

# The $O(\alpha_s^3 n_f T_F^2 C_{A,F})$ Contributions to the Gluonic Massive Operator Matrix Elements

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## Abstract

The  $O(\alpha_s^3 n_f T_F^2 C_{A,F})$  terms to the massive gluonic operator matrix elements are calculated for general values of the Mellin variable  $N$  using a new summation technique. These twist-2 matrix elements occur as transition functions in the variable flavor number scheme at NNLO. The calculation uses sum-representations in generalized hypergeometric series turning into harmonic sums. The analytic continuation to complex values of  $N$  is provided.

# 1 Introduction

Heavy quark contributions to the deep inelastic scattering structure functions play a crucial role in the QCD analyses to determine the parton distribution functions and the strong coupling constant  $\alpha_s(M_Z^2)$  in a consistent manner, cf. [1]. The heavy flavor corrections were calculated at NLO in semianalytic form in [2]<sup>1</sup>. To avoid contributions of higher twist the analysis has to be restricted to large enough values of  $Q^2$ . It has been shown in [4] that for  $Q^2 \gtrsim 10 m^2$ , with  $m$  the heavy quark mass, the heavy flavor contributions to the structure function  $F_2(x, Q^2)$  are rather accurately described using the asymptotic representation in which all power corrections  $\propto (m^2/Q^2)^k, k \in \mathbb{N}_+$  are neglected. In this case the heavy flavor Wilson coefficients can be calculated analytically. They are given by convolutions of massive operator matrix elements (OMEs) and the massless Wilson coefficients, cf. Ref. [4, 5]. The massless Wilson coefficients are known to 3-loop order [6]. At NLO the massive OMEs were calculated in [4, 7–12] in the unpolarized and polarized case, including the  $O(\alpha_s^2 \varepsilon)$  contributions, and in [13] for transversity. The heavy flavor corrections for charged current reactions are available at one loop and in the asymptotic case at two-loops [14, 15].

At 3-loop order a series of moments has been calculated for all massive OMEs for  $N = 2 \dots 10(14)$  contributing in the fixed and variable flavor scheme, [5]. The 3-loop heavy flavor corrections to  $F_L(x, Q^2)$  in the asymptotic case were calculated in [16]. First results for general values of  $N$  have been obtained for the OMEs with operator insertions on the quark lines in case for the color factors  $n_f T_F^2 C_{A,F}$  [17] and 3-loop ladder topologies [18]. First  $T_F^2 C_{A,F}$ -contributions at general  $N$  were calculated in [19] for two heavy quark lines carrying the same mass. Furthermore, the moments  $N = 2, 4, 6$  in case of the OMEs contributing to the structure function  $F_2(x, Q^2)$  with two different heavy quark masses were computed in [19, 20]. In all the above cases the massive OMEs are calculated for external massless partons which are on-shell. The case of massive on-shell external lines has been treated in [21] recently.

In the present paper the 3-loop corrections of  $O(n_f T_F^2 C_{A,F})$  to the massive OMEs with local operator insertions on the gluonic lines,  $A_{gq,Q}$  and  $A_{gg,Q}$ , at general values of  $N$  are calculated, which constitute the first 3-loop gluonic contributions being obtained. Together with the corresponding terms with the insertions on the quark lines, [17], these contributions complete all terms corresponding to the case of one massless and one massive fermion line at 3-loop order. In the present case we used a new fast technology being implemented into the package **Sigma** [22] to evaluate  $\sim 2400$  sums in a combined manner, reducing them to few ( $O(30)$ ), but very large terms instead. This approach, realized in a computer algebra code, is of general interest for many other 2- and 3-loop calculations in particle physics and constitutes the method of choice in using larger computer storage. The computational time could be reduced significantly compared to the earlier calculation [17]. These matrix elements contribute to the transition functions needed to describe the parton densities in the variable flavor number scheme (VFNS). In this scheme it is possible to define heavy quark distribution functions assuming that there exists only one heavy quark and all other quarks can be dealt with as massless in the sense of an effective field theory approach. These distributions can be used for effective calculations in some processes at hadron colliders. The picture holds to 2-loop orders. Starting with the 3-loop corrections, [19, 20], dia-

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<sup>1</sup>A fast and precise numerical implementation in Mellin space has been given in [3].

grams containing quarks of two different masses contribute even to the universal corrections. Since  $m_c^2/m_b^2 \approx 1/10$  is not a small number, the original VFNS-picture does not necessarily hold in practice. Here we deal with the  $O(n_f T_F^2 C_{A,F})$  contributions which are in accordance with the VFNS. In Section 2 the main formalism is lined out. The calculation is performed in  $D = 4 + \varepsilon$  dimensions and uses representations in terms of generalized hypergeometric functions. They lead to multiple sum representations, which are solved using modern summation technologies encoded in the package `Sigma` [22]. The results of the calculation are given in Section 3 both in Mellin- $N$  and in  $x$ -space, and Section 4 contains the conclusions.

## 2 Parton distribution functions in the VFNS

The neutral current Born cross section of unpolarized deep inelastic scattering (DIS) is given by [23]

$$\frac{d^2\sigma_B^{\text{NC}}}{dxdy} = \frac{2\pi\alpha^2}{xyQ^2} \left\{ \left[ 2(1-y) - 2xy\frac{M^2}{S} + \left( 1 + 4x^2\frac{M^2}{Q^2} \right) \frac{y^2}{1+R(x, Q^2)} \right] \mathcal{F}_2(x, Q^2) + xy(1-y)\mathcal{F}_3(x, Q^2) \right\}, \quad (1)$$

neglecting lepton mass contributions. Here  $x$  and  $y$  denote the Bjorken variables and  $-q^2 = Q^2 = xyS$ , with  $q^2$  the 4-momentum transfer. The structure functions  $\mathcal{F}_i(x, Q^2)$  contain electro-weak effects due to  $Z$ -boson exchange and differ for lepton and anti-lepton-nucleon scattering, cf. [23], and

$$R(x, Q^2) = \left( 1 + 4x^2\frac{M^2}{Q^2} \right) \frac{\mathcal{F}_2(x, Q^2)}{2x\mathcal{F}_1(x, Q^2)} - 1. \quad (2)$$

In the limit  $M_Z^2 \gg Q^2$  the electromagnetic terms in (1) dominate and only the two structure functions  $F_{1,2}(x, Q^2)$  contribute, with

$$2xF_1(x, Q^2) = F_2(x, Q^2) - F_L(x, Q^2), \quad (3)$$

where  $F_L$  is the longitudinal structure function. Both structure functions contain light and heavy quark contributions. The  $y$ -dependence of the differential scattering cross section is used to separate the structure functions [24] and allows precise measurements of the structure function  $F_2(x, Q^2)$ . In the twist-2 approximation, referring to the fixed flavor number scheme, they are given by

$$F_2(x, Q^2, n_f) = F_2^{m=0}(x, Q^2, n_f) + F_{2,Q}^{\text{massive}}(x, Q^2, n_f, m). \quad (4)$$

Here  $F_2^{m=0}(x, Q^2)$  denotes the well-known massless contribution and the massive contribution in the presence of a single massive quark reads [5]

$$\begin{aligned} F_{2,Q}^{\text{massive}}(x, Q^2, n_f, m) &= \sum_{k=1}^{n_f} e_k^2 \left\{ L_{2,q}^{\text{NS}} \left( n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \left[ f_k(x, \mu^2, n_f) + f_{\bar{k}}(x, \mu^2, n_f) \right] \right. \\ &\quad \left. + \tilde{L}_{2,q}^{\text{PS}} \left( n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, n_f) + \tilde{L}_{2,g}^{\text{S}} \left( n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, n_f) \right\} \\ &\quad + e_Q^2 \left[ H_{2,q}^{\text{PS}} \left( n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, n_f) \right. \\ &\quad \left. + H_{2,g}^{\text{S}} \left( n_f, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, n_f) \right]. \quad (5) \end{aligned}$$

Here  $f_k(x, \mu^2, n_f)$ ,  $\Sigma(x, \mu^2, n_f)$ ,  $G(x, \mu^2, n_f)$  denote the  $k$ th quark, singlet-quark, and gluon densities, respectively with

$$\Sigma(x, n_f, \mu^2) = \sum_{k=1}^{n_f} [f_k(x, n_f, \mu^2) + f_{\bar{k}}(x, n_f, \mu^2)] . \quad (6)$$

The Wilson coefficients  $\tilde{L}_{2,q}^{\text{PS}}(n_f, Q^2/m^2, m^2/\mu^2)$  and  $\tilde{L}_{2,g}^{\text{S}}(n_f, Q^2/m^2, m^2/\mu^2)$  have been calculated completely for general values of  $N$  in [17].

The renormalization group implies the following representation for the set of  $(n_f + 1)$  (massless) parton densities expressed in terms of  $n_f$  parton densities [8] :

$$\begin{aligned} f_k(n_f + 1, \mu^2, m^2, N) + f_{\bar{k}}(n_f + 1, \mu^2, m^2, N) &= A_{qq,Q}^{\text{NS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot [f_k(n_f, \mu^2, N) \\ &\quad + f_{\bar{k}}(n_f, \mu^2, N)] \\ &\quad + \tilde{A}_{qq,Q}^{\text{PS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot \Sigma(n_f, \mu^2, N) \\ &\quad + \tilde{A}_{gg,Q} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot G(n_f, \mu^2, N), \end{aligned} \quad (7)$$

$$\begin{aligned} f_Q(n_f + 1, \mu^2, m^2, N) + f_{\bar{Q}}(n_f + 1, \mu^2, m^2, N) &= A_{Qq}^{\text{PS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot \Sigma(n_f, \mu^2, N) \\ &\quad + A_{Qg} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot G(n_f, \mu^2, N) . \end{aligned} \quad (8)$$

Here  $f_Q(f_{\bar{Q}})$  are the heavy quark densities. The flavor singlet, non-singlet and gluon densities for  $(n_f + 1)$  flavors are given by

$$\begin{aligned} \Sigma(n_f + 1, \mu^2, m^2, N) &= \left[ A_{qq,Q}^{\text{NS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) + n_f \tilde{A}_{qq,Q}^{\text{PS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) \right. \\ &\quad \left. + A_{Qq}^{\text{PS}} \left( n_f, \frac{\mu^2}{m^2}, N \right) \right] \cdot \Sigma(n_f, \mu^2, N) \\ &\quad + \left[ n_f \tilde{A}_{gg,Q} \left( n_f, \frac{\mu^2}{m^2}, N \right) + A_{Qg} \left( n_f, \frac{\mu^2}{m^2}, N \right) \right] \cdot G(n_f, \mu^2, N) \end{aligned} \quad (9)$$

$$\begin{aligned} \Delta(n_f + 1, \mu^2, m^2, N) &= f_k(n_f + 1, \mu^2, m^2, N) + f_{\bar{k}}(n_f + 1, \mu^2, m^2, N) \\ &\quad - \frac{1}{n_f + 1} \Sigma(n_f + 1, \mu^2, m^2, N) \end{aligned} \quad (10)$$

$$\begin{aligned} G(n_f + 1, \mu^2, m^2, N) &= A_{gq,Q} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot \Sigma(n_f, \mu^2, N) \\ &\quad + A_{gg,Q} \left( n_f, \frac{\mu^2}{m^2}, N \right) \cdot G(n_f, \mu^2, N) . \end{aligned} \quad (11)$$

Any relation between the  $(n_f + 1)$ - and  $n_f$ -parton density can only contain **universal**, i.e. process-independent, quantities.

Note that the new parton densities depend on the renormalized heavy quark mass  $m^2$ . As outlined above, the corresponding relations for the operator matrix elements depend on the

mass-renormalization scheme, with  $m = m(a_s(\mu^2))$  in the  $\overline{\text{MS}}$  scheme, which we will apply below. These equations describe the transition of one heavy quark becoming light at the time referring to the scale  $\mu^2$ .

The matching scales  $\mu^2$  are often chosen as  $\mu^2 = m^2$ . The comparison of the results in complete calculations to those in which flavor thresholds are matched in the VFNS allows in principle to determine the relevant matching scale. In an analysis of the various deep-inelastic structure function sum rules [25] it has been shown that the scale  $\mu^2$  turns out to be significantly different of  $m^2$ . This is not unexpected since mass effects do not turn into the behaviour of the massless case close to the production threshold.

The resummation of large logs, as being performed in the VFNS, has to be performed at very high scales. As has been shown in [26] this is not the case in the kinematic range at HERA. A smooth transition from the threshold region to asymptotic scales has been proposed in terms of the BMSN-scheme [8],

$$F_2^{c\bar{c}}(x, Q^2, n_f = 4) = F_2^{c\bar{c}, \text{FFNS}}(x, Q^2, n_f = 3) + F_2^{c\bar{c}, \text{asympt}}(x, Q^2, n_f = 4) - F_2^{c\bar{c}, \text{asympt}}(x, Q^2, n_f = 3), \quad (12)$$

which is found to be in excellent agreement with the HERA data [27]. There is a series of other proposals to match between the threshold and asymptotic region [28–30], partly with a faster transition to the massless case. Here precise data on  $F_2^{c\bar{c}}(x, Q^2)$  are helpful to distinguish between different descriptions. We would like to mention that a correct treatment of the heavy flavor corrections is of instrumental importance in the QCD analysis of the complete structure functions  $F_2(x, Q^2)$ , which has been measured to a precision of  $O(1\%)$  [31].

### 3 The $O(\alpha_s^3 n_f T_F^2)$ contributions to $A_{gq, Q}$ and $A_{gg, Q}$

The OMEs  $A_{gq, Q}$  and  $A_{gg, Q}$  are expectation values  $\langle j | O_g | j \rangle$ ,  $i, j = q, g$  of the gluonic operator

$$O_{g, \mu_1, \dots, \mu_N} = 2i^{N-2} \text{SSp}[F_{\mu_1 \alpha} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^\alpha] - \text{trace terms}. \quad (13)$$

between massless on-shell external states. The corresponding massive OMEs  $A_{gq, Q}$ ,  $A_{gg, Q}$  were calculated to  $O(\alpha_s^2)$  in [8] and including also terms linear in  $\varepsilon$  in [9] correcting the previous result.

The renormalized expressions  $A_{gq, Q}$  and  $A_{gg, Q}$  to  $O(\alpha_s^3)$  were derived in [5]. In the  $\overline{\text{MS}}$  scheme with the heavy quark mass  $m$  on-shell they are given by :

$$\begin{aligned} A_{gq, Q}^{(3), \overline{\text{MS}}} &= -\frac{\gamma_{gq}^{(0)}}{24} \left\{ \gamma_{gq}^{(0)} \hat{\gamma}_{gq}^{(0)} + \left( \gamma_{gq}^{(0)} - \gamma_{gg}^{(0)} + 10\beta_0 + 24\beta_{0, Q} \right) \beta_{0, Q} \right\} \ln^3 \left( \frac{m^2}{\mu^2} \right) + \frac{1}{8} \left\{ 6\gamma_{gq}^{(1)} \beta_{0, Q} \right. \\ &+ \hat{\gamma}_{gq}^{(1)} \left( \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} - 4\beta_0 - 6\beta_{0, Q} \right) + \gamma_{gq}^{(0)} \left( \hat{\gamma}_{qq}^{(1), \text{NS}} + \hat{\gamma}_{qq}^{(1), \text{PS}} - \hat{\gamma}_{gg}^{(1)} + 2\beta_{1, Q} \right) \left. \right\} \ln^2 \left( \frac{m^2}{\mu^2} \right) \\ &+ \frac{1}{8} \left\{ 4\hat{\gamma}_{gq}^{(2)} + 4a_{gq, Q}^{(2)} \left( \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} - 4\beta_0 - 6\beta_{0, Q} \right) + 4\gamma_{gq}^{(0)} \left( a_{qq, Q}^{(2), \text{NS}} + a_{Qq}^{(2), \text{PS}} - a_{gg, Q}^{(2)} \right. \right. \\ &+ \beta_{1, Q}^{(1)} \left. \right\} + \gamma_{gq}^{(0)} \zeta_2 \left( \gamma_{gq}^{(0)} \hat{\gamma}_{gq}^{(0)} + \left[ \gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} + 12\beta_{0, Q} + 10\beta_0 \right] \beta_{0, Q} \right) \left. \right\} \ln \left( \frac{m^2}{\mu^2} \right) \\ &+ \bar{a}_{gq, Q}^{(2)} \left( \gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} + 4\beta_0 + 6\beta_{0, Q} \right) + \gamma_{gq}^{(0)} \left( \bar{a}_{gg, Q}^{(2)} - \bar{a}_{Qq}^{(2), \text{PS}} - \bar{a}_{qq, Q}^{(2), \text{NS}} \right) - \gamma_{gq}^{(0)} \beta_{1, Q}^{(2)} \\ &- \frac{\gamma_{gq}^{(0)} \zeta_3}{24} \left( \gamma_{gq}^{(0)} \hat{\gamma}_{gq}^{(0)} + \left[ \gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} + 10\beta_0 \right] \beta_{0, Q} \right) - \frac{3\gamma_{gq}^{(1)} \beta_{0, Q} \zeta_2}{8} + 2\delta m_1^{(-1)} a_{gq, Q}^{(2)} \\ &+ \delta m_1^{(0)} \hat{\gamma}_{gq}^{(1)} + 4\delta m_1^{(1)} \beta_{0, Q} \gamma_{gq}^{(0)} + a_{gq, Q}^{(3)}, \quad (14) \end{aligned}$$

$$\begin{aligned}
A_{gg,Q}^{(3),\overline{\text{MS}}} &= \frac{1}{48} \left\{ \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \left( \gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 6\beta_0 - 4n_f \beta_{0,Q} - 10\beta_{0,Q} \right) - 4 \left( \gamma_{gg}^{(0)} \left[ 2\beta_0 + 7\beta_{0,Q} \right] \right. \right. \\
&\quad \left. \left. + 4\beta_0^2 + 14\beta_{0,Q}\beta_0 + 12\beta_{0,Q}^2 \right) \beta_{0,Q} \right\} \ln^3 \left( \frac{m^2}{\mu^2} \right) + \frac{1}{8} \left\{ \hat{\gamma}_{gg}^{(0)} \left( \gamma_{gg}^{(1)} + (1 - n_f) \hat{\gamma}_{gg}^{(1)} \right) \right. \\
&\quad \left. + \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(1)} + 4\gamma_{gg}^{(1)} \beta_{0,Q} - 4\hat{\gamma}_{gg}^{(1)} [\beta_0 + 2\beta_{0,Q}] + 4[\beta_1 + \beta_{1,Q}] \beta_{0,Q} \right. \\
&\quad \left. + 2\gamma_{gg}^{(0)} \beta_{1,Q} \right\} \ln^2 \left( \frac{m^2}{\mu^2} \right) + \frac{1}{16} \left\{ 8\hat{\gamma}_{gg}^{(2)} - 8n_f a_{gg,Q}^{(2)} \hat{\gamma}_{gg}^{(0)} - 16a_{gg,Q}^{(2)} (2\beta_0 + 3\beta_{0,Q}) \right. \\
&\quad \left. + 8\gamma_{gg}^{(0)} a_{Qg}^{(2)} + 8\gamma_{gg}^{(0)} \beta_{1,Q}^{(1)} + \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \zeta_2 \left( \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 4n_f \beta_{0,Q} + 6\beta_{0,Q} \right) \right. \\
&\quad \left. + 4\beta_{0,Q} \zeta_2 \left( \gamma_{gg}^{(0)} + 2\beta_0 \right) \left( 2\beta_0 + 3\beta_{0,Q} \right) \right\} \ln \left( \frac{m^2}{\mu^2} \right) + 2(2\beta_0 + 3\beta_{0,Q}) \bar{a}_{gg,Q}^{(2)} \\
&\quad + n_f \hat{\gamma}_{gg}^{(0)} \bar{a}_{gg,Q}^{(2)} - \gamma_{gg}^{(0)} \bar{a}_{Qg}^{(2)} - \beta_{1,Q}^{(2)} \gamma_{gg}^{(0)} + \frac{\gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \zeta_3}{48} \left( \gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 2[2n_f + 1] \beta_{0,Q} \right. \\
&\quad \left. - 6\beta_0 \right) + \frac{\beta_{0,Q} \zeta_3}{12} \left( [\beta_{0,Q} - 2\beta_0] \gamma_{gg}^{(0)} + 2[\beta_0 + 6\beta_{0,Q}] \beta_{0,Q} - 4\beta_0^2 \right) \\
&\quad - \frac{\hat{\gamma}_{gg}^{(0)} \zeta_2}{16} \left( \gamma_{gg}^{(1)} + \hat{\gamma}_{gg}^{(1)} \right) + \frac{\beta_{0,Q} \zeta_2}{8} \left( \hat{\gamma}_{gg}^{(1)} - 2\gamma_{gg}^{(1)} - 2\beta_1 - 2\beta_{1,Q} \right) + \frac{\delta m_1^{(-1)}}{4} \left( 8a_{gg,Q}^{(2)} \right. \\
&\quad \left. + 24\delta m_1^{(0)} \beta_{0,Q} + 8\delta m_1^{(1)} \beta_{0,Q} + \zeta_2 \beta_{0,Q} \beta_0 + 9\zeta_2 \beta_{0,Q}^2 \right) + \delta m_1^{(0)} \left( \beta_{0,Q} \delta m_1^{(0)} + \hat{\gamma}_{gg}^{(1)} \right) \\
&\quad + \delta m_1^{(1)} \left( \hat{\gamma}_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \gamma_{gg}^{(0)} + 4\beta_{0,Q} \beta_0 + 8\beta_{0,Q}^2 \right) - 2\delta m_2^{(0)} \beta_{0,Q} + a_{gg,Q}^{(3)}. \quad (15)
\end{aligned}$$

Here  $\delta m_i^{(k)}$  are expansion coefficients of the unrenormalized mass,  $\beta_i, \beta_{i,Q}$  are coefficients of the  $\beta$ -functions (including mass effects),  $\zeta_k$  is the Riemann- $\zeta$  function with  $k \in \mathbb{N} \setminus \{0, 1\}$ ,  $a_{ij}^{(2)}, \bar{a}_{ij}^{(2)}$  are two loop contributions to order  $\varepsilon^0$  and  $\varepsilon^1$  respectively, and  $\gamma_{ij}, \hat{\gamma}_{ij}$  are the anomalous dimensions, and quantities with a hat or a tilde are defined by

$$\hat{f} = f(n_f + 1) - f(n_f), \quad \tilde{f} = \frac{1}{n_f} f, \quad (16)$$

see Ref. [5]. The unreormalized OME  $\hat{A}_{gg,Q}^{(3)}$  also receives contributions from the vacuum polarization insertions on the external lines

$$\hat{\Pi}_{\mu\nu}^{ab}(p^2, \hat{m}^2, \mu^2, \hat{a}_s^2) = i\delta^{ab} [-g_{\mu\nu} p^2 + p_\mu p_\nu] \sum_{k=1}^{\infty} \hat{a}_s^k \hat{\Pi}^{(k)}(p^2, \hat{m}^2, \mu^2) \quad (17)$$

$$\hat{\Pi}^{(k)} \equiv \hat{\Pi}^{(k)}(0, \hat{m}^2, \mu^2) \quad (18)$$

such that

$$\hat{A}_{gg,Q}^{(3)} = \hat{A}_{gg,Q}^{(3),1\text{PI}} - \hat{\Pi}^{(3)} - \hat{A}_{gg,Q}^{(2),1\text{PI}} \hat{\Pi}^{(1)} - 2\hat{A}_{gg,Q}^{(1)} \hat{\Pi}^{(2)} + \hat{A}_{gg,Q}^{(1)} \hat{\Pi}^{(1)} \hat{\Pi}^{(1)} \quad (19)$$

$$\equiv \frac{a_{gg,Q}^{(3,0)}}{\varepsilon^3} + \frac{a_{gg,Q}^{(3,1)}}{\varepsilon^2} + \frac{a_{gg,Q}^{(3,2)}}{\varepsilon} + a_{gg,Q}^{(3)}. \quad (20)$$

All contributions to (14,15) but the constant terms  $a_{ij,Q}^{(3)}$  are known [4, 7–9, 12, 32]. In particular, all the logarithmic contributions have already been obtained for general values of the Mellin variable  $N$ , [33].

In the following we calculate the contributions  $O(a_s^3 n_f T_F^2 C_{F,A})$  to the massive gluonic OMEs. The Feynman diagrams are generated by **QGRAF** [34] and the extension allowing to include local operators [5]. The color-algebra is performed using [35]. For a large part of the calculation we use **FORM** [36]. The momentum integrals are performed introducing a Feynman parameterization. The Feynman parameter integrals are then rewritten in terms of hypergeometric functions ( ${}_2F_{1,3}F_2$ ), which are represented in terms of absolutely convergent series. The resulting sums, which may still contain finite sums due to binomial expansions, are then processed applying the symbolic summation technology, which is encoded in the package **Sigma** [22] and making use of a large number of algorithms for processing multi sums using the package **EvaluateMultiSums** [37, 38]. Additionally it is very useful to reduce such sums to a smaller number of ‘key sums’, by synchronization of the summation ranges and algebraic reduction of the summands. This step helped to reduce the size of the terms from 2GByte to 7.6MByte and the number of sums from 2419 to 29. The algorithms for this step are implemented in the package **SumProduction** [39]. Details of the corresponding technique are described in [37, 38]. The corresponding expressions have simplified using mutual relations and methods applicable to the respective classes of sums encoded in the package **HarmonicSums** [40]. The results for the individual diagrams have been checked comparing to the moments obtained in [5] using the code **MATAD** [41]. The constant contributions  $a_{gg,Q}^{(3)}$ ,  $j = q, g$  to (14,15) read :

$$\begin{aligned}
a_{gg,Q}^{(3),n_f T_F^2} &= C_F T_F^2 n_f \left\{ -\frac{16(N^2 + N + 2)}{9(N-1)N(N+1)} \left( \frac{1}{3} S_1^3 + S_2 S_1 + \frac{2}{3} S_3 + 14\zeta_3 + 3S_1 \zeta_2 \right) \right. \\
&+ \frac{16(8N^3 + 13N^2 + 27N + 16)}{27(N-1)N(N+1)^2} (3\zeta_2 + S_1^2 + S_2) \\
&- \frac{32(35N^4 + 97N^3 + 178N^2 + 180N + 70)}{27(N-1)N(N+1)^3} S_1 \\
&\left. + \frac{32(1138N^5 + 4237N^4 + 8861N^3 + 11668N^2 + 8236N + 2276)}{243(N-1)N(N+1)^4} \right\}. \tag{21}
\end{aligned}$$

$$\begin{aligned}
a_{gg,Q}^{(3),n_f T_F^2} &= n_f T_F^2 \left\{ C_A \frac{1}{(N-1)(N+2)} \left[ \frac{4P_1}{27N^2(N+1)^2} S_1^2 + \frac{8P_2}{729N^3(N+1)^3} S_1 \right. \right. \\
&+ \frac{160}{27}(N-1)(N+2)\zeta_2 S_1 - \frac{448}{27}(N-1)(N+2)\zeta_3 S_1 + \frac{P_3}{729N^4(N+1)^4} \\
&- \left. \frac{2P_4}{27N^2(N+1)^2} \zeta_2 + \frac{56(3N^4 + 6N^3 + 13N^2 + 10N + 16)}{27N(N+1)} \zeta_3 - \frac{4P_5}{27N^2(N+1)^2} S_2 \right] \\
&+ C_F \frac{1}{(N-1)(N+2)} \left[ \frac{112(N^2 + N + 2)^2}{27N^2(N+1)^2} S_1^3 - \frac{16P_6}{27N^3(N+1)^3} S_1^2 \right. \\
&+ \frac{32P_7}{81N^4(N+1)^4} S_1 + \frac{16(N^2 + N + 2)^2}{3N^2(N+1)^2} \zeta_2 S_1 + \frac{16(N^2 + N + 2)^2}{3N^2(N+1)^2} S_2 S_1 \\
&- \frac{32P_8}{243N^5(N+1)^5} - \frac{16P_9}{9N^3(N+1)^3} \zeta_2 + \frac{448(N^2 + N + 2)^2}{9N^2(N+1)^2} \zeta_3 + \frac{16P_{10}}{9N^3(N+1)^3} S_2 \\
&\left. - \frac{160(N^2 + N + 2)^2}{27N^2(N+1)^2} S_3 \right] \left. \right\}, \tag{22}
\end{aligned}$$

where the polynomials  $P_i$  are given by

$$P_1 = 16N^5 + 41N^4 + 2N^3 + 47N^2 + 70N + 32 \quad (23)$$

$$P_2 = 6944N^8 + 26480N^7 + 23321N^6 - 15103N^5 - 39319N^4 - 27001N^3 - 11178N^2 - 2016N + 864 \quad (24)$$

$$P_3 = 4809N^{10} + 24045N^9 - 182720N^8 - 854414N^7 - 1522031N^6 - 1472927N^5 - 758234N^4 - 126080N^3 - 1152N^2 - 50688N - 24192 \quad (25)$$

$$P_4 = 3N^6 + 9N^5 + 307N^4 + 599N^3 + 746N^2 + 448N + 96 \quad (26)$$

$$P_5 = 40N^6 + 112N^5 - 3N^4 - 166N^3 - 301N^2 - 210N - 96 \quad (27)$$

$$P_6 = 44N^6 + 123N^5 + 386N^4 + 543N^3 + 520N^2 + 248N + 24 \quad (28)$$

$$P_7 = 205N^8 + 856N^7 + 3169N^6 + 6484N^5 + 7310N^4 + 4722N^3 + 1534N^2 + 48N - 72 \quad (29)$$

$$P_8 = 1976N^{10} + 9385N^9 + 24088N^8 + 38989N^7 + 50214N^6 + 53872N^5 + 35219N^4 + 6890N^3 - 4233N^2 - 2844N - 756 \quad (30)$$

$$P_9 = 14N^6 + 33N^5 + 59N^4 + 39N^3 + 55N^2 + 20N - 12 \quad (31)$$

$$P_{10} = 4N^6 + 3N^5 - 50N^4 - 129N^3 - 100N^2 - 56N - 24 . \quad (32)$$

Here  $S_{b,\bar{a}} \equiv S_{b,\bar{a}}(N) = \sum_{n=1}^N (\text{sign}(b))^n S_{\bar{a}}(N)/n^{|b|}$ ;  $S_0 = 1$  denote the harmonic sums [42] which only occur as single harmonic sums in the present calculation.

It is convenient to express the renormalized OMEs  $A_{gj,Q}$ ,  $j = q, g$  also referring to the heavy quark mass in the  $\overline{\text{MS}}$  scheme, cf. [5]. The OMEs  $A_{gg,Q}^{(3),n_f T_F^2}$  and  $A_{gg,Q}^{(3),n_f T_F^2}$  read :

$$\begin{aligned} A_{gg,Q,C_F T_F^2 n_f}^{(3),\overline{\text{MS}}} &= C_F n_f T_F^2 \left\{ \frac{32(N^2 + N + 2)}{9(N-1)N(N+1)} \ln^3 \left( \frac{\bar{m}^2}{\mu^2} \right) \right. \\ &+ \left[ -\frac{16(N^2 + N + 2)}{3(N-1)N(N+1)} (S_1^2 + S_2) + \frac{32(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^2} S_1 \right. \\ &+ \left. \left. + \frac{32(19N^4 + 81N^3 + 86N^2 + 80N + 38)}{27(N-1)N(N+1)^3} \right] \ln \left( \frac{\bar{m}^2}{\mu^2} \right) \right. \\ &+ \left[ \frac{32(N^2 + N + 2)}{27(N-1)N(N+1)} (S_1^3 + 3S_2 S_1 + 2S_3 - 24\zeta_3) \right. \\ &- \frac{32(8N^3 + 13N^2 + 27N + 16)}{27(N-1)N(N+1)^2} (S_1^2 + S_2) \\ &+ \frac{64(4N^4 + 4N^3 + 23N^2 + 25N + 8)}{27(N-1)N(N+1)^3} S_1 \\ &+ \left. \left. + \frac{64(197N^5 + 824N^4 + 1540N^3 + 1961N^2 + 1388N + 394)}{243(N-1)N(N+1)^4} \right] \right\} \quad (33) \end{aligned}$$

$$\begin{aligned} A_{gg,Q}^{(3),n_f T_F^2, \overline{\text{MS}}} &= n_f T_F^2 \left\{ \left( C_F \frac{64(N^2 + N + 2)^2}{9(N-1)N^2(N+1)^2(N+2)} \right. \right. \\ &+ \left. \left. C_A \left[ \frac{128(N^2 + N + 1)}{27(N-1)N(N+1)(N+2)} - \frac{64}{27} S_1 \right] \right) \ln^3 \left( \frac{\bar{m}^2}{\mu^2} \right) \right\} \end{aligned}$$

(34)

$$-C_F \frac{16}{3} \ln^2 \left( \frac{\bar{m}^2}{\mu^2} \right) + \left( C_A \frac{1}{(N-1)(N+2)} \left[ -\frac{4P_{11}}{81N^3(N+1)^3} \right. \right.$$

(35)

$$\left. -\frac{16P_{12}}{81N^2(N+1)^2} S_1 \right] + C_F \frac{1}{(N-1)(N+2)} \left[ \frac{16(N^2+N+2)^2}{N^2(N+1)^2} \left( S_1^2 - \frac{5}{3} S_2 \right) \right. \\ \left. -\frac{4P_{13}}{9N^4(N+1)^4} - \frac{32P_{14}}{3N^3(N+1)^3} S_1 \right] \ln \left( \frac{\bar{m}^2}{\mu^2} \right)$$

$$+ C_A \frac{1}{(N-1)(N+2)} \left[ -\frac{4P_{15}}{27N^2(N+1)^2} S_1^2 - \frac{8P_{16}}{729N^3(N+1)^3} S_1 \right. \\ \left. + \frac{512}{27} (N-1)(N+2) \zeta_3 S_1 - \frac{2P_{17}}{729N^4(N+1)^4} - \frac{1024(N^2+N+1)}{27N(N+1)} \zeta_3 \right.$$

$$\left. + \frac{4P_{18}}{27N^2(N+1)^2} S_2 \right]$$

$$+ C_F \frac{1}{(N-1)(N+2)} \left[ \frac{64(N^2+N+2)^2}{9N^2(N+1)^2} \left( -\frac{1}{3} S_1^3 - 8\zeta_3 + \frac{4}{3} S_3 \right) \right.$$

$$\left. + \frac{32P_{19}}{27N^3(N+1)^3} S_1^2 - \frac{64P_{20}}{81N^4(N+1)^4} S_1 - \frac{32P_{21}}{243N^5(N+1)^5} \right.$$

$$\left. - \frac{32P_{22}}{3N^3(N+1)^3} S_2 \right] \Bigg\} ,$$

(36)

with the polynomials

$$P_{11} = 297N^8 + 1188N^7 + 640N^6 - 2094N^5 - 1193N^4 + 2874N^3 + 5008N^2 \\ + 3360N + 864 \quad (37)$$

$$P_{12} = 136N^6 + 390N^5 + 19N^4 - 552N^3 - 947N^2 - 630N - 288 \quad (38)$$

$$P_{13} = 15N^{10} + 75N^9 - 48N^8 - 866N^7 - 2985N^6 - 6305N^5 - 8206N^4 - 7656N^3 \\ - 4648N^2 - 1600N - 288 \quad (39)$$

$$P_{14} = 5N^5 + 52N^4 + 109N^3 + 90N^2 + 48N + 16 \quad (40)$$

$$P_{15} = 4N^5 + 17N^4 + 14N^3 + 71N^2 + 70N + 32 \quad (41)$$

$$P_{16} = 3008N^8 + 11600N^7 + 9197N^6 - 10255N^5 - 27739N^4 - 24745N^3 - 12474N^2 \\ - 2016N + 864 \quad (42)$$

$$P_{17} = 4185N^{10} + 20925N^9 + 1892N^8 - 117118N^7 - 222151N^6 - 176863N^5 - 41446N^4 \\ + 22304N^3 - 1296N^2 - 18432N - 6912 \quad (43)$$

$$P_{18} = 16N^6 + 52N^5 - 3N^4 - 106N^3 - 277N^2 - 210N - 96 \quad (44)$$

$$P_{19} = 10N^6 + 30N^5 + 109N^4 + 168N^3 + 155N^2 + 76N + 12 \quad (45)$$

$$P_{20} = 38N^8 + 206N^7 + 962N^6 + 2246N^5 + 2509N^4 + 1542N^3 + 509N^2 + 24N - 36 \quad (46)$$

$$P_{21} = 123N^{12} + 738N^{11} + 691N^{10} - 3526N^9 - 14521N^8 - 29458N^7 - 39189N^6 \\ - 37672N^5 - 21920N^4 - 3914N^3 + 2856N^2 + 1872N + 432 \quad (47)$$

$$P_{22} = 2N^6 + 4N^5 + N^4 - 10N^3 - 5N^2 - 4N - 4 . \quad (48)$$

As has been noted before [5], the above results are free of  $\zeta_2$ , which is common to all massive OMEs, and hence is a particular feature of representing also the mass in the  $\overline{\text{MS}}$  scheme. Furthermore we note, that the  $\ln^2(\bar{m}^2/\mu^2)$ -contribution to  $A_{gg,Q,C_F T_F^2 n_f}^{(3),\overline{\text{MS}}}$  is particularly simple, while the corresponding contribution to  $A_{qq,Q,C_F T_F^2 n_f}^{(3),\overline{\text{MS}}}$  vanishes.

As a by-product of the calculation we obtain the corresponding contributions to the anomalous dimensions from the single pole term  $1/\varepsilon$  resp. the linear logarithmic contribution, cf. (14,15),

$$\hat{\gamma}_{gg}^{(2),n_f} = n_f T_F^2 C_F \left( -\frac{64(N^2 + N + 2)}{3(N-1)N(N+1)} (S_1^2 + S_2) + \frac{128(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^2} S_1 - \frac{128(4N^4 + 4N^3 + 23N^2 + 25N + 8)}{9(N-1)N(N+1)^3} \right), \quad (49)$$

$$\begin{aligned} \hat{\gamma}_{gg}^{(2),n_f} = & n_f T_F^2 C_A \left[ -\frac{32P_{23}}{27(N-1)N^2(N+1)^2(N+2)} S_1 - \frac{8P_{24}}{27(N-1)N^3(N+1)^3(N+2)} \right] \\ & + n_f T_F^2 C_F \left[ \frac{64(N^2 + N + 2)^2}{3(N-1)N^2(N+1)^2(N+2)} (S_1^2 - 3S_2) \right. \\ & \left. + \frac{128P_{25}}{9(N-1)N^3(N+1)^3(N+2)} S_1 - \frac{16P_{26}}{27(N-1)N^4(N+1)^4(N+2)} \right], \quad (50) \end{aligned}$$

where

$$P_{23} = 8N^6 + 24N^5 - 19N^4 - 78N^3 - 253N^2 - 210N - 96 \quad (51)$$

$$P_{24} = 87N^8 + 348N^7 + 848N^6 + 1326N^5 + 2609N^4 + 3414N^3 + 2632N^2 + 1088N + 192 \quad (52)$$

$$P_{25} = 4N^6 + 3N^5 - 50N^4 - 129N^3 - 100N^2 - 56N - 24, \quad (53)$$

$$P_{26} = 33N^{10} + 165N^9 + 256N^8 - 542N^7 - 3287N^6 - 8783N^5 - 11074N^4 - 9624N^3 - 5960N^2 - 2112N - 288. \quad (54)$$

Eqs. (49,50) confirm previous results in [32] by a first direct diagrammatic calculation, here in the massive case.

The leading singlet eigenvalue for the gluonic anomalous dimensions  $\gamma_{gj}^{(3)}$ ,  $j = q, g$  in form of

$$\gamma_{gg}^{(2),n_f^2} + \frac{\gamma_{gq}^{(2),n_f^2} \gamma_{qg}^{(0)}}{\gamma_{gg}^{(0),n_f} n_f} \quad (55)$$

has been calculated in [43] for the leading  $n_f$  contribution,  $\propto n_f^2$ . We also confirm this result by a direct massive calculation.

Usually the calculation in  $N$ -space is being performed multiplying the massive OMEs and the parton distributions analytically<sup>2</sup>, cf. e.g. [44]. The corresponding analytic continuations of harmonic sums up to weight  $w=8$  are given in [45]. Only a single numerical contour integral around the singularities has to be performed, allowing for very fast implementations.

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<sup>2</sup>For Mellin-space representations of a wide class of parton densities see [15].

The OMEs (33,34) can also be given in  $x$ -space directly for codes operating in  $x$ -space only. They are given by :

$$\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2, \overline{\text{MS}}}(x) = & C_F n_f T_F^2 \left\{ \left( \frac{32x}{9} + \frac{64}{9x} - \frac{64}{9} \right) \ln^3 \left( \frac{\bar{m}^2}{\mu^2} \right) + \left[ \left( -\frac{16x}{3} - \frac{32}{3x} + \frac{32}{3} \right) H_1^2 \right. \right. \\
& + \left. \left( \frac{256x}{9} + \frac{320}{9x} - \frac{320}{9} \right) H_1 + \frac{608x}{27} + \frac{2176}{27x} - \frac{2176}{27} \right] \ln \left( \frac{\bar{m}^2}{\mu^2} \right) \\
& + \left( \frac{32x}{27} + \frac{64}{27x} - \frac{64}{27} \right) H_1^3 + \left( -\frac{256x}{27} - \frac{320}{27x} + \frac{320}{27} \right) H_1^2 \\
& + \left( \frac{256x}{27} - \frac{128}{27x} + \frac{128}{27} \right) H_1 + \left( -\frac{256x}{9} - \frac{512}{9x} + \frac{512}{9} \right) \zeta_3 + \frac{12608x}{243} \\
& \left. + \frac{24064}{243x} - \frac{24064}{243} \right\} \tag{56}
\end{aligned}$$

$$\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2, \overline{\text{MS}}}(x) = & n_f T_F^2 \left\{ \left( C_A \left[ -\frac{64x^2}{27} + \frac{64x}{27} - \frac{64}{27(x-1)_+} - \frac{128}{27} + \frac{64}{27x} \right] + C_F \left[ -\frac{256x^2}{27} \right. \right. \right. \\
& \left. \left. - \frac{64x}{9} + \frac{128}{9}(1+x)H_0 + \frac{64}{9} + \frac{256}{27x} \right] \right) \ln^3 \left( \frac{\bar{m}^2}{\mu^2} \right) \\
& - \frac{16}{3} C_F \delta(1-x) \ln^2 \left( \frac{\bar{m}^2}{\mu^2} \right) + \left[ C_A \left[ -\frac{608x^2}{27} - \frac{16}{81}(144\zeta_2 - 85)x \right. \right. \\
& + \frac{32}{3}(1+x)H_0^2 - \frac{44}{3}\delta(1-x) - \frac{16}{81}(144\zeta_2 + 149) + \left( -\frac{832x^2}{27} + \frac{16x}{27} \right. \\
& \left. \left. - \frac{800}{27} \right) H_0 + \left( -\frac{832x^2}{27} + \frac{208x}{9} - \frac{176}{9} + \frac{832}{27x} \right) H_1 \right. \\
& + \frac{256}{9}(1+x)H_{0,1} - \frac{2176}{81(x-1)_+} + \frac{224}{27x} \left. \right] + C_F \left[ \frac{32}{3}(1+x)H_0^3 \right. \\
& + \left( -\frac{256x^2}{9} - \frac{688x}{9} - \frac{592}{9} \right) H_0^2 + \left( -\frac{64x^2}{9} - \frac{64}{9}(12\zeta_2 + 5)x - \frac{64}{9}(12\zeta_2 \right. \\
& \left. \left. - 41) \right) H_0 + \left( -\frac{512x^2}{9} - \frac{128x}{3} + \frac{128}{3} + \frac{512}{9x} \right) H_1 H_0 + \frac{256}{3}(1+x)H_{0,1}H_0 \right. \\
& + \left( -\frac{64x^2}{3} - 16x + 16 + \frac{64}{3x} \right) H_1^2 - \frac{20}{3}\delta(1-x) + \frac{64}{27}x^2(18\zeta_2 - 7) + \frac{64}{9}x(3\zeta_2 \\
& + 3\zeta_3 - 28) + \frac{64}{9}(6\zeta_2 + 3\zeta_3 + 10) + \left( -\frac{64x^2}{9} - \frac{416x}{3} + \frac{736}{3} - \frac{896}{9x} \right) H_1 \\
& + \left( \frac{128x^2}{9} + \frac{64x}{3} - \frac{256}{3} - \frac{512}{9x} \right) H_{0,1} - \frac{256}{3}(1+x)H_{0,0,1} + 64(1+x)H_{0,1,1} \\
& \left. \left. + \frac{3904}{27x} \right] \right] \ln \left( \frac{\bar{m}^2}{\mu^2} \right) + C_A \zeta_3 \left( \frac{512x^2}{27} - \frac{512x}{27} + \frac{512}{27(x-1)_+} + \frac{1024}{27} - \frac{512}{27x} \right)
\end{aligned}$$

$$\begin{aligned}
& + C_F \zeta_3 \left( \frac{2048x^2}{27} + \frac{512x}{9} - \frac{1024}{9}(1+x)H_0 - \frac{512}{9} - \frac{2048}{27x} \right) + C_A \left[ \frac{128}{81}(1+x)H_0^3 \right. \\
& + \left( -\frac{208x^2}{81} + \frac{812x}{81} + \frac{320}{81} \right) H_0^2 + \left( -\frac{8624x^2}{243} - \frac{8}{81}(48\zeta_2 - 199)x - \frac{16}{27}(8\zeta_2 + 19) \right. \\
& \left. \left. - \frac{64}{27(x-1)} \right) H_0 + \left( -\frac{416x^2}{81} + \frac{56x}{27} - \frac{88}{27} + \frac{416}{81x} \right) H_1 H_0 + \left( \frac{64}{27}\zeta_2 - \frac{310}{27} \right) \delta(1-x) \right. \\
& + \frac{128}{27}(1+x)H_{0,1}H_0 + \left( \frac{208x^2}{81} - \frac{20x}{9} + \frac{44}{27} - \frac{208}{81x} \right) H_1^2 \\
& - \frac{416}{729}x^2(9\zeta_2 + 113) - \frac{8}{729}(2088\zeta_2 - 864\zeta_3 + 6055) - \frac{8}{729}x(2601\zeta_2 - 864\zeta_3 - 4883) \\
& + \left( -\frac{8624x^2}{243} + \frac{2600x}{81} - \frac{872}{81} + \frac{4592}{243x} \right) H_1 + \left( \frac{832x^2}{81} + \frac{2144x}{81} + \frac{2120}{81} - \frac{416}{81x} \right) H_{0,1} \\
& \left. - \frac{128}{27}(1+x)(H_{0,0,1} + H_{0,1,1}) - \frac{24064}{729(x-1)_+} + \frac{32320}{729x} \right] \\
& + C_F \left[ \frac{32}{27}(1+x)H_0^4 + \left( -\frac{128x^2}{81} + \frac{256x}{81} + \frac{64}{81} \right) H_0^3 + \left( -\frac{2176x^2}{81} - \frac{32}{81}(18\zeta_2 + 107)x \right. \right. \\
& \left. \left. - \frac{32}{81}(18\zeta_2 - 1) \right) H_0^2 + \left( -\frac{128x^2}{27} - \frac{32x}{9} + \frac{32}{9} + \frac{128}{27x} \right) H_1 H_0^2 + \frac{64}{9}(1+x)H_{0,1}H_0^2 \right. \\
& + \left( \frac{128x^2}{27} + \frac{32x}{9} - \frac{32}{9} - \frac{128}{27x} \right) H_1^2 H_0 + \left( -\frac{128}{243}(18\zeta_2 - 1)x^2 - \frac{64}{243}(333\zeta_2 - 108\zeta_3 - 410)x \right. \\
& \left. \left. - \frac{64}{243}(225\zeta_2 - 108\zeta_3 - 1292) \right) H_0 + \left( -\frac{4352x^2}{81} - \frac{320x}{9} + \frac{704}{9} + \frac{896}{81x} \right) H_1 H_0 \right. \\
& + \left( \frac{512x^2}{27} + \frac{2560x}{27} + \frac{1408}{27} - \frac{256}{27x} \right) H_{0,1}H_0 - \frac{128}{9}(1+x)(H_{0,0,1}H_0 + H_{0,1,1}H_0) \\
& + \left( \frac{256x^2}{81} + \frac{64}{27}(x-1) - \frac{256}{81x} \right) H_1^3 + \left( -\frac{1472x^2}{81} + \frac{64}{9}(x-1) + \frac{1472}{81x} \right) H_1^2 \\
& + \left( \frac{1024}{9}\zeta_2^2 - \frac{1312}{81} \right) \delta(1-x) - \frac{64}{405}x(63\zeta_2^2 + 145\zeta_2 - 120\zeta_3 + 1720) + \frac{64}{729}x^2(414\zeta_2 \\
& - 108\zeta_3 - 1165) - \frac{64}{405} \left( 63\zeta_2^2 - 215\zeta_2 - 30\zeta_3 - 1675 \right) - \left( \frac{128}{243}(18\zeta_2 - 1)x^2 \right. \\
& \left. + \frac{64}{27}[(3\zeta_2 + 44)x - (3\zeta_2 + 80)] - \frac{128(18\zeta_2 - 163)}{243x} \right) H_1 \\
& + \left( \frac{1408x^2}{81} + \frac{128}{81}[(9\zeta_2 + 37)x + (9\zeta_2 - 71)] - \frac{896}{81x} \right) H_{0,1} + \left( -\frac{512x^2}{27} - \frac{2560x}{27} \right. \\
& \left. - \frac{1408}{27} + \frac{256}{27x} \right) H_{0,0,1} + \left( \frac{256x^2}{27} + \frac{1664x}{27} + \frac{1664}{27} + \frac{256}{27x} \right) H_{0,1,1} + \frac{128}{9}(1+x) \left( H_{0,0,0,1} \right. \\
& \left. + H_{0,0,1,1} - 2H_{0,1,1,1} \right) + \frac{79744}{729x} \left. \right\}, \tag{57}
\end{aligned}$$

with the harmonic polylogarithms  $H_{\vec{a}} \equiv H_{\vec{a}}(x)$  over the alphabet  $\mathfrak{A} = \{0, 1, -1\}$  [46]. They can be expressed in terms of elementary functions and the Nielsen integrals [47] :  $H_0(x) = \ln(x)$ ,  $H_1(x) = -\ln(1-x)$ ,  $H_{0,1}(x) = \text{Li}_2(x)$ ,  $H_{0,0,1}(x) = \text{Li}_3(x)$ ,  $H_{0,1,1}(x) = S_{1,2}(x)$ ,  $H_{0,0,0,1}(x) = \text{Li}_4(x)$ ,  $H_{0,0,1,1}(x) = S_{2,2}(x)$  and  $H_{0,1,1,1}(x) = S_{1,3}(x)$ , with

$$S_{n,p}(x) = \frac{(-1)^{(n+p-1)}}{(n-1)!p!} \int_0^1 \frac{dy}{y} \ln^{(n-1)}(y) \ln^p(1-xy) , \quad (58)$$

$$\text{Li}_n(x) = S_{n-1,n}(x) . \quad (59)$$

Here  $\text{Li}_n(x)$  denotes the polylogarithm. All higher functions but  $S_{2,2}(x)$  can be reduced to polylogarithms by the argument relation  $x \rightarrow (1-x)$ . Numerical implementations of the functions  $S_{n,p}(x)$  were given in [48].

At small values of  $x$  the functions  $A_{gq(g),Q}^{(3),n_f T_F^2, \overline{\text{MS}}}(x)$  are singular as  $\propto 1/x$ , or in  $N$ -space like  $\propto 1/(N-1)$ , unlike the quarkonic contributions given in [17] with a leading pole  $\propto 1/N$ . One notices that the number of functions needed in  $x$ -space to express  $A_{gq(g),Q}^{(3),n_f T_F^2, \overline{\text{MS}}}$  is larger than in  $N$ -space, as has been found also in other analyses, cf. [7, 11, 49], requesting very careful numeric implementations.

## 4 Conclusions

We have calculated the contributions  $O(\alpha_s^3 n_f T_F^2 C_{A,F})$  to the massive OMEs with local operator insertions on gluonic lines and vertices at general values of the Mellin variable  $N$ , as first complete parts for the gluonic operator matrix elements at 3-loop order. These matrix elements are needed to describe the transition functions in the VFNS. In the calculation representations of the Feynman diagrams by generalized hypergeometric functions play an essential role. They allow the  $\varepsilon$ -expansion into nested sums, which can be solved using modern summation technologies. The number of these sums is very large, although their structures exhibit similarities. One may synchronize these sums, leading to a low number, however, with voluminous intermediate terms. The solution of the latter sums turns out to be more economic. The final results in  $N$  space can be expressed by rational functions in  $N$  and single harmonic sums up to  $S_3(N)$ . We also derived the corresponding  $x$ -space results, which have a more involved structure and depend on six Nielsen integrals.

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