INTRODUCTION

The experimental \cite{1} and theoretical \cite{2} determinations of the anomalous magnetic moment of the muon $a_{\mu}$ have both reached an accuracy that is better than six parts per million. This high precision reveals a discrepancy of over 3 standard deviations ($3\sigma$), which raises the possibility of physics beyond the standard model. However, the dominant error in the theory computation is due to hadronic effects that are currently not calculated but are instead either separately measured or simply modeled. This obscures the significance of the $3\sigma$ effect and makes it difficult to improve the accuracy of the Standard Model calculation.

In this Letter, we present a reliable lattice QCD determination of the leading-order hadronic correction for the muon, $a_{\mu}^{\text{hvp}}$, which is the single largest source of error in the theory calculation of $a_{\mu}$. Additionally, we calculate the leading-order corrections $a_{e}^{\text{hvp}}$ for the electron and $a_{\tau}^{\text{hvp}}$ for the tau, achieving an accuracy of better than 3% for each. This was accomplished by introducing a modification of the existing method that results in a significantly more well-controlled calculation. After examining all sources of systematic error and performing our own extraction of the two-flavor contribution to the experimental measurements, we find agreement for all three charged leptons in the standard model.

Our current computation is performed in two-flavor QCD, but the technique presented in this work is readily generalized to a realistic four-flavor calculation that is already under way \cite{3}. The precision of our calculation and the prospects for improving it demonstrate that lattice QCD can realistically provide a first-principles determination of the leading-order hadronic contributions to the magnetic moments of the standard model leptons.

LEADING-ORDER HADRONIC CORRECTION

The anomalous magnetic moment $a_{l}$ of a lepton $l$ can be written as a perturbative expansion in the electromagnetic coupling $\alpha$. Contributions from QCD first occur at the order $\alpha^2$ and can be written as \cite{4}

$$a_{l}^{\text{hvp}} = \alpha^2 \int_0^\infty dQ^2 \frac{1}{Q^2} w(Q^2/m_l^2)^2 \Pi_{\mathcal{R}}(Q^2),$$

(1)

where $m_l$ is the mass of the lepton, $Q$ is the Euclidean momentum and $w(Q^2/m_l^2)$ is a known function. The combination $\Pi_{\mathcal{R}}(Q^2) = \Pi(Q^2) - \Pi(0)$ is the renormalized hadronic vacuum polarization function $\Pi(Q^2)$, which is defined shortly. The weight function $w(Q^2/m_l^2)$ vanishes as $(Q^2)^{-2}$ for large $Q^2$. This ensures that the integral above is dominated by the low $Q^2$ region, making it clear that $a_{l}^{\text{hvp}}$ must be evaluated nonperturbatively.

EXPERIMENTAL DETERMINATION

The electron and muon magnetic moments have been measured in dedicated experiments \cite{1,6}. To compare to the standard model prediction, the leading-order hadronic correction is determined by using unitarity and causality to relate the expression in Eq.\ref{eq:1} to

$$a_{l}^{\text{hvp}} = \alpha^2 \int_0^\infty ds \frac{1}{s} w'(s/m_l^2)^2 R(s),$$

(2)

where $w'$ is another known weight function and $R(s)$ is the ratio of the hadronic cross section $\sigma(e^+e^-\rightarrow\text{hadrons})$ to the leptonic cross section $\sigma(e^+e^-\rightarrow\mu^+\mu^-)$. The determination of $R(s)$ relies on the results of many experiments, and the integral in Eq.\ref{eq:2} has been evaluated by several
groups, most recently [2–5]. Additionally, there are higher-order corrections, including the so-called light-by-light contribution, which is difficult to measure and is modeled instead.

Our calculation is performed in QCD with only up and down quarks, so we need to extract the two-flavor contribution to $a_l^{hvp}$. Inevitably, this introduces some ambiguity. For the purposes of comparing to our current two-flavor calculation, we adopt the simple procedure of rescaling the contribution to the integral in Eq. 2 from the energy regions between quark thresholds by the value of $\sum_2 Q_f^2$, where the sum runs over only the active quark flavors for that region and the electric charges of the quarks are $eQ_f$. This neglects the very small changes due to the running of the QCD coupling, it ignores small off-diagonal contributions proportional to $Q_f Q_j$, and it disregards any complications at the flavor thresholds. These are all caveats that we must accept in the current comparisons but that will be eliminated in our ongoing four-flavor computation.

Using the results from [9,10], we extract the two-flavor contributions to $a_l^{hvp}$ along the lines just described, giving $a_{e,N_f=2}^{hvp,\text{ex}} = 1.547 \times 10^{-12}$ and $a_{\tau,N_f=2}^{hvp,\text{ex}} = 5.660 \times 10^{-8}$, and $a_{\tau,N_f=2}^{hvp,\text{ex}} = 2.638 \times 10^{-6}$. The errors result from propagating just those of [9,10]. The systematic error due to extracting the two-flavor contribution is likely larger than these uncertainties and must be taken into consideration when comparing our calculation to these estimates.

**LATTICE QCD CALCULATION**

The leading-order hadronic correction, $a_l^{hvp}$, is the order $\alpha^2$ contribution in a perturbative QED expansion of $a_l$ but it must be treated nonperturbatively in QCD. To this order in the QED coupling, the QCD corrections only modify the photon propagator. These contributions can be formally summed to all orders giving the hadronic vacuum polarization tensor,

$$\Pi_{\mu\nu}(Q) = \int d^4x \ e^{iQ \cdot x} \langle \Omega | T J_{\mu}(X) J_{\nu}(0) | \Omega \rangle .$$

The current $J_{\mu} = \sum_f Q_f \bar{q}_f \gamma_{\mu} q_f$ is the hadronic component of the electromagnetic current and the sum runs over all relevant quark flavors. The current $J_{\mu}$ is conserved, consequently this correlation function satisfies a Ward identity that allows us to write $\Pi_{\mu\nu}$ in terms of a single scalar function of $Q^2$ as

$$\Pi_{\mu\nu}(Q) = (Q_{\mu} Q_{\nu} - Q^2 \delta_{\mu\nu}) \Pi(Q^2) .$$

Note that both $\Pi_{\mu\nu}$ and $\Pi$ are calculated directly in Euclidean space without any analytic continuation.

We use standard lattice QCD techniques to calculate the vacuum-to-vacuum matrix element $\langle \Omega | T J_{\mu}(x) J_{\nu}(y) | \Omega \rangle$. The functional integral that is implicit in this correlation function is evaluated stochastically using the results of the European Twisted Mass Collaboration [11]. We have used two lattice spacings, $a = 0.079$ and $a = 0.063$ fm, to examine lattice cutoff corrections. Two finite-volume studies were performed to check for finite-size effects. The up and down quark masses $m_q$ are equal and are parametrized in terms of the pseudoscalar meson mass $m_{PS}$, with $m_q \propto m_{PS}^2$ in the chiral limit. As is common, we use heavier-than-physical quark masses and then take the limit as $m_{PS}$ approaches the physical pion mass $m_{\pi}$. This was done by studying the dependence on $m_{PS}$ over the range from 650 to 290 MeV. The so-called disconnected diagrams, ignored in all previous calculations, were included for almost half of the ensembles used in this work and are accounted for as a systematic error along with those from the continuum, infinite-volume and physical quark-mass limits. The additional details are standard and deferred to a later publication.

Apart from variations in how the lattice calculation of $\Pi(Q^2)$ is matched to a smooth function, the method used so far in all calculations [12–14] proceeds by numerically integrating Eq. 1 directly to form $a_l^{hvp}$. In our calculation, we parametrize $\Pi(Q^2)$ over the entire range of $Q^2$ that is determined from our lattice computation. The presence of the lattice cutoff and the restriction to finite volume induce an ultraviolet cutoff $Q_{av}^2$ proportional to $1/a^2$ and an infrared cutoff proportional to $1/L^2$. Extrapolating the functional form for $\Pi(Q^2)$ to $Q^2 = 0$ in the continuum, infinite-volume and physical quark-mass limits, the systematic error caused by extrapolating to $Q^2 = 0$ is eliminated as $L$ is taken large and the error due to truncating the integral at $Q_{av}^2$ is removed as $a$ goes to zero. Thus both effects are automatically accounted for as part of the corresponding systematic errors.

Our results for the muon using this method, which we deem the standard method, are shown as the lowest set of points in Fig. 1. Consistent with all other lattice calculations of $a_l^{hvp}$, we find that the values calculated at $m_{PS}$ heavier than $m_{\pi}$ are significantly lower than the experimentally measured value and apparently rise rapidly only when $m_{PS}$ approaches quite near the physical value $m_{\pi}$.

We attribute this behavior to the contributions of the lowest-lying vector mesons. The rho, omega, and phi mesons account for over 80% of the fully measured $a_l^{hvp}$ [10]. Any description of the vector-meson contribution to $a_l^{hvp}$ will depend on the mass $m_V$ and a variety of dimensionless couplings. Without loss of generality, we focus on just those models based on $m_V$ and the electromagnetic coupling $g_V$ with $(\langle \Omega | T J_{\mu}(V) V_{\nu} | \Omega \rangle = m_V g_V \epsilon_{\mu\nu} / \sqrt{2}$. The coupling shows up quadratically and dimensionless analysis then results in a vector-meson contribution of
FIG. 1: Comparison of methods for $a_{\mu}^{\text{hvp}}$. The upper set of points are the results for $a_{\mu}^{\text{hvp}}$ using $H = m_V$, the middle set use $H = f_V$, and the lower set correspond to the standard method, formally $H = 1$. The two lines are linear extrapolations of $a_{\mu}^{\text{hvp}}$ and the curve is the phenomenological extrapolation of $a_{\mu}^{\text{hvp}}$. The three methods agree at the physical point, denoted by the dashed line, and agree with the estimated two-flavor contribution to the experimental value.

$\frac{M}{f^2_{\text{hvp}}} = \frac{g_V^2 f(m^2_V/m^2_{T_V})}{m^2_T}. \quad$ Additionally, $f(m^2_V/m^2_{T_V})$ should vanish for $m_T \to 0$ and $m_V \to \infty$. Thus on rather general grounds we expect $a_{\mu,\text{hvp}}^{\text{phys}} \approx C g_V^2 m^2_T/m^2_{T_V}$, with a model-dependent constant $C$.

These expectations can be combined with our lattice calculation of $m_V$ and $g_V$. As shown in Fig. 2, we find that $m_V$ decreases moderately with decreasing $m_{PS}$ but the values from our calculation are still rather high compared to the experimental result $m_{\pi}$. Thus at some point a rapid decrease in $m_V$ must occur. In contrast, $G_V$, not shown but well fit by $G_V = 0.29(1) - 0.09(2) m^2_{PS}$, has a mild dependence on $m_{PS}$ and extrapolates smoothly to the experimental value $a_{\mu}$. When combined with the model expectation $a_{\mu,\text{hvp}}^{\text{phys}} \sim g_V^2 m^2_{T_V}$, the behavior of $a_{\mu}^{\text{hvp}}$ in Fig. 1 becomes plausible. The values of $a_{\mu}^{\text{hvp}}$ are lower than the experimental value and vary moderately for the region of $m_{PS}$ covered in our calculation. Only at lighter values of $m_{PS}$ do we expect a sharp increase in $a_{\mu}^{\text{hvp}}$.

We can make these observations more precise, at the expense of introducing model dependence, by considering the tree-level form for the vector-meson contribution $a_{\mu,\text{hvp}}^{\text{phys}}$ as given from effective field theory [13]. This gives a specific result for $f(m^2_V/m^2_{T_V})$ that we combine with the calculation of $m_V$ and $g_V$ to construct a model-dependent extrapolation of the results for $a_{\mu}^{\text{hvp}}$. Additionally, constraining $m_V$ to approach $m_{\pi}$ as shown in Fig. 2 gives the lowest-lying curve in Fig. 1. The apparent agreement with the physical value for $a_{\mu}^{\text{hvp}}$ increases the plausibility that our explanation is correct. However, this construction does not provide a reliable means of extrapolating our results to the physical $m_{\pi}$ but instead serves to illustrate the apparently strong $m_{PS}$ dependence in the standard method.

The difficulties encountered in the standard method can be traced to the occurrence of two distinct scales, $m_T$ and $m_V$. Apart from any model, this is relevant because $a_{\mu}^{\text{hvp}}$ is made dimensionless at the expense of introducing an external scale $m_T$ that is completely unrelated to the scales of QCD. Based on this observation, we define the following class of observables:

$$a_{\mu,\text{hvp}}(m^2_{PS}) = \frac{\alpha^2}{2 \pi} \int_0^\infty dQ^2 \frac{1}{Q^2} \left((Q^2/m^2_V)(H^2_{\text{phys}}/H^2)\right) \Pi_R(Q^2)$$

where $H$ is any hadronic quantity, understood to be a function of $m_{PS}$, and $H_{\text{phys}}$ is its physical value. The natural choice for our calculation is $H = m_V$, but any choice produces a new modified quantity that has the same physical limit as $a_{\mu}^{\text{hvp}}$. This follows simply by construction because $H(m_{PS} \to m_{\pi}) = H_{\text{phys}}$. The standard method can be formally reproduced by the choice $H = 1$, but choosing a dimensionful scale has the additional advantage that the explicit dependence on the lattice spacing is eliminated. At the same time, the renormalization condition that defines the physical limit is now given by the dimensionless ratio $m_{T}/H_{\text{phys}}$ rather than $m_{T}$ alone.

The calculation of $a_{\mu}^{\text{hvp}}$ using $H = m_V$ and $H = f_V$, the vector-meson decay constant given by $f_V = m_V/G_V$, are shown in Fig. 1. All three extrapolations agree with each other and with the estimated two-flavor contribution to the experimental measurement of $a_{\mu}^{\text{hvp}}$. The results for the new method show a significantly milder dependence on $m_{PS}$. This can be understood using the model considerations earlier. Specifically for $H = m_V$, we expect a vector-meson contribution of $a_{\mu}^{\text{phys}} \approx C g_V^2 m^2_T/m^2_{\pi}$, in which only the mild $m_{PS}$ dependence of $g_V$ now enters. The demonstration that $H = f_V$ results in similar improvements illustrates that any quantity sensitive to $m_V$...
The quoted errors are due to the stochastic integration uncertainty. Without regard to any particular model or the experimental measurements, we can examine the relative merits of the standard and modified methods. Using the muon as an example, the shift between linear and quadratic extrapolations for $a_\mu^{\text{hvp}}$ (using $H = m_N$) is 1.7%, which is only a 0.6 $\sigma$ effect. The same results for $a_\mu^{\text{hvp}}$ are 17% and 3.5 $\sigma$, indicating the presence of noticeably more curvature in the standard approach. In this case, cubic fits are required and give an extrapolated value of $4.1 \times 10^{-8}$, which agrees with the more precise value of $5.72 \times 10^{-8}$ that results from extrapolating $a_\mu^{\text{hvp}}$. The same pattern holds for the electron and tau; thus, the lattice calculation itself provides direct evidence that our modified method has a smoother approach to the physical limit leading to a more accurate calculation.

Taking the modified method with $H = m_N$ as our definition of $a_T^{\text{hvp}}$, we calculate all three $l = e, \mu$ and $\tau$. These results are shown in Fig. 3, and the extrapolated values at the physical point are

$$a_{e, N_f=2}^{\text{hvp}} = 1.513 (43) \times 10^{-12},$$
$$a_{\mu, N_f=2}^{\text{hvp}} = 5.72 (16) \times 10^{-8},$$
$$a_{\tau, N_f=2}^{\text{hvp}} = 2.650 (54) \times 10^{-6}.$$  

The quoted errors are due to the stochastic integration only. We do not find any statistically meaningful uncertainties due to lattice artifacts, finite-size effects, the extrapolation in $m_{PS}$ or the exclusion of the disconnected diagrams. At some higher precision these effects will be relevant, but there is no sign that they are significant at the few percent level of our current calculation.

**CONCLUSIONS AND OUTLOOK**

We have performed the first lattice QCD calculation of the leading-order QCD correction to the anomalous magnetic moments $a_l^{\text{hvp}}$ that included dynamical quarks, examined lattice artifacts, checked finite-size effects and studied the disconnected diagrams. We examined the pitfalls of the standard method for calculating $a_l^{\text{hvp}}$ and introduced a modification that creates a dimensionless quantity $a_T^{\text{hvp}}$ composed of hadronic scales only. This quantity has the same physical limit as $a_l^{\text{hvp}}$ but has a mild approach to that limit that is now well controlled. This allowed us to calculate the leading-order correction for all three charged leptons with an accuracy better than 3%, reproducing our estimate of the two-flavor contributions to the experimental measurements.

The calculation was done using two-flavor QCD, which is the most significant systematic error. To resolve this, we are currently starting a four-flavor calculation. This will eliminate any ambiguity regarding the extraction of the two-flavor experimental value. When combined with further anticipated improvements, the modified method presented here should produce a result precise enough to replace the experimentally estimated $a_l^{\text{hvp}}$ with a complete first-principles QCD calculation and eliminate this source of ambiguity in the current $3\sigma$ discrepancy in $a_\mu$.

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