. In the next section we reduce the multi-fold MB transform to the two-fold MB transform by making a use of the loop reduction trick described in the previous section.

The iterative integral relation for the triangle ladder diagram given in Refs. [1, 2, 5],

$$C^{(n)}(p_1^2, p_2^2, p_3^2) = \int d^4 r_n \frac{C^{(n-1)}((p_1 + r_n)^2, (p_2 - r_n)^2, p_3^2)}{(p_1 + r_n)^2 (p_2 - r_n)^2 r_n^2},$$

results in an integral recursive relation for the MB transforms of the UD functions

$$\begin{aligned} \Phi^{(n+1)}\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right) &= p_3^2 \int d^4 r \Phi^{(n)}\left(\frac{(p_1 + r)^2}{p_3^2}, \frac{(p_2 - r)^2}{p_3^2}\right) \frac{1}{(p_1 + r)^2 (p_2 - r)^2 r^2} = \\ p_3^2 \int d^4 r \oint_C dz_2 dz_3 \mathcal{M}^{(n)}(z_2, z_3) &\left(\frac{(p_1 + r)^2}{p_3^2}\right)^{z_2} \left(\frac{(p_2 - r)^2}{p_3^2}\right)^{z_3} \frac{1}{(p_1 + r)^2 (p_2 - r)^2 r^2} = \\ \int d^4 r \oint_C dz_2 dz_3 \mathcal{M}^{(n)}(z_2, z_3) &\frac{1}{((p_1 + r)^2)^{1-z_2} ((p_2 - r)^2)^{1-z_3} r^2} \frac{1}{(p_3^2)^{z_2+z_3-1}} = \\ \pi^2 \oint_C dz_2 dz_3 \mathcal{M}^{(n)}(z_2, z_3) &J(1-z_3, 1-z_2, 1) \frac{1}{(p_3^2)^{z_2+z_3-1}} = \\ \pi^2 \oint_C dz_2 dz_3 du dv x^u y^v \mathcal{M}^{(n)}(z_2, z_3) &D^{(u,v)}[1-z_3, 1-z_2, 1], \end{aligned}$$

where $D^{(u,v)}[1-z_3, 1-z_2, 1]$ is defined above as the MB transform of the corresponding three-point integral, according to description done in the previous subsection ¹.

Thus, one can derive the first recursive formula for the MB transforms, which relates the transforms of two neighbour UD functions [1, 2]

$$\mathcal{M}^{(n+1)}(u, v) = \pi^2 \oint_C dz_2 dz_3 \mathcal{M}^{(n)}(z_2, z_3) D^{(u,v)}[1-z_3, 1-z_2, 1].$$

The function $D^{(u,v)}[1-z_3, 1-z_2, 1]$ is a nontrivial combination of the Euler gamma functions in numerator and denominator,

$$D^{(u,v)}[1-z_3, 1-z_2, 1] = \frac{\Gamma(z_2 - u) \Gamma(z_3 - v) \Gamma(1 - z_2 - z_3 + u + v)}{\Gamma(1 + z_2 + z_3) \Gamma(1 - z_2) \Gamma(1 - z_3)} \Gamma(-u) \Gamma(-v) \Gamma(1 + u + v).$$

This formula is written from definition (4).

¹Factor π^2 appears due to usage of the original measure $d^4 r$ in the momentum space.

4 Reduction of the multi-fold MB transforms

The formulas of the previous subsection are in some sense consequence of the ladder-like topology of the diagram. We did not make any integration in the complex planes of MB parameters.

Consider the diagram on the l.h.s. of fig. (9). We can repeat the trick of the previous section and obtain the following MB representation for it

$$\begin{aligned}
& \frac{1}{p_3^2} \int Dr \oint dz_2 dz_3 \left(\frac{(p_1 + r)^2}{p_3^2} \right)^{z_2} \left(\frac{(p_2 - r)^2}{p_3^2} \right)^{z_3} \frac{D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3]}{[(p_1 + r)^2]^{1+\varepsilon_2} [(p_2 - r)^2]^{1+\varepsilon_1} [r^2]^{1+\varepsilon_3}} = \\
& \frac{1}{p_3^2} \oint_C Dr dz_2 dz_3 \frac{1}{[p_2^3]^{z_2+z_3}} \frac{D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3]}{[(p_1 + r)^2]^{1+\varepsilon_2-z_2} [(p_2 - r)^2]^{1+\varepsilon_1-z_3} [r^2]^{1+\varepsilon_3}} = \\
& \frac{1}{p_3^2} \oint_C dz_2 dz_3 \frac{1}{[p_3^2]^{z_2+z_3}} J(1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3) D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \\
& \frac{1}{(p_3^2)^2} \oint_C dz_2 dz_3 du dv x^u y^v D^{(u, v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] \times \\
& D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] \quad (6)
\end{aligned}$$

This formula is derived in analogy with the MB transforms of the UD functions in the previous section. In detail, the procedure looks like follows. First, we calculate the MB transform of the leftmost triangle integral, this yields another triangle integral with indices depending on the complex variables of the previous MB transform. This procedure will be used in all the constructions below.

The general strategy of the present investigation consists in the expanding of the r.h.s. in terms of ε_i . The coefficients of this expansion will contain ψ functions and its derivatives. In the r.h.s. on fig. (9) we cannot put all the values of ε_i immediately equal to zero without expanding these r.h.s. in terms of ε_i and observing that poles in ε_i disappear. Thus, instead of a Laurent series we obtain a Taylor series. Comparing the coefficients in front of the different products of powers of ε_i we can derive an infinite number of new relations for the two-fold MB integrals over the complex variables z_2 and z_3 .

4.1 Description of the momentum integral

Notation to be used

$$\int_n (\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

corresponds to the momentum integral with incoming momenta p_1, p_2, p_3 as they are depicted on the l.h.s. of Fig.(9), with the integral measure defined in (3), but with n loops². The ε -terms $\varepsilon_1, \varepsilon_2, \varepsilon_3$ appear in the indices of lines for the first eight propagators on the left side of the diagram, namely, in the indices $1 + \varepsilon_1, 1 + \varepsilon_2, 1 + \varepsilon_1$ on the upper side of the diagram, $1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_2$ on the lower side of the diagram, and $1 + \varepsilon_3$ on the first two rungs. The rest of lines have indices equal to 1. These positions and values of the indices are indicated in the third line of Fig.(1) (in the position space).

²with amputated external legs

4.2 Two-fold MB transform for the two-rung ladder

As a first step to our formulas, we reproduce Eq.(25) of Ref.[1]. This is already done in Section 2, however we make it again just to introduce the notation that will be used for the higher rung diagrams.

We consider a diagrammatic relation that can be obtained by integral convolution of the leftmost external point of the diagrams in the first line of fig. (1) with the line that has index $2 - \varepsilon_3$. In such a way the Dirac δ -function is produced which eliminates one of the integrations and the leftmost point on the l.h.s. is converted to the external vertex, that is, it is not a vertex of integration longer. The formula which can be used for this purpose is identity (2).

Then, we take the Fourier transform of each one of the diagrams in the diagrammatic relation obtained in that way. By keeping all the factors that appear after making the Fourier transform on both the parts of this new diagrammatic equation, we come to

$$\begin{aligned} J^3 \frac{\Gamma(1 + \varepsilon_3)}{\Gamma(1 - \varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} (p_2^2)^{1-\varepsilon_2}} \int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = \\ J \left[\frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1 + \varepsilon_3)}{\Gamma(1 - \varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} p_2^2 (p_3^2)^{1+\varepsilon_3}} J(1, 1, 1 + \varepsilon_1) + \right. \\ \left. \frac{J}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(1 + \varepsilon_3)}{\Gamma(1 - \varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} (p_2^2)^{1-\varepsilon_2} (p_3^2)^{1+\varepsilon_3}} J(1, 1, 1 - \varepsilon_3) + \right. \\ \left. \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1 + \varepsilon_3)}{\Gamma(1 - \varepsilon_3)} \frac{1}{p_1^2 (p_2^2)^{1-\varepsilon_2} (p_3^2)^{1+\varepsilon_3}} J(1, 1, 1 + \varepsilon_2) \right]. \end{aligned}$$

After a simple algebra we derive an equation

$$\begin{aligned} \int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = \frac{J^{-1}}{(p_3^2)^{1+\varepsilon_3}} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{(p_2^2)^{\varepsilon_2}} J(1, 1, 1 + \varepsilon_1) + \right. \\ \left. \frac{1}{\varepsilon_1 \varepsilon_2} J(1, 1, 1 - \varepsilon_3) + \frac{1}{\varepsilon_1 \varepsilon_3} \frac{1}{(p_1^2)^{\varepsilon_1}} J(1, 1, 1 + \varepsilon_2) \right], \end{aligned}$$

from which by changing the signs of all the values ε_i we obtain

$$\begin{aligned} \int_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{J}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{(p_2^2)^{-\varepsilon_2}} J(1, 1, 1 - \varepsilon_1) + \right. \\ \left. \frac{1}{\varepsilon_1 \varepsilon_2} J(1, 1, 1 + \varepsilon_3) + \frac{1}{\varepsilon_1 \varepsilon_3} \frac{1}{(p_1^2)^{-\varepsilon_1}} J(1, 1, 1 - \varepsilon_2) \right]. \end{aligned} \quad (7)$$

This is exactly Eq.(25) of Ref.[1]. This equation has been derived diagrammatically in the previous section. In that case the Fourier transform of the first line of fig. (1) has been done. This approach to reproduce Eq.(25) of Ref.[1] can be considered as a cross-check of the procedure used in the previous section to produce fig. (9).

At this moment we start to use the matter developed in the previous section. First of all, we rewrite the r.h.s. of Eq.(7) in the MB transformed representation

$$\begin{aligned} \int_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{J}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{(p_2^2)^{\varepsilon_2}}{(p_3^2)^{1-\varepsilon_1}} \oint_C du dv x^u y^v D^{(u,v)} [1 - \varepsilon_1] + \right. \\ \left. \frac{1}{\varepsilon_1 \varepsilon_2} \frac{1}{(p_3^2)^{1+\varepsilon_3}} \oint_C du dv x^u y^v D^{(u,v)} [1 + \varepsilon_3] + \frac{1}{\varepsilon_1 \varepsilon_3} \frac{(p_1^2)^{\varepsilon_1}}{(p_3^2)^{1-\varepsilon_2}} \oint_C du dv x^u y^v D^{(u,v)} [1 - \varepsilon_2] \right], \end{aligned} \quad (8)$$

where we have introduced a notation

$$D^{(u,v)}[1+\nu] \equiv D^{(u,v)}[1, 1, 1+\nu].$$

Eq.(8) can be written as

$$\begin{aligned} \int_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \frac{J}{(p_3^2)^2} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \left(\frac{p_2^2}{p_3^2} \right)^{\varepsilon_2} \oint_C du dv x^u y^v D^{(u,v)}[1-\varepsilon_1] + \right. \\ &\frac{1}{\varepsilon_1 \varepsilon_2} \oint_C du dv x^u y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1 \varepsilon_3} \left(\frac{p_1^2}{p_3^2} \right)^{\varepsilon_1} \oint_C du dv x^u y^v D^{(u,v)}[1-\varepsilon_2] \Big] = \\ &\frac{J}{(p_3^2)^2} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \oint_C du dv x^u y^{v+\varepsilon_2} D^{(u,v)}[1-\varepsilon_1] + \right. \\ &\frac{1}{\varepsilon_1 \varepsilon_2} \oint_C du dv x^u y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1 \varepsilon_3} \oint_C du dv x^{u+\varepsilon_1} y^v D^{(u,v)}[1-\varepsilon_2] \Big] = \\ &\frac{J}{(p_3^2)^2} \oint_C du dv x^u y^v \left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right] \equiv \\ &\frac{1}{(p_3^2)^2} \oint_C du dv x^u y^v M_2^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3]. \end{aligned} \quad (9)$$

Here we shift the variable of integration in the complex plane. That means, the contour of integration C still passes between the left and the right poles. Positions of the poles in the plane of two complex variables will be changed with that trick but their nature (left or right) remains unchanged.

On the other hand, we have formula (6), from which we obtain

$$\begin{aligned} \oint_C dz_2 dz_3 D^{(u,v)}[1+\varepsilon_1-z_3, 1+\varepsilon_2-z_2, 1+\varepsilon_3] D^{(z_2, z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] = \\ J \left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right] = M_2^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3]. \end{aligned} \quad (10)$$

This formula is valid for any u, v and presents by itself a nontrivial result which can be used in practical applications of the MB integration. This is the two-fold MB transform of two-loop integral, and it is valid not only in the limit of all $\varepsilon_i \rightarrow 0$ but for any ε_i . Expanding this formula in terms of ε_i we obtain infinite number of new relations. The explicit form of $M_2^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ is calculated in the next section in the limit of vanishing ε_i , the result is Eq. (16),

$$\begin{aligned} \oint_C dz_2 dz_3 D^{(u,v)}[1-z_3, 1-z_2, 1] D^{(z_2, z_3)}[1, 1, 1] = \\ \Gamma^2(1+u+v) \Gamma^2(-u) \Gamma^2(-v) \times \\ \left[\frac{1}{2} (\psi'(-v) + \psi'(-u)) - \frac{1}{2} (\psi(-v) - \psi(-u))^2 - \frac{3}{2} (\Gamma(1-\varepsilon) \Gamma(1+\varepsilon))_{\varepsilon=0}^{(2)} \right]. \end{aligned}$$

4.3 Two-fold MB transform for the three-rung ladder

We can derive a new formula for the two-fold MB transformation going to higher loops in the ladder diagrams. We start to work with integral $\int_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ which appears on the l.h.s. in the third line

of Fig.(1) and take the Fourier transform of each one of the diagrams of that line. The Fourier transform of the l.h.s. can be written as

$$\left(\frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_3)}\right)^2 \frac{1}{p_1^2} \frac{1}{p_2^2} \int_3(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3)$$

and the whole diagrammatic relation becomes as

$$J^3 \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2} \int_3(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^2 \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} p_2^2 (p_3^2)^{1-\varepsilon_2}} \int_2(-\varepsilon_1) +$$

$$\frac{J^3}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} (p_2^2)^{1-\varepsilon_2} p_3^2} \int_2(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \frac{J^2}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 (p_2^2)^{1-\varepsilon_2} (p_3^2)^{1-\varepsilon_1}} \int_2(-\varepsilon_2).$$

We kept all the factors that appear after making the Fourier transform on both the parts of the diagrammatic equation. Here we use a brief notation

$$\int_n(-\varepsilon_1) \equiv \int_n(-\varepsilon_1, 0, \varepsilon_1), \quad \int_n(-\varepsilon_2) \equiv \int_n(0, -\varepsilon_2, \varepsilon_2).$$

After a little algebra we obtain an expansion of the three-loop integral in terms of the two-loop integrals,

$$\int_3(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^{-1} \left[\frac{1}{(p_1^2)^{-\varepsilon_1} (p_3^2)^{1-\varepsilon_2}} \frac{1}{\varepsilon_2 \varepsilon_3} \int_2(-\varepsilon_1) + \right.$$

$$\left. \frac{J}{(p_1^2)^{-\varepsilon_1} (p_2^2)^{-\varepsilon_2} p_3^2} \frac{1}{\varepsilon_1 \varepsilon_2} \int_2(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \frac{1}{(p_2^2)^{-\varepsilon_2} (p_3^2)^{1-\varepsilon_1}} \frac{1}{\varepsilon_1 \varepsilon_3} \int_2(-\varepsilon_2) \right].$$

By changing the signs of all values ε_i we obtain an equation

$$\int_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = J \left[\frac{1}{(p_1^2)^{\varepsilon_1} (p_3^2)^{1+\varepsilon_2}} \frac{1}{\varepsilon_2 \varepsilon_3} \int_2(\varepsilon_1) + \right.$$

$$\left. \frac{J^{-1}}{(p_1^2)^{\varepsilon_1} (p_2^2)^{\varepsilon_2} p_3^2} \frac{1}{\varepsilon_1 \varepsilon_2} \int_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{1}{(p_2^2)^{\varepsilon_2} (p_3^2)^{1+\varepsilon_1}} \frac{1}{\varepsilon_1 \varepsilon_3} \int_2(\varepsilon_2) \right].$$

The previous equation can be re-written for the MB transforms, and we obtain the result

$$\int_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = J \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{(p_1^2)^{\varepsilon_1} (p_3^2)^{3+\varepsilon_2}} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1) + \right.$$

$$\left. J^{-1} \frac{1}{\varepsilon_1 \varepsilon_2} \frac{1}{(p_1^2)^{\varepsilon_1} (p_2^2)^{\varepsilon_2} (p_3^2)^3} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \right.$$

$$\left. \frac{1}{\varepsilon_1 \varepsilon_3} \frac{1}{(p_2^2)^{\varepsilon_2} (p_3^2)^{3+\varepsilon_1}} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_2) \right] =$$

$$J \left[\frac{1}{\varepsilon_2 \varepsilon_3} \left(\frac{p_3^2}{p_1^2} \right)^{\varepsilon_1} \frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1) + \right.$$

$$\left. J^{-1} \frac{1}{\varepsilon_1 \varepsilon_2} \left(\frac{p_3^2}{p_1^2} \right)^{\varepsilon_1} \left(\frac{p_3^2}{p_2^2} \right)^{\varepsilon_2} \frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \right]$$

$$\begin{aligned}
& + \frac{1}{\varepsilon_1 \varepsilon_3} \left(\frac{p_3^2}{p_2^2} \right)^{\varepsilon_2} \frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_2) \Big] = \\
\frac{J}{(p_3^2)^{3-\varepsilon_3}} & \left[\frac{1}{\varepsilon_2 \varepsilon_3} \oint_C du dv x^{u-\varepsilon_1} y^v M_2^{(u,v)}(\varepsilon_1) + J^{-1} \frac{1}{\varepsilon_1 \varepsilon_2} \oint_C du dv x^{u-\varepsilon_1} y^{v-\varepsilon_2} M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \right. \\
& \left. + \frac{1}{\varepsilon_1 \varepsilon_3} \oint_C du dv x^u y^{v-\varepsilon_2} M_2^{(u,v)}(\varepsilon_2) \right] = \\
\frac{J}{(p_3^2)^{3-\varepsilon_3}} & \oint_C du dv x^u y^v \left[\frac{1}{\varepsilon_2 \varepsilon_3} M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1 \varepsilon_2} M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \right. \\
& \left. + \frac{1}{\varepsilon_1 \varepsilon_3} M_2^{(u, v+\varepsilon_2)}(\varepsilon_2) \right] \equiv \frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C du dv x^u y^v M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3).
\end{aligned}$$

Here we have shifted the variable of integration in the complex plane. That means, the contour of integration C still passes between the left and the right poles. Positions of the poles in the planes of two complex variables will be changed but their nature (to belong to the set of left poles or to the set of the right poles) cannot be changed with such a trick.

On the other side, in complete analogy with Eq.(6) we obtain

$$\begin{aligned}
& \int_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
\frac{1}{(p_3^2)^2} \int Dr & \oint dz_2 dz_3 \left(\frac{(p_1+r)^2}{p_3^2} \right)^{z_2} \left(\frac{(p_2-r)^2}{p_3^2} \right)^{z_3} \frac{M_2^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{[(p_1+r)^2]^{1+\varepsilon_1} [(p_2-r)^2]^{1+\varepsilon_2} r^2} = \\
\frac{1}{(p_3^2)^2} \int Dr & \oint_C dz_2 dz_3 \frac{1}{[p_3^2]^{z_2+z_3}} \frac{M_2^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{[(p_1+r)^2]^{1+\varepsilon_1-z_2} [(p_2-r)^2]^{1+\varepsilon_2-z_3} r^2} = \\
\oint_C dz_2 dz_3 & \frac{1}{[p_3^2]^{2+z_2+z_3}} J(1+\varepsilon_2-z_3, 1+\varepsilon_1-z_2, 1) M_2^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
\frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C & dz_2 dz_3 du dv x^u y^v D^{(u,v)}[1+\varepsilon_2-z_3, 1+\varepsilon_1-z_2, 1] M_2^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3). \quad (11)
\end{aligned}$$

Thus, we derive that

$$\begin{aligned}
M_3^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3] &= \oint_C dz_2 dz_3 D^{(u,v)}[1+\varepsilon_2-z_3, 1+\varepsilon_1-z_2, 1] M_2^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
J \left[\frac{1}{\varepsilon_2 \varepsilon_3} M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1 \varepsilon_2} M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{1}{\varepsilon_1 \varepsilon_3} M_2^{(u, v+\varepsilon_2)}(\varepsilon_2) \right]. \quad (12)
\end{aligned}$$

This formula is valid for any u, v and for any $\varepsilon_1, \varepsilon_2$. Expanding this formula in terms of ε_i we can obtain infinite number of new integral relations. In the next section we calculate the limit $\varepsilon_i \rightarrow 0$ of $M_3^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3]$.

4.4 Two-fold MB transform for the four-rung ladder

In the next paragraphs of this section we consider the case of the four-loop momentum triangle ladder diagram. It looks like the diagram on the l.h.s. of the third line of Fig.(1) but with one more rung. To get that graphical representation, the diagrams in the third line of fig. (1) are integrated

with three more propagators, the index of each propagator is equal to 1. The Fourier transform of the l.h.s. of the diagrammatic relation obtained in such a way contains the integral $\int_4(\varepsilon_1, \varepsilon_2, \varepsilon_3)$,

$$\left(\frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_3)}\right)^2 \frac{1}{p_1^2} \frac{1}{p_2^2} \int_4(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3),$$

and we derive the identity

$$J^3 \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2} \int_4(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^2 \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2 (p_3^2)^{1-\varepsilon_2}} \int_3(-\varepsilon_1) +$$

$$\frac{J^3}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2 p_3^2} \int_3(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \frac{J^2}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2 (p_3^2)^{1-\varepsilon_1}} \int_3(-\varepsilon_2).$$

After a little algebra we obtain an expansion of the four-loop integral in terms of the three-loop integrals,

$$\int_4(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^{-1} \left[\frac{1}{(p_3^2)^{1-\varepsilon_2}} \frac{1}{\varepsilon_2 \varepsilon_3} \int_3(-\varepsilon_1) + \frac{J}{p_3^2} \frac{1}{\varepsilon_1 \varepsilon_3} \int_3(-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \right.$$

$$\left. \frac{1}{(p_3^2)^{1-\varepsilon_1}} \frac{1}{\varepsilon_1 \varepsilon_3} \int_3(-\varepsilon_2) \right],$$

from which by changing the signs of all the values ε_i we obtain an equation

$$\int_4(\varepsilon_1, \varepsilon_2, \varepsilon_3) = J \left[\frac{1}{(p_3^2)^{1+\varepsilon_2}} \frac{1}{\varepsilon_2 \varepsilon_3} \int_3(\varepsilon_1) + \frac{J^{-1}}{p_3^2} \frac{1}{\varepsilon_1 \varepsilon_3} \int_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \right.$$

$$\left. \frac{1}{(p_3^2)^{1+\varepsilon_1}} \frac{1}{\varepsilon_1 \varepsilon_3} \int_3(\varepsilon_2) \right].$$

This equation can be re-written for the MB transforms

$$\int_4(\varepsilon_1, \varepsilon_2, \varepsilon_3) = J \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{(p_3^2)^{1+\varepsilon_2} (p_3^2)^{3+\varepsilon_1}} \oint_C du dv x^u y^v M_3^{(u,v)}(\varepsilon_1) + \right.$$

$$J^{-1} \frac{1}{\varepsilon_1 \varepsilon_2} \frac{1}{(p_3^2)^{4-\varepsilon_3}} \oint_C du dv x^u y^v M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) +$$

$$\left. \frac{1}{\varepsilon_1 \varepsilon_3} \frac{1}{(p_3^2)^{1+\varepsilon_1} (p_3^2)^{3+\varepsilon_2}} \oint_C du dv x^u y^v M_3^{(u,v)}(\varepsilon_2) \right] =$$

$$\frac{J}{(p_3^2)^{4-\varepsilon_3}} \oint_C du dv x^u y^v \left[\frac{1}{\varepsilon_2 \varepsilon_3} M_3^{(u,v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1 \varepsilon_2} M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{1}{\varepsilon_1 \varepsilon_3} M_3^{(u,v)}(\varepsilon_2) \right] \equiv$$

$$\frac{1}{(p_3^2)^{4-\varepsilon_3}} \oint_C du dv x^u y^v M_4^{(u,v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3].$$

On the other hand, in analogy to Eq.(6),

$$\int_4(\varepsilon_1, \varepsilon_2, \varepsilon_3) =$$

$$\frac{1}{(p_3^2)^{3-\varepsilon_3}} \int Dr \oint dz_2 dz_3 \left(\frac{(p_1+r)^2}{p_3^2} \right)^{z_2} \left(\frac{(p_2-r)^2}{p_3^2} \right)^{z_3} \frac{M_3^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{(p_1+r)^2 (p_2-r)^2 r^2} =$$

$$\begin{aligned}
& \frac{1}{(p_3^2)^{3-\varepsilon_3}} \int Dr \oint_C dz_2 dz_3 \frac{1}{[p_3^2]^{z_2+z_3}} \frac{M_3^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{[(p_1+r)^2]^{1-z_2} [(p_2-r)^2]^{1-z_3} r^2} = \\
& \frac{1}{(p_3^2)^{3-\varepsilon_3}} \oint_C dz_2 dz_3 \frac{1}{[p_3^2]^{z_2+z_3}} J(1-z_3, 1-z_2, 1) M_3^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
& \frac{1}{(p_3^2)^{4-\varepsilon_3}} \oint_C dz_2 dz_3 \oint_C du dv x^u y^v D^{(u, v)}[1-z_3, 1-z_2, 1] M_3^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3). \tag{13}
\end{aligned}$$

We obtain

$$\begin{aligned}
& \oint_C dz_2 dz_3 D^{(u, v)}[1-z_3, 1-z_2, 1] M_3^{(z_2, z_3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
& J \left[\frac{1}{\varepsilon_2 \varepsilon_3} M_3^{(u, v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1 \varepsilon_2} M_3^{(u, v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{1}{\varepsilon_1 \varepsilon_3} M_3^{(u, v)}(\varepsilon_2) \right] = M_4^{(u, v)}[\varepsilon_1, \varepsilon_2, \varepsilon_3]. \tag{14}
\end{aligned}$$

New integral relations can be derived by expanding this formula in terms of ε_i , and this formula is valid for any u, v .

5 One-fold MB transforms

In this section we collect the useful formulas for the one-fold MB transforms,

$$\begin{aligned}
& \oint_C dz x^z \Gamma^2(-z) \Gamma^2(1+z) = -\frac{\ln x}{1-x}, \\
& \oint_C dz x^z \{ \Gamma^2(-z) \Gamma(z+1) \Gamma^*(z) \} = -\left(\frac{1}{2} \ln^2 x + 2\zeta(2) \right) + \ln(1-x) \ln x + \text{Li}_2(x), \\
& \oint_C dz x^z \Gamma^2(-z) \Gamma^2(1+z) \psi(1+z) = \\
& \frac{1}{(1-x)} [-\psi(1) \ln x + \ln(1-x) \ln x - \zeta(2) + \text{Li}_2(x)].
\end{aligned}$$

It is possible to work with the MB transforms by making a use of such tricks like derivation with respect to parameter and integration by parts in the complex plane. Here we demonstrate several examples. First of all, after the derivation with respect to x of

$$\frac{1}{1+x} = \oint_C dz x^z \Gamma(-z) \Gamma(1+z)$$

we obtain

$$\frac{x}{(1+x)^2} = \oint_C dz x^z \Gamma(1-z) \Gamma(1+z).$$

The result can be checked directly by the counting of the residues. Now, let us check that the integral of the total derivative is equal to zero. Indeed, we have

$$0 = \oint_C dz \frac{d}{dz} \{ x^z \Gamma(-z) \Gamma(1+z) \} = \ln x \oint_C dz x^z \Gamma(-z) \Gamma(1+z) +$$

$$\oint_C dz x^z \Gamma(-z) \Gamma(1+z) (-\psi(-z) + \psi(1+z)) = \frac{\ln x}{1+x} - \oint_C dz x^z \frac{\Gamma(-z) \Gamma(1+z)}{z} + \oint_C dz x^z \Gamma(-z) \Gamma(1+z) (\psi(1+z) - \psi(1-z)). \quad (15)$$

We calculate all the integrals on the r.h.s. explicitly and show that their sum is zero. The contribution of residues leads to

$$\begin{aligned} \oint_C dz x^z \Gamma(-z) \Gamma(1+z) \psi(1+z) &= \psi(1) - x\psi(2) + x^2\psi(3) + \dots, \\ \oint_C dz x^z \Gamma(-z) \Gamma(1+z) \psi(1-z) &= \psi(1) - x\psi(1) + x^2\psi(2) - x^3\psi(3) + \dots - \ln x + \frac{\ln x}{1+x}, \\ \oint_C dz x^z \frac{\Gamma(-z) \Gamma(1+z)}{z} &= \ln x - \ln(1+x). \end{aligned}$$

Substituting these results in (15) we reproduce zero in the l.h.s. of it.

6 Explicit results of the two-fold MB transform

In order to use formulas obtained for the two-fold integration explicitly, we need to calculate the r.h.s. of the equations in the limit of vanishing ε_i . First, we consider the result for $\int_2(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. There are two ways to derive the result in this limit for the r.h.s.

6.1 Belokurov-Usyukina expansion for the higher rung ladder

According to Eqs. (9) and (5), we have to find the limit

$$\begin{aligned} \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \\ \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} J \left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right] &= \\ \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left[\frac{1}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(-u) \Gamma(-v) \Gamma(\varepsilon_1 + \varepsilon_2 - u) \Gamma(\varepsilon_1 + \varepsilon_2 - v) \Gamma^2(1 - \varepsilon_1 - \varepsilon_2 + u + v)}{\Gamma(1 - \varepsilon_3) \Gamma(1 + \varepsilon_3)} \right. &= \\ - \frac{1}{\varepsilon_2(\varepsilon_1 + \varepsilon_2)} \frac{\Gamma(-u) \Gamma(\varepsilon_2 - v) \Gamma(\varepsilon_1 - u) \Gamma(\varepsilon_1 + \varepsilon_2 - v) \Gamma^2(1 - \varepsilon_1 - \varepsilon_2 + u + v)}{\Gamma(1 - \varepsilon_1) \Gamma(1 + \varepsilon_1)} &= \\ - \frac{1}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{\Gamma(\varepsilon_1 - u) \Gamma(-v) \Gamma(\varepsilon_1 + \varepsilon_2 - u) \Gamma(\varepsilon_2 - v) \Gamma^2(1 - \varepsilon_1 - \varepsilon_2 + u + v)}{\Gamma(1 - \varepsilon_2) \Gamma(1 + \varepsilon_2)} \Big] &= \\ \Gamma^2(1 + u + v) \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left[\frac{1}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(-u) \Gamma(-v) \Gamma(\varepsilon_1 + \varepsilon_2 - u) \Gamma(\varepsilon_1 + \varepsilon_2 - v)}{\Gamma(1 - \varepsilon_3) \Gamma(1 + \varepsilon_3)} \right. &= \\ - \frac{1}{\varepsilon_2(\varepsilon_1 + \varepsilon_2)} \frac{\Gamma(-u) \Gamma(\varepsilon_2 - v) \Gamma(\varepsilon_1 - u) \Gamma(\varepsilon_1 + \varepsilon_2 - v)}{\Gamma(1 - \varepsilon_1) \Gamma(1 + \varepsilon_1)} &= \\ - \frac{1}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{\Gamma(\varepsilon_1 - u) \Gamma(-v) \Gamma(\varepsilon_1 + \varepsilon_2 - u) \Gamma(\varepsilon_2 - v)}{\Gamma(1 - \varepsilon_2) \Gamma(1 + \varepsilon_2)} \Big]. & \end{aligned}$$

After a simple calculus, the limit is

$$\lim_{\varepsilon_1 \rightarrow 0, \varepsilon_2 \rightarrow 0} M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \Gamma^2(1+u+v)\Gamma^2(-u)\Gamma^2(-v) \times \left[\frac{1}{2} (\psi'(-v) + \psi'(-u)) - \frac{1}{2} (\psi(-v) - \psi(-u))^2 - \frac{3}{2} (\Gamma(1-\varepsilon)\Gamma(1+\varepsilon))_{\varepsilon=0}^{(2)} \right]. \quad (16)$$

The last term is a value of Riemann zeta function multiplied by the first UD function. This result can be represented in other forms. For example, by integrating by parts in the complex u and v planes we reproduce result (18) of the next subsection. However, this form is important since it stands in the right hand side of Eq. (12) in the limit of vanishing ε_i .

6.2 Integration by parts in the complex plane

Also, it can be helpful to work with the two-fold MB transforms by using integration by parts technique in the complex plane which is described in the previous section. Indeed, as we have derived in Eq. (9)

$$\begin{aligned} \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left[\frac{1}{\varepsilon_1 \varepsilon_2} \oint_C du dv x^u y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} \oint_C du dv x^u y^v D^{(u,v)}[1-\varepsilon_1] + \right. \\ \left. \frac{1}{\varepsilon_1 \varepsilon_3} x^{\varepsilon_1} \oint_C du dv x^u y^v D^{(u,v)}[1-\varepsilon_2] \right] = \\ \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v J \left[\frac{1}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} D^{(u,v)}[1-\varepsilon_1] + \frac{1}{\varepsilon_1 \varepsilon_2} D^{(u,v)}[1+\varepsilon_3] \right. \\ \left. + \frac{1}{\varepsilon_1 \varepsilon_3} x^{\varepsilon_1} D^{(u,v)}[1-\varepsilon_2] \right]. \quad (17) \end{aligned}$$

From definition (5), we write the integrand explicitly

$$\begin{aligned} \frac{1}{\varepsilon_1 \varepsilon_2} D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} D^{(u,v)}[1-\varepsilon_1] + \frac{1}{\varepsilon_1 \varepsilon_3} x^{\varepsilon_1} D^{(u,v)}[1-\varepsilon_2] = \\ \Gamma(-u)\Gamma(-v) \left[\frac{1}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(\varepsilon_1 + \varepsilon_2 - u)\Gamma(\varepsilon_1 + \varepsilon_2 - v)\Gamma^2(1 - \varepsilon_1 - \varepsilon_2 + u + v)}{\Gamma(1 + \varepsilon_1 + \varepsilon_2)\Gamma(1 - \varepsilon_1 - \varepsilon_2)} + \right. \\ \left. - \frac{1}{\varepsilon_2(\varepsilon_1 + \varepsilon_2)} y^{\varepsilon_2} \frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right. \\ \left. - \frac{1}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} x^{\varepsilon_1} \frac{\Gamma(\varepsilon_2 - u)\Gamma(\varepsilon_2 - v)\Gamma^2(1 - \varepsilon_2 + u + v)}{\Gamma(1 + \varepsilon_2)\Gamma(1 - \varepsilon_2)} \right]. \end{aligned}$$

The brackets in this formula can be re-written in the following form

$$\begin{aligned} \frac{1}{\varepsilon_1 \varepsilon_2} \left[\frac{\Gamma(\varepsilon_1 + \varepsilon_2 - u)\Gamma(\varepsilon_1 + \varepsilon_2 - v)\Gamma^2(1 - \varepsilon_1 - \varepsilon_2 + u + v)}{\Gamma(1 + \varepsilon_1 + \varepsilon_2)\Gamma(1 - \varepsilon_1 - \varepsilon_2)} \right. \\ \left. - y^{\varepsilon_2} \frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\varepsilon_1^2} \left[\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} - x^{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) \right] + o(\varepsilon_2) = \\
& \quad \frac{1}{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) [\psi(-u) + \psi(-v) - 2\psi(1 + u + v) - \ln y] \\
& \quad + \left(\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right)_{\varepsilon_1=0}^{(2)} \\
& \quad - \ln y \left(\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v) \right)'_{\varepsilon_1=0} \\
& + \frac{1}{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) [\psi(-u) + \psi(-v) - 2\psi(1 + u + v) - \ln x] \\
& \quad + \frac{1}{2} \left(\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right)_{\varepsilon_1=0}^{(2)} \\
& \quad - \frac{1}{2} \ln^2 x \Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) + o(\varepsilon_2) + o(\varepsilon_1).
\end{aligned}$$

First of all, we demonstrate that all the poles in ε_i disappear. The corresponding contribution taking into account the factor $\Gamma(-u)\Gamma(-v)$ has the form

$$\begin{aligned}
& \frac{1}{\varepsilon_1} \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1 + u + v) [(\psi(-u) + \psi(-v) - 2\psi(1 + u + v) - \ln y) \\
& \quad + (\psi(-u) + \psi(-v) - 2\psi(1 + u + v) - \ln x)] = \\
& \quad \frac{1}{\varepsilon_1} \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1 + u + v) [(2\psi(-u) - 2\psi(1 + u + v) - \ln x) \\
& \quad + (2\psi(-v) - 2\psi(1 + u + v) - \ln y)] = \\
& \quad - \frac{1}{\varepsilon_1} \oint_C du dv (\partial_u + \partial_v) \{x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1 + u + v)\} = 0.
\end{aligned}$$

Thus, we have derived

$$\begin{aligned}
& \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
& \oint_C du dv x^u y^v \Gamma(-u)\Gamma(-v) \left[\frac{3}{2} \left(\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right)_{\varepsilon_1=0}^{(2)} \right. \\
& \quad - \ln y \left(\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v) \right)'_{\varepsilon_1=0} \\
& \quad \left. - \frac{1}{2} \ln^2 x \Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) \right].
\end{aligned}$$

As to the last two terms, we can consider the integral transformations

$$\begin{aligned}
& \oint_C du dv x^u y^v \Gamma(-u)\Gamma(-v) \left[-\ln y \left(\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v) \right)'_{\varepsilon_1=0} \right. \\
& \quad \left. - \frac{1}{2} \ln^2 x \Gamma(-u)\Gamma(-v)\Gamma^2(1 + u + v) \right] = \\
& - \ln y \oint_C du dv x^u y^v (\psi(-u) + \psi(-v) - 2\psi(1 + u + v)) \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1 + u + v)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\ln^2 x \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& \frac{1}{2}\ln y \oint_C du dv x^u y^v (\partial_u + \partial_v) \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) \\
& -\frac{1}{2}\ln^2 x \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& -\frac{1}{2}\ln y(\ln x + \ln y) \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) \\
& -\frac{1}{2}\ln^2 x \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v).
\end{aligned}$$

Thus, the result for integral (17) is

$$\begin{aligned}
& \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\
& \frac{3}{2} \oint_C du dv x^u y^v \Gamma(-u)\Gamma(-v) \left(\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v)}{\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1)} \right)_{\varepsilon_1=0}^{(2)} \\
& -\frac{1}{2}(\ln^2 x + \ln x \ln y + \ln^2 y) \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& \frac{3}{2} \oint_C du dv x^u y^v \Gamma(-u)\Gamma(-v) (\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v))_{\varepsilon_1=0} \\
& -\frac{1}{2}(\ln^2 x + \ln x \ln y + \ln^2 y) \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) \\
& -\frac{3}{2}(\Gamma(1 + \varepsilon_1)\Gamma(1 - \varepsilon_1))_{\varepsilon_1=0}^{(2)} \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v).
\end{aligned}$$

This formula can be developed further and be transformed to the form without derivatives of the Euler ψ function in the integrand,

$$\begin{aligned}
& \oint_C du dv x^u y^v \Gamma(-u)\Gamma(-v) (\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1 - \varepsilon_1 + u + v))_{\varepsilon_1}^{(2)} = \\
& \oint_C du dv x^u y^v [-\partial_u \psi(-u) - \partial_v \psi(-v) + \partial_u \psi(1+u+v) \\
& + \partial_v \psi(1+u+v) + (\psi(-u) + \psi(-v) - 2\psi(1+u+v))^2] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& \oint_C du dv x^u y^v [(\ln x - 2\psi(-u) + 2\psi(1+u+v))(\psi(-u) - \psi(1+u+v)) + \\
& + (\ln y - 2\psi(-v) + 2\psi(1+u+v))(\psi(-v) - \psi(1+u+v)) \\
& + (\psi(-u) + \psi(-v) - 2\psi(1+u+v))^2] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& \oint_C du dv x^u y^v [\ln x (\psi(-u) - \psi(1+u+v)) + \ln y (\psi(-v) - \psi(1+u+v)) \\
& - (\psi(-u) - \psi(-v))^2] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) = \\
& \frac{1}{2}(\ln^2 x + \ln^2 y) \oint_C du dv x^u y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)
\end{aligned}$$

$$- \oint_C du dv x^u y^v (\psi(-u) - \psi(-v))^2 \Gamma^2(-u) \Gamma^2(-v) \Gamma^2(1+u+v).$$

The first term is proportional to the first UD function, and the result for integral (17) is

$$\begin{aligned} & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \left(\frac{1}{4} \ln^2 \frac{x}{y} - \frac{3}{2} (\Gamma(1-\varepsilon) \Gamma(1+\varepsilon))_{\varepsilon=0}^{(2)} \right) \oint_C du dv x^u y^v \Gamma^2(-u) \Gamma^2(-v) \Gamma^2(1+u+v) - \\ & - \frac{3}{2} \oint_C du dv x^u y^v (\psi(-u) - \psi(-v))^2 \Gamma^2(-u) \Gamma^2(-v) \Gamma^2(1+u+v). \end{aligned} \quad (18)$$

This result is related to formula (16) via integration by parts in the complex planes of u and v .

6.3 Explicit two-fold MB transform for the three-rung ladder

We need to derive the same limit for $M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We show that the result is reduced to the Euler ψ function and its derivatives. According to Eq. (12),

$$\begin{aligned} & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left[\frac{1}{\varepsilon_1 \varepsilon_2} M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{J}{\varepsilon_2 \varepsilon_3} M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) + \frac{J}{\varepsilon_1 \varepsilon_3} M_2^{(u, v+\varepsilon_2)}(\varepsilon_2) \right] = \\ & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left[\frac{1}{\varepsilon_1 \varepsilon_2} M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) - \frac{J}{\varepsilon_2(\varepsilon_1 + \varepsilon_2)} M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) \right. \\ & \quad \left. - \frac{J}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} M_2^{(u, v+\varepsilon_2)}(\varepsilon_2) \right] = \\ & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left(\frac{1}{\varepsilon_1 \varepsilon_2} \left[M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) - J M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) \right] + \frac{1}{\varepsilon_1^2} \left[M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) - M_2^{(u, v)} \right] \right). \end{aligned} \quad (19)$$

We use the notation

$$(\Gamma(-\varepsilon_i - u) \Gamma(\varepsilon_i - v))_0^{(2)}, \quad (\Gamma(-\varepsilon_i - u) \Gamma(\varepsilon_i - v))_0^{(3)}, \quad (\Gamma(-\varepsilon_i - u) \Gamma(\varepsilon_i - v))_0^{(4)}$$

for the corresponding derivatives with respect to ε_i at the point $\varepsilon_i = 0$. We obtain for the second term of (19) the expansion in terms of ε_1

$$\begin{aligned} & M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) - M_2^{(u, v)} = \\ & \Gamma^2(1+u+v) \left[\frac{\varepsilon_1}{6} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(3)} \Gamma(-u) \Gamma(-v) \right. \\ & \quad \left. - \frac{\varepsilon_1}{2} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))'_0 (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(2)} \right. \\ & \quad \left. + \varepsilon_1 a_0^{(2)} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))'_0 \Gamma(-u) \Gamma(-v) + \frac{\varepsilon_1^2}{24} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(4)} \Gamma(-u) \Gamma(-v) \right. \\ & \quad \left. - \frac{\varepsilon_1^2}{6} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(3)} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))'_0 \right] \end{aligned}$$

$$+a_0^{(2)}\varepsilon_1^2\left(\frac{3}{4}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(2)}\Gamma(-u)\Gamma(-v)\right. \\ \left.-\frac{1}{2}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0\right)+a_0^{(4)}\varepsilon_1^2\frac{5}{24}\Gamma^2(-u)\Gamma^2(-v)\Big]+o(\varepsilon_1^2).$$

We use the notation

$$a(\varepsilon)=[\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)]^{-1}, \quad a_0^{(n)}=(a(\varepsilon))_{\varepsilon=0}^{(n)}.$$

Going back to the first term of Eq. (19) we derive, by using Eq. (9), that

$$M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ J \left[\frac{1}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(-\varepsilon_1-u)\Gamma(-\varepsilon_2-v)\Gamma(\varepsilon_2-u)\Gamma(\varepsilon_1-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_3)\Gamma(1+\varepsilon_3)} \right. \\ - \frac{1}{\varepsilon_2(\varepsilon_1+\varepsilon_2)} \frac{\Gamma(-\varepsilon_1-u)\Gamma(-v)\Gamma(-u)\Gamma(\varepsilon_1-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_1)} \\ \left. - \frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)} \frac{\Gamma(-u)\Gamma(-\varepsilon_2-v)\Gamma(\varepsilon_2-u)\Gamma(-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_2)} \right]. \quad (20)$$

A simple calculus gives that

$$M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) - JM_2^{(u+\varepsilon_1, v)}(\varepsilon_1) = \\ \varepsilon_2 \Gamma^2(1+u+v) \left[-\frac{1}{6}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(3)}\Gamma(-u)\Gamma(-v) \right. \\ + \frac{1}{2}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v))_0^{(2)} \\ - a_0^{(2)}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0\Gamma(-u)\Gamma(-v) \\ - \frac{\varepsilon_1}{24}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(4)}\Gamma(-u)\Gamma(-v) \\ + \frac{\varepsilon_1}{4}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(2)}(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v))_0^{(2)} \\ + a_0^{(2)}\varepsilon_1\left(\frac{1}{4}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(2)}\Gamma(-u)\Gamma(-v) \right. \\ \left. - (\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))'_0\right) + a_0^{(4)}\varepsilon_1\frac{5}{24}\Gamma^2(-u)\Gamma^2(-v) + o(\varepsilon_1) \Big] + o(\varepsilon_2).$$

Thus, we obtain the limit for $M_3^{(u, v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ when the ε -terms are vanishing

$$\lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} M_3^{(u, v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \left(\frac{1}{\varepsilon_1 \varepsilon_2} \left[M_2^{(u+\varepsilon_1, v+\varepsilon_2)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) - JM_2^{(u+\varepsilon_1, v)}(\varepsilon_1) \right] \right. \\ \left. + \frac{1}{\varepsilon_1^2} \left[M_2^{(u+\varepsilon_1, v)}(\varepsilon_1) - M_2^{(u, v)} \right] \right) = \\ \Gamma^2(1+u+v) \left[\frac{1}{4}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(2)}(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v))_0^{(2)} \right.$$

$$\begin{aligned}
& -\frac{1}{6} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(3)} (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))'_0 \\
& + a_0^{(2)} \left(-\frac{3}{2} ((\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))'_0)^2 - \Gamma(-u) \Gamma(-v) (\Gamma(-\varepsilon_1 - u) \Gamma(\varepsilon_1 - v))_0^{(2)} \right) \\
& + a_0^{(4)} \frac{5}{12} \Gamma^2(-u) \Gamma^2(-v) \Big]. \quad (21)
\end{aligned}$$

7 Conclusions

In this paper we have explicitly calculated a new type of the two-fold MB integrals. The integrals of that type have been obtained due to recursive relations of the momentum loop integrals which are derived from the Belokurov-Usyukina loop reduction method. The recursion creates an infinite system of the integral relations, which allowed us to represent the MB transform of momentum $(L+1)$ -loop integral as a linear combination of the MB transforms of momentum L -loop integrals. All the integrals have been reduced to the Appell's hypergeometric function corresponding to the one-loop momentum massless integral. In order to derive those relations, we needed to shift the complex variables of the MB transforms on the right hand sides of the integral relations. This shift is necessary in order to organize the same dependence on the parameters x and y in the integrands on the on the left and on the right hand sides. As the result, in the integral relations representation (16) appears while representation (18) does not appear.

At the first site, the results (18) and (21) are difficult to analyse, since on the right hand sides of them certain products of higher order derivatives of the Euler gamma functions

$$\frac{d^n}{d\varepsilon^n} \Gamma(-\varepsilon - u) \Gamma(\varepsilon - v)|_{\varepsilon=0} \quad (22)$$

appear. These derivatives can be re-written as

$$\frac{d^n}{d\varepsilon^n} \Gamma(-\varepsilon - u) \Gamma(\varepsilon - v)|_{\varepsilon=0} = (\partial_u - \partial_v)^n \Gamma(-u) \Gamma(-v), \quad (23)$$

that makes them more convenient for use in the integration by parts.

However, the integration by parts in the planes of complex variables u and v can serve to us just to prove the coincidence of two, at first view, different representations. The reason why the representation for MB transforms of UD functions should be a simple expansion in terms of derivatives (22) or (23) is in a recursive construction of the MB images. The recursive construction has nothing to do with quantum field theory or with theory of polylogarithms. We will prove this observation further. For this purpose representation (18) is more useful than representation (16) used in the integral relations³.

As the first step, we consider the integral relation⁴ that can be found in Ref. [1]

$$J(1, 1, 1 - \varepsilon) = \frac{1}{2} (p_3^2)^\varepsilon [x^\varepsilon J(1 - \varepsilon, 1 + \varepsilon, 1) + y^\varepsilon J(1 + \varepsilon, 1 - \varepsilon, 1)],$$

or, equivalently,

$$\oint_C du dv x^u y^v D^{(u,v)}[1, 1, 1 - \varepsilon] =$$

³ Actually, beyond the four rung in the ladder the difference between two types of representation disappears

⁴ This relation can easily be derived via shifts in the planes of complex integration variables u and v in the MB transforms

$$\frac{1}{2} \oint_C du dv x^u y^v \left(x^\varepsilon D^{(u,v)}[1-\varepsilon, 1+\varepsilon, 1] + y^\varepsilon D^{(u,v)}[1+\varepsilon, 1-\varepsilon, 1] \right). \quad (24)$$

The benefits of representation (24) are in more simple dependence on the variable ε , namely, it presents in the numerator in two factors only,

$$D^{(u,v)}[1-\varepsilon, 1+\varepsilon, 1] = \frac{\Gamma(-u)\Gamma(-v)\Gamma(-\varepsilon-u)\Gamma(\varepsilon-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}.$$

For example, the equation for M_2 can be re-written as

$$\begin{aligned} & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v \left[\frac{1}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} D^{(u,v)}[1-\varepsilon_1] + \frac{1}{\varepsilon_1 \varepsilon_2} D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1 \varepsilon_3} x^{\varepsilon_1} D^{(u,v)}[1-\varepsilon_2] \right] = \\ & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v \left[\frac{a(\varepsilon_1)}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} [x^{\varepsilon_1} \Gamma(-u-\varepsilon_1) \Gamma(-v+\varepsilon_1) + y^{\varepsilon_1} \Gamma(-u+\varepsilon_1) \Gamma(-v-\varepsilon_1)] \right. \\ & \quad \left. + \frac{a(\varepsilon_3)}{\varepsilon_1 \varepsilon_2} [x^{-\varepsilon_3} \Gamma(-u+\varepsilon_3) \Gamma(-v-\varepsilon_3) + y^{-\varepsilon_3} \Gamma(-u-\varepsilon_3) \Gamma(-v+\varepsilon_3)] \right. \\ & \quad \left. + \frac{a(\varepsilon_2)}{\varepsilon_1 \varepsilon_3} x^{\varepsilon_1} [x^{\varepsilon_2} \Gamma(-u-\varepsilon_2) \Gamma(-v+\varepsilon_2) + y^{\varepsilon_2} \Gamma(-u+\varepsilon_2) \Gamma(-v-\varepsilon_2)] \right] \times \\ & \quad \times \Gamma(-u) \Gamma(-v) \Gamma^2(1+u+v). \end{aligned}$$

In the limit of vanishing ε_i the value of this expression contains derivatives (22) only, that is,

$$\begin{aligned} & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \oint_C du dv x^u y^v \Gamma(-u) \Gamma(-v) \Gamma^2(1+u+v) \left[\frac{3}{2} (a(\varepsilon) \Gamma(-u-\varepsilon) \Gamma(-v+\varepsilon))_0^{(2)} + \right. \\ & \quad \left. \frac{3}{2} \ln \frac{x}{y} (a(\varepsilon) \Gamma(-u-\varepsilon) \Gamma(-v+\varepsilon))'_0 + \frac{1}{4} \ln^2 \frac{x}{y} \Gamma(-u) \Gamma(-v) \right]. \end{aligned}$$

This expression does not have a form of (16) or (18), however it can be transformed to that form by using the integration by parts in the complex planes of u and v .

In integral relation (12) for M_3 , which is

$$\begin{aligned} & M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \frac{1}{\varepsilon_1 \varepsilon_2} x^{-\varepsilon_1} y^{-\varepsilon_2} M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{J}{\varepsilon_2 \varepsilon_3} x^{-\varepsilon_1} M_2^{(u,v)}(\varepsilon_1) + \frac{J}{\varepsilon_1 \varepsilon_3} y^{-\varepsilon_2} M_2^{(u,v)}(\varepsilon_2), \end{aligned}$$

we substitute the structure found in (24), and again in the limit of vanishing ε_i we obtain an expansion in terms of derivatives (22), that is,

$$\begin{aligned} & \lim_{\varepsilon_2 \rightarrow 0, \varepsilon_1 \rightarrow 0} \oint_C du dv x^u y^v M_3^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \\ & \oint_C du dv x^u y^v \Gamma(-u) \Gamma(-v) \Gamma^2(1+u+v) \left[\frac{5}{12} (a(\varepsilon) \Gamma(-u-\varepsilon) \Gamma(-v+\varepsilon))_0^{(4)} + \right. \end{aligned}$$

$$\begin{aligned} & \frac{5}{6} \ln \frac{x}{y} (a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon))_0^{(3)} + \frac{1}{2} \ln^2 \frac{x}{y} (a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon))_0^{(2)} \\ & + \frac{1}{12} \ln^3 \frac{x}{y} (a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon))_0' \Big]. \end{aligned} \quad (25)$$

By using the integration by part in the complex plane of u and v , we obtain coincidence with Eq. (21).

Following the recursive procedure for higher values of n , we will obtain the higher derivatives of type (22) multiplied by powers of $\ln \frac{x}{y}$. At present, it is difficult to calculate the coefficients in front of the expansion terms. However, it is clear that the finiteness of the limit of vanishing ε_i has nothing to do with the polylogarithms, since instead of the Euler Γ function we can write any other smooth function in construction (24) and the limit remains finite. Thus, the infinite sum in the limit of vanishing ε_i of the quantities $M_n^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ constructed from (24) has underlying integrable structure which can be uncovered by identifying coefficients in front of the terms in the expansion (25) for higher n . That integrable structure has nothing common with the MB transforms of polylogarithms and is based on the properties that can be studied by the basic methods of mathematical analysis.

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