**Motivation**

- Irregular structure of the QCD vacuum
  - fractal distribution of the topological charge density
  - low-dimensional defects
- Effective model for sQGP
  - effective Lagrangian
  - two-component fluid
- Axion-like field within QCD

**Bosonization with a finite cut-off**

Euclidean functional integral for SU(Nc) × Uem(1) is given by

\[
\int D\psi \bar{\psi} \exp \left( -\int d^4x \left( \bar{\psi} \left( \partial - i m \right) \psi + \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \right) \right),
\]

where we define the Dirac operator as

\[
\mathcal{D} = -i(\partial + A + g \mathcal{G} + \gamma_5 \mathcal{A})
\]

- Integrate out quarks below a cut-off Dirac eigenvalue Λ
- add gauge-invariant terms to the Lagrangian to match the chiral anomaly
- consider a pure gauge \( A_{\mu} = \partial_\mu \theta \) for the auxiliary axial field
- and the chiral limit \( m \to 0 \)

The total effective Euclidean Lagrangian reads as

\[
\mathcal{L}_{\text{E}}^{(1)} = \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \partial_\mu A_\mu
\]

\[
= \frac{N_c}{2} \delta^{\mu\nu} \partial_\mu \partial_\nu + \frac{g^2}{4 \pi^2} \bar{\psi} \gamma_\mu G_{\mu\nu} \psi + \frac{N_c}{8 \pi^2} \theta F_{\mu\nu} F^{\mu\nu}
\]

\[
= \frac{24 \pi^2}{\Lambda^2} (\partial_\mu \partial_\nu \theta)^2 - \frac{N_c}{12 \pi^2} (\partial_\mu \partial_\nu \theta)^2
\]

So we get an axion-like field with decay constant \( f = \frac{2\Lambda}{\pi} \sqrt{\frac{N_c}{3}} \) and a negligible mass \( m_\theta^2 = \lim_{v \to 0} \frac{\partial^2 \mathcal{L}_{\text{E}}^{(1)}}{\partial v^2} \equiv \chi(T)/T^2 \). We tend to interpret it as a quasiparticle moving along the low-dimensional defects!

**Interpretation of the scale Λ**

From the quartic Lagrangian at \( N_c = N_f = 1 \) we get

\[
\rho_5 = \frac{1}{2} \left( \frac{\Lambda}{\pi} \right)^2 \mu_5 + \frac{1}{3 \sqrt{2}} \mu_3^2
\]

- Free quarks (see 0808.3382): \( \Lambda = \sqrt{\frac{3}{2}} \left( 1 + \frac{3}{2} \cdot \mu_3 \right) \)
- Free quarks and strong B-field: \( \Lambda = 2 \sqrt{\frac{1}{\mu_3 B}} \)
- Dynamical lattice fermions (1105.0385): \( \Lambda \approx 3 \text{GeV} \gg \Lambda_{\text{QCD}} \)

**Change in entropy and higher order gradient corrections**

The terms \( r_{\mu\nu}, u_{\mu\nu} \) and \( v_5 \) denote higher-order gradient corrections and obey the Landau conditions

\[
u_{\mu} r_{\mu\nu} = 0, \quad u_{\mu} u_{\mu} = 0, \quad u_{\mu} v_5 = 0
\]

Using both hydrodynamic equations and constitutive relations one can derive

\[
u_{\mu}(s u_{\mu\nu} - \frac{\mu}{T} u_{\mu\nu} - \frac{\mu}{T} v_5) = -\frac{1}{T}(\partial_\mu u_\nu) r_{\mu\nu} - \nu^\nu(\partial_\mu \nu^\mu - \frac{1}{T} E_{\mu}) - v_5^\nu \partial_\nu \frac{\rho_5}{T}
\]

so the entropy production is always nonnegative. This fact tells us that

- we don’t need to add any additional terms to the entropy current like in Son and Surowka (0906.5044)
- we don’t need any additional corrections to the currents.

- in absence of dissipative corrections we obtain \( \nu_{\mu}(s u_{\mu\nu}) = 0 \), i.e.
  - only the “normal” component contributes to the entropy current, while the “superfluid” component has zero entropy.

**Fermionic spectrum at finite temperature**

\[
\nu(\lambda) = \frac{1}{2} \left( \frac{T}{2\pi} \right)^2 \sqrt{\frac{3}{2}} \left( \frac{3}{2} \cdot \mu_3 \right)
\]

- there are two separated parts of the spectrum at intermediate temperatures!
- a strong external magnetic field does not destroy the picture
- all the chiral properties are described by the near zero modes

**Hydrodynamic equations**

Equations of motion for the quadratic effective Minkowski Lagrangian

\[
\partial_\mu \partial_\mu \theta = \frac{C}{4\pi^2} F_{\mu\nu} F^{\mu\nu} + \frac{g^2}{32\pi^2} G_\pi G_{\pi},
\]

\[
\partial_\mu F^{\mu\nu} = -\frac{i}{2} \mathcal{C}(\partial_\nu \theta) F^{\mu\nu},
\]

\[
\partial_\nu F^{\mu\nu} = 0.
\]

Varying the quadratic Lagrangian with respect to axial-vector \( A_{\mu} = \partial_\mu \theta \) we obtain the axial current \( j_5^{\mu} = F_{5\nu}^\mu (\text{curi-free}) \). Conservation law \( \partial_{\mu}(T^{\mu\nu} + \rho^{\mu\nu}) = 0 \) makes it possible to express divergency of the fluid energy-momentum tensor \( T^{\mu\nu} \) via the one of the electromagnetic stress-energy tensor \( \Theta^{\mu\nu} = F_{\lambda\nu}^\mu F_{\lambda\mu} + \frac{1}{4} \rho^{\mu\nu} G_{\pi} G_{\pi} \).

In summary, the hydrodynamic equations are

\[
\partial_\mu T^{\mu\nu} = F^{\lambda\nu} (j_\lambda + C F_{\lambda\nu} (\partial_\nu \theta)) = F^{\lambda\nu} (j_\lambda + \tilde{f}_{\lambda\nu}^\mu),
\]

\[
\partial_\mu \tilde{f}_{\mu\nu}^\mu = \frac{C}{4\pi^2} F_{\mu\nu} F^{\mu\nu} - \frac{g^2}{32\pi^2} G_\pi G_{\pi}, \quad \partial_\nu \tilde{f}_{\mu\nu}^\mu = 0.
\]

plus the Josephson equation \( u^{\mu} \partial_\mu \theta + \mu_5 = 0 \).

Corresponding constitutive relations in gradient expansion are

\[
T^{\mu\nu} = (x + P) u^{\mu} u^{\nu} + P g^{\mu\nu} + \tilde{f}_{\mu\nu}^\lambda \partial_\lambda \theta + \tau^{\mu\nu},
\]

\[
\tilde{f}_{\mu\nu}^\mu = -\tilde{f}_{\nu\mu}^\mu + \nu_5^\nu.
\]

The stress-energy tensor \( T^{\mu\nu} \) consists of two parts, an ordinary fluid component and a pseudoscalar “superfluid” component. This modifies the equation of state by adding to the r.h.s. a new \( \theta \)-dependent term

\[
dP = sdT + \rho d\mu - \frac{1}{2} d\left( \frac{3}{2} \cdot \mu_3 \right)\partial_\nu \theta
\]

**Phenomenological output**

Electric and magnetic fields in the fluid rest frame are defined as

\[
E^\mu = F^{\mu\nu} u_\nu, \quad B^\mu = F^{\mu\nu} u_\nu \equiv \frac{1}{2} \mu_3 u_\nu F_{\alpha\beta}^\mu
\]

An additional electric current, induced by \( \theta \)-field

\[
\tilde{f}_{\mu\nu}^\mu = C \tilde{F}_{\mu\nu} \partial_\mu \theta = -C \mu_5 B_\lambda + C \epsilon_{\lambda\mu\nu\rho} u^\rho \partial_\nu E_\lambda - u_\nu (\partial_\rho \theta \cdot B)
\]

- I term: Chiral Magnetic Effect (electric current along B-field)
- II term: Chiral Electric Effect (electric current transverse to E-field and both to normal and superfluid velocities)
- III term: Chiral Dipole Wave (dipole moment induced by B-field)
- The field \( \theta(\vec{x},t) \) itself: Chiral Magnetic Wave (propagating imbalance between the number of left- and right-handed quarks)