Asymptotic behavior of oscillating cyclic integrals

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"The Riemann-Hilbert approach to a generalized sine kernel", Comm. Math. Phys. **291**, 691-761, (2009).
N. Kitanine, K. K. Kozlowski, J.M. Maillet, N. A. Slavnov and V. Terras

" Fine structure of the asymptotic expansion of cyclic integrals", J. Math. Phys. 50, 095205, (2009).
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Outline

Examples of multiple integrals

ODE $\rightsquigarrow$ Integral representations for solutions

$$z\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}+(c-z)\frac{\mathrm{d}y}{\mathrm{d}z}-ay=0 \quad \rightsquigarrow \quad \psi(a,c;z)=\frac{1}{\Gamma(a)}\int_0^{+\infty}\mathrm{e}^{-zt}t^{a-1}(1-t)^{c-a-1}\,\mathrm{d}t\,.$$

Multiple integrals provide solutions

- Eigenfunctions of the open Toda chains (Gutzwiller, Kharchev, Lebedev, S.Tian-Shansky).

$$\Psi(\boldsymbol{x})=\int_{\mathscr{C}}\mathrm{d}\gamma\,\mathcal{G}_N(\gamma)\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N-1}(x_n-x_{n+1})\sum_{\ell=1}^{N}\gamma_{n\ell}\right\}\,.$$

- Solution of $q-KZ$ equations (Smirnov).

$$G_n(\beta_1,\dots\beta_{2n})_{\epsilon_1\dots\epsilon_{2n}}=\int_{\mathscr{C}}\mathrm{d}\lambda\,\mathcal{F}^{(2n)}_{\epsilon_1,\dots,\epsilon_{2n}}\begin{pmatrix}\lambda_1,\dots,\lambda_{2n}\\ \beta_1,\dots,\beta_{2n}\end{pmatrix}$$

- Elementary blocks in XXZ: (Jimbo, Miki, Miwa, Nakayashiki, Kitanine, Maillet, Terras). Correlators of integrable models described in terms of multiple integrals.

The understanding of multiple integrals

Integral representations $\equiv$ Special functions

$$\psi\,(a,c;z) \in \mathcal{O}\,(\mathbb{C}\setminus\mathbb{R}_-) \qquad \psi\,(a,c;z) = z^{-a}\left(1+\mathrm{O}\left(z^{-1}\right)\right) \quad z\to\infty\,.$$

Steps of characterization

I Determination of particular values, study of regularity points

- ★ Elementary blocks at small n (Boos, Korepin, Smirnov) and (Sato, Shiroishi, Takahashi).

II Behavior at the irregular points (Asymptotics)

- ★ Saddle-point method, Mellin-Barnes type expansions:

$$\Psi_E\,(\boldsymbol{x}) = \sum_{\sigma\in\mathfrak{S}_N} \phi\,(\boldsymbol{\gamma}_\sigma)\,\mathrm{e}^{\frac{i}{\hbar}(\gamma_\sigma,\boldsymbol{x})} \qquad \text{for} \quad x_{k+1}\gg x_k$$

- ★ Separation of integrals, determinant analysis:

$$\tau\,(N) = \int_{\mathscr{C}} \mathrm{d}^N\lambda\,\mathcal{F}_N\,(\{\lambda\})\cdot\prod_{i<j}^{N}\left(\lambda_i-\lambda_j\right)^2\cdot\prod_{j=1}^{N}\mathrm{e}^{-NV(\lambda_j)} = \mathrm{det}_N\left[c_{i-j}\right] \quad \text{when}\ \Delta=0$$

- No method work when **too much coupling** in the integrand/ **singular** functions.

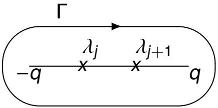
Oscillating cyclic integrals for the XXZ chain

- Generating function of current-current correlators in non-linear Schrödinger model

$$\langle e^{\beta Q_x} \rangle = \sum_{n=0}^{+\infty} \frac{1}{p!} \sum_{\{\ell_j\}} \mathfrak{I}_{\ell_1}^{(x)} \otimes \cdots \otimes \mathfrak{I}_{\ell_s}^{(x)} \left[\mathcal{F}_p \right]$$

- $\mathfrak{I}_n^{(x)}$ cyclic integral in $2n$ variables

$$\mathfrak{I}_n^{(x)} \left[\mathcal{G}_n \right] = \oint_{\Gamma} \frac{d^n z}{(2i\pi)^n} \int_{-q}^{q} \frac{d^n \lambda}{(2i\pi)^n} \mathcal{G}_n \begin{pmatrix} \lambda_1, \dots, \lambda_n \\ z_1, \dots, z_n \end{pmatrix} \prod_{j=1}^{n} \frac{\exp\left\{ix\left[\lambda_j - z_j\right]\right\}}{\left(\lambda_j - z_j\right)\left(\lambda_{j+1} - z_j\right)}$$



★ Poles $z_j = \lambda_j$ and $z_j = \lambda_{j+1} \Rightarrow$ Combinatorics

★ Denominator $\dfrac{1}{\lambda_{j+1} - \lambda_j} \Rightarrow$ Asymtptotic analysis in λ.

Example : $\mathcal{G}_n$ independent of $\{z\}$:

$$\mathfrak{I}_n^{(x)} \left[\mathcal{G}_n \right] = \int_{-q}^{q} d^n \lambda \, \mathcal{G}_n \left(\lambda_1, \dots, \lambda_n \right) \; V\left(\lambda_1, \lambda_2\right) V\left(\lambda_2, \lambda_3\right) \dots V\left(\lambda_n, \lambda_1\right)$$

cycle of kernels $\qquad V\left(\lambda, \mu\right) = \dfrac{e^{i\frac{x}{2}(\lambda-\mu)} - e^{-i\frac{x}{2}(\lambda-\mu)}}{2i\pi\left(\lambda - \mu\right)}$

Asymptotics behavior of cyclic integrals

- The $x \to +\infty$ asymptotic behavior of oscillating cyclic integrals

$$\mathfrak{I}_n^{(x)}[\mathcal{G}_n] = x \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi} \mathcal{G}_n\begin{pmatrix} \lambda, \dots, \lambda \\ \lambda, \dots, \lambda \end{pmatrix} + \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2i\pi} \frac{\partial}{\partial \epsilon} \mathcal{G}_n\begin{pmatrix} \lambda, \dots, \lambda \\ \lambda + \epsilon, \lambda, \dots, \lambda \end{pmatrix}_{\epsilon=0}$$

$$+ b_n \left\{ \mathcal{G}_n\begin{pmatrix} q, \dots, q \\ q, \dots, q \end{pmatrix} + \mathcal{G}_n\begin{pmatrix} -q, \dots, -q \\ -q, \dots, -q \end{pmatrix} \right\} \ln x + H_0[\mathcal{G}_n] + C[\mathcal{G}_n]$$

- The corrections $C[\mathcal{G}_n]$ computable order-by-order
- General structure

$$C[\mathcal{G}_n] \equiv \sum_{\ell \geq 1} \frac{1}{x^\ell} H_\ell^{(\mathrm{no})}(\ln x)[\mathcal{G}_n] + \mathrm{e}^{ix2q} \sum_{\ell \geq 2} \frac{1}{x^\ell} H_\ell^{(\mathrm{osc})}(\ln x)[\mathcal{G}_n] + \dots$$

- Application to the long-distance asymptotics of current-current correlators.

Fredholm determinants and cyclic integrals

- Fredholm determinants are generating functions of multiple integrals

$$\ln\det\left[I+\gamma K\right]=\gamma\int_{-q}^{q}K\left(\lambda,\lambda\right)\mathrm{d}\lambda+\dots+\left(-1\right)^{n}\frac{\gamma^{n}}{n}\int_{-q}^{q}\mathrm{d}^{n}\lambda\, K\left(\lambda_1,\lambda_2\right)\dots K\left(\lambda_n,\lambda_1\right)+\dots$$

$$\frac{\left(-1\right)^{n-1}}{\left(n-1\right)!}\cdot\frac{\partial^{n}}{\partial\gamma^{n}}\ln\det\left[I+\gamma K\right]_{|\gamma=0}=\int_{-q}^{q}\mathrm{d}^{n}\lambda\, K\left(\lambda_1,\lambda_2\right)\dots K\left(\lambda_n,\lambda_1\right)$$

More general cyclic integrals pure product *symmetric* functions:

$$\mathcal{F}_n\left(\{\lambda\}_1^n\right)=\prod_{p=1}^{n}\varphi\left(\lambda_p\right) \quad .$$

Still reduces to a Fredholm determinant

$$\int_{-q}^{q}\mathrm{d}^{n}\lambda\left\{\prod_{p=1}^{n}\varphi\left(\lambda_p\right)\right\}\cdot K\left(\lambda_1,\lambda_2\right)\dots K\left(\lambda_n,\lambda_1\right)=\int_{-q}^{q}\mathrm{d}^{n}\lambda\cdot\prod_{p=1}^{n}\left\{\varphi\left(\lambda_p\right)K\left(\lambda_p,\lambda_{p+1}\right)\right\}$$

$$=\frac{\left(-1\right)^{n-1}}{\left(n-1\right)!}\partial_{\gamma}^{n}\ln\det\left[I+\gamma\,\varphi\,K\right]_{|\gamma=0}$$

Expanding of coupled cyclic integrals

- Coupled cyclic integrals of *symmetric* functions:

$$\mathfrak{I}_n\left[\mathcal{F}_n\right]=\int_{-q}^{q}\mathrm{d}^n\lambda\,\mathcal{F}_n\left(\{\lambda\}_1^n\right)\prod_{p=1}^{n}K\left(\lambda_p,\lambda_{p+1}\right)$$

Density theorem:

☞ $\mathcal{F}_n$ *symmetric* recasts as a uniformly convergent sequence of pure product functions

$$\mathcal{F}_n\left(\{\lambda\}_1^n\right)=\sum_{\ell=1}^{+\infty}\prod_{p=1}^{n}\varphi_\ell\left(\lambda_p\right)$$

Formally $$\mathfrak{I}_n\left[\mathcal{F}_n\right]=\frac{(-1)^{n-1}}{(n-1)!}\sum_{\ell=1}^{+\infty}\partial_\gamma^n\ln\det\left[I+\gamma\,\varphi_\ell\,K\right]_{|\gamma=0}$$

Two ingredients are necessary

- manageable expression for $\ln\det\left[I+\gamma\varphi_\ell K\right]$;
- compute the sum over ℓ;
- Check that $\mathrm{O}\left(x^{-N}\right)$ remain stable.

Cyclic integrals

- Asymptotic analysis of cyclic integrals

$$\mathfrak{I}_n^{(x)}\left[\mathcal{F}_n\right] = \oint_{\Gamma} \frac{\mathrm{d}^n z}{(2i\pi)^n} \int_{-q}^{q} \frac{\mathrm{d}^n \lambda}{(2i\pi)^n} \mathcal{F}_n\begin{pmatrix} \{\lambda\}_1^n \\ \{z\}_1^n \end{pmatrix} \prod_{j=1}^{n} \frac{\exp\left\{ix\left[\lambda_j - z_j\right]\right\}}{\left(\lambda_j - z_j\right)\left(\lambda_{j+1} - z_j\right)}$$

the pure product case

$$\mathcal{F}_n\begin{pmatrix} \{\lambda\}_1^n \\ \{z\}_1^n \end{pmatrix} = \prod_{p=1}^{n} \varphi\left(\lambda_p\right) \mathrm{e}^{g(z_p)}$$

Simple combinatorics of poles $\Rightarrow$ cyclic integral of a generalized sine kernel

$$\mathfrak{I}_n^{(x)}\left[\prod_{p=1}^{n} \varphi\left(\lambda_p\right)\mathrm{e}^{g(z_p)}\right] = \int_{-q}^{q} \mathrm{d}^n\lambda \prod_{j=1}^{n} V_{(F,g)}^{(x)}\left(\lambda_j, \lambda_{j+1}\right) = \frac{(-1)^{n-1}}{(n-1)!} \partial_\gamma^n \ln\det\left[I + \gamma\, V_{(F,g)}^{(x)}\right]_{|\gamma=0}$$

$$V_{(F,g)}^{(x)}(\lambda,\mu) = \sqrt{F(\lambda)\,F(\mu)}\; \frac{\mathrm{e}^{ix\frac{\lambda-\mu}{2} + \frac{g(\lambda)-g(\mu)}{2}} - \lambda \leftrightarrow \mu}{2i\pi(\lambda-\mu)} \qquad F(\lambda) = \varphi(\lambda)\,\mathrm{e}^{g(\lambda)}$$

general case $\Longrightarrow$ density of pure product function

$$\mathcal{F}_n\left(\begin{array}{c}\{\lambda\}_1^n\\ \{z\}_1^n\end{array}\right)=\sum_{\ell=1}^{+\infty}\prod_{p=1}^{n}\varphi_\ell\left(\lambda_p\right)\mathrm{e}^{g_\ell(z_p)}$$

☞ $$\mathfrak{I}_n^{(x)}\left[\mathcal{F}_n\right]=\frac{(-1)^{n-1}}{(n-1)!}\sum_{\ell=1}^{+\infty}\partial_\gamma^n \ln\det\left[I+\gamma\, V_{(F_\ell,g_\ell)}^{(x)}\right]|_{\gamma=0} \qquad F(\lambda)=\varphi_\ell(\lambda)\,\mathrm{e}^{g_\ell(\lambda)}$$

Main steps to obtain the large x asymptotics

- Analyse the x asymptotic behavior of a generalized sine kernel determinant

$$\ln\det\left[I+V_{(F,g)}^{(x)}\right]=\int_{-q}^{q}\{x-ig'(\lambda)\}\ln\left[1+\gamma F(\lambda)\right]\frac{\mathrm{d}\lambda}{2\pi}+\mathrm{O}\left(\ln x\right) \quad ;$$

- Compute the γ derivatives in asymptotic formula, Compute $\sum_{\ell=1}^{+\infty}$.

The leading Asymptotics

- Density procedure for the leading term
  - Take n^{th} derivative of the leading asymptotics of $\ln\det\left[I+\gamma V^{(x)}_{(F,g)}\right]$

$$x\int_{-q}^{q}\ln\left[1+\gamma F(\lambda)\right]\frac{d\lambda}{2\pi} \quad \underset{\frac{(-1)^{n-1}}{(n-1)}\partial_\gamma^n}{\longrightarrow} \quad x\int_{-q}^{q}F^n(\lambda)\frac{d\lambda}{2\pi}$$

  - Specialize to density case for $\mathcal{F}_n$: $\quad F(\lambda)\rightarrowtail F_\ell(\lambda)=\varphi_\ell(\lambda)\,e^{g_\ell(\lambda)}$
  - Perform the summation over ℓ

$$\mathfrak{I}_n^{(x)}\left[\mathcal{F}_n\right]=\frac{(-1)^{n-1}}{(n-1)!}\sum_{\ell=1}^{+\infty}\partial_\gamma^n\ln\det\left[I+\gamma\, V^{(x)}_{(F_\ell,g_\ell)}\right]\Big|_{\gamma=0}\sim x\sum_{\ell=1}^{+\infty}\int_{-q}^{q}\frac{d\lambda}{2\pi}\left\{\varphi_\ell(\lambda)\,e^{g_\ell(\lambda)}\right\}^n$$

  - Recognize the function $\mathcal{F}_n$

$$\mathcal{F}_n\begin{pmatrix}\{\lambda\}_1^n\\ \{z\}_1^n\end{pmatrix}=\sum_{\ell=1}^{+\infty}\prod_{p=1}^{n}\varphi_\ell(\lambda_p)\,e^{g_\ell(z_p)}\Rightarrow\sum_{\ell=1}^{+\infty}\overbrace{\left\{\varphi_\ell(\lambda)\,e^{g_\ell(\lambda)}\right\}^n}^{\prod_{p=1}^{n}\varphi_\ell(\lambda)e^{g_\ell(\lambda)}}=\mathcal{F}_n\begin{pmatrix}\overbrace{\{\lambda,\dots,\lambda\}}^{n}\\ \underbrace{\{\lambda,\dots,\lambda\}}_{n}\end{pmatrix}$$

$$\mathfrak{I}_n^{(x)}\left[\mathcal{F}_n\right]\sim x\int_{-q}^{q}\mathcal{F}_n\begin{pmatrix}\overbrace{\{\lambda,\dots,\lambda\}}^{n}\\ \underbrace{\{\lambda,\dots,\lambda\}}_{n}\end{pmatrix}\frac{d\lambda}{2\pi}$$

Handling of the derivatives and remainders

- Handling of derivative terms

$$\int_{-q}^{q} g_\ell{}'(\lambda) \ln\left(1+\gamma F(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi} \quad \underset{\frac{(-1)^{n-1}}{(n-1)}\partial_\gamma^n}{\longrightarrow} \quad \int_{-q}^{q} g_\ell{}'(\lambda)\, \mathrm{e}^{n g_\ell(\lambda)} \varphi_\ell{}^n(\lambda) \frac{\mathrm{d}\lambda}{2\pi}$$

$$\sum_{\ell=1}^{+\infty} \int_{-q}^{q} g_\ell{}'(\lambda)\, \mathrm{e}^{n g_\ell(\lambda)} \varphi_\ell{}^n(\lambda) \frac{\mathrm{d}\lambda}{2\pi} = \int_{-q}^{q} \sum_{\ell=1}^{+\infty} \varphi_\ell(\lambda)\, \partial_\epsilon \mathrm{e}^{g_\ell(\lambda+\epsilon)} |_{\epsilon=0} \prod_{\ell=2}^{n} \mathrm{e}^{g_\ell(\lambda)} \varphi_\ell(\lambda) \frac{\mathrm{d}\lambda}{2\pi}$$

$$= \int_{-q}^{q} \partial_\epsilon \mathcal{F}_n \Big(\begin{matrix} \overbrace{\{\lambda,\dots,\lambda\}}^{n} \\ \lambda+\epsilon, \underbrace{\{\lambda,\dots,\lambda\}}_{n-1} \end{matrix} \Big)_{\epsilon=0} \frac{\mathrm{d}\lambda}{2\pi}$$

The remainders are stable under summations

GSK asymptotics $\ln\det\left[I + V^{(x)}_{(F,g)}\right] = \mathcal{A}^{(0)}[F,g;x] + \mathcal{R}^{(x)}[F,g]$

Linear structure of the remainder

$$\partial_\gamma^n \mathcal{R}^{(x)}[F,g]_{|\gamma=0} = \oint_{\mathscr{C}_{\pm q}} \mathrm{d}^n z\, \mathcal{R}_n^{(x)}(z_1,\dots,z_n) \cdot \prod_{\ell=1}^{n} \varphi(z_\ell)\, \mathrm{e}^{g(z_\ell)}$$

$$\mathcal{I}_n[\mathcal{F}_n] = \mathcal{H}_n^{(M)}[\mathcal{F}_n] + \mathrm{O}\left(\frac{\ln^n x}{x^{M+1}}\right)$$

The pure sine kernel

Integral operator on $[\,-1\,;\,1\,]$ with kernel $S^{(x)}(\lambda,\mu) = \gamma \dfrac{\sin x\,(\lambda-\mu)}{\pi\,(\lambda-\mu)}$

Some applications

- gap probability in random matrix theory (Gaudin, Mehta);
- correlation functions : XX0, imp. Bose gas (Colomo, Izergin, Korepin, Tognetti, Its, Varzugin);

Investigation of the Asymptotics

$$\ln\det\left[I + S^{(x)}\right]_{\gamma=1} = -\frac{x^2}{2} - \frac{1}{4}\ln x + c_0 + \frac{c_1}{x} + \dots$$

- Heuristic structure of the asymptotics (Gaudin, Mehta, Dyson)
- Leading asymptotics (Widom)
- $\gamma = 1$ asymptotics + constant (Deift, Its, Zhou, Ehrhardt, Krasovsky)
- PV equation for $x\partial_x \ln\det\left[I + S^{(x)}\right]_{\gamma=1}$ (Jimbo, Miwa, Mori, Sato)
- $|\gamma| < 1$ leading asymptotics (Budylin,Buslayev, Deift, Its, Zhou, Cheianov, Zvonarev)
- $|\gamma| < 1$ sub-leading asymptotics (McCoy, Tang, Perk, Schrok)

The generalized sine kernel: Asymptotics

GSK ☞ Integral operator $I+V$ on $[-q\,;q]$

$$V^{(x)}_{F,g}(\lambda,\mu)=\gamma\sqrt{F(\lambda)\,F(\mu)}\,\frac{\sin\{x[\lambda-\mu]/2+i[g(\mu)-g(\lambda)]/2\}}{\pi(\lambda-\mu)}$$

Asymptotic behavior of the GSK determinant

$$\det\left[I+V^{(x)}_{F,g}\right]\sim\mathcal{A}^{(0)}[\nu,g;x]$$

$$\mathcal{A}^{(0)}[\nu,g;x]=-ix\int_{-q}^{q}\nu(\lambda)\,\mathrm{d}\lambda+\left(\nu^2(q)+\nu^2(-q)\right)\ln x-\int_{-q}^{q}\nu(\lambda)\,g'(\lambda)\,\mathrm{d}\lambda+C_1[\nu]+\mathrm{O}(1)$$

$$\nu(\lambda)=\frac{i}{2\pi}\ln\left(1+\gamma F(\lambda)\right)$$

★ The asymptotics *brake* the 1-ν periodicity of $\det\left[I+V^{(x)}_{F,g}\right]$

$$F(\lambda)=\mathrm{e}^{-2i\pi\nu(\lambda)}-1$$

The sub-leading corrections

- Leading asymptotics brake the ν-periodicity of the determinant . However

$$\det\left[I+V_{F,g}^{(x)}\right]=\mathcal{A}^{(0)}\left[\nu,g;x\right]\left(1+\frac{N_1\left(\ln x\right)[\nu]}{x}+\mathrm{o}\left(\frac{1}{x}\right)\right)$$
$$+\mathcal{A}^{(0)}\left[\nu+1,g;x\right]\left(1+\mathrm{o}\left(1\right)\right)+\mathcal{A}^{(0)}\left[\nu-1,g;x\right]\left(1+\mathrm{o}\left(1\right)\right)$$

$$\mathcal{A}^{(0)}\left[\nu+1,g;x\right]=C_{\pm}\left[\nu\right]\frac{\mathrm{e}^{\mp ix2q}}{x^{2\pm(\nu(q)+\nu(-q))+2}}\mathcal{A}^{(0)}\left[\nu,g;x\right]$$

Conjecture

The non oscillating corrections are of the type

$$\mathcal{A}\left[\nu\right]=\mathcal{A}^{(0)}\left[\nu,g;x\right]\left\{1+\frac{N_1\left(\ln x\right)[\nu]}{x}+\dots+\frac{N_M\left(\ln x\right)[\nu]}{x^M}+\dots\right\}$$

The asymptotic expansion restore order by order the lost ν-periodicity

$$\det\left[I+V_{F,g}^{(x)}\right]\sim\mathcal{A}\left[\nu\right]+\mathcal{A}\left[\nu+1\right]+\mathcal{A}\left(\nu-1\right)+\dots+\underbrace{\mathcal{A}\left[\nu+n\right]}_{\mathrm{e}^{-in2qx}}+\mathcal{A}\left(\nu-n\right)+\dots$$

- first sub-leading corrections
- pure sine kernel corrections (McCoy, Tang)

Integrable integral operators

Algebra of operator $I+V$ with "nice" kernel (Its, Izergin, Korepin, Slavnov).

GSK : $$V^{(x)}_{F,g}(\lambda,\mu)=\gamma\sqrt{F(\lambda)F(\mu)}\frac{e_+(\lambda)e_-(\mu)-e_+(\mu)e_-(\lambda)}{2i\pi(\lambda-\mu)}$$ OK ✔

The inverse $I-R$ is in the algebra $\Rightarrow$ $$R(\lambda,\mu)=\gamma\sqrt{F(\lambda)F(\mu)}\frac{f_+(\lambda)f_-(\mu)-f_+(\mu)f_-(\lambda)}{2i\pi(\lambda-\mu)}$$

Construction of $f_\pm$ boils down to a Riemann-Hilbert Problem :

- χ is an analytic 2×2 matrix on $\mathbb{C}\setminus[-q;q]$,
- $\chi_+G=\chi_-$, $\quad G=\begin{pmatrix}1-\gamma F & \gamma F e_+^2\\ -\gamma F e_-^2 & 1+\gamma F\end{pmatrix}$

$$\chi(z)\begin{pmatrix}e_+(z)\\ e_-(z)\end{pmatrix}=\begin{pmatrix}f_+(z)\\ f_-(z)\end{pmatrix}$$

RHP data $\leftarrow$ $$\begin{cases}\partial_\gamma \ln\det\left[I+V^{(x)}_{F,g}\right] = \displaystyle\int_{-q}^{q}\frac{\mathrm{d}\lambda}{\gamma}R(\lambda,\lambda)\\ \partial_x \ln\det\left[I+V^{(x)}_{F,g}\right] = \displaystyle\oint_\Gamma\frac{\mathrm{d}\lambda}{4\pi}\,\mathrm{tr}\left[\partial_\lambda\chi(\lambda)\,\sigma^z\chi^{-1}(\lambda)\right]\end{cases}$$

Asympototic analysis: Non-linear steepest descent (Deift, Zhou)

★ find a set of transformations leading to a jump matrix $I_2+\mathrm{o}(1)$ uniformly.

Transformations of the Riemann-Hilbert problem

$$(\chi, G) \quad \hookrightarrow \quad \left(\Xi, G_{\xi} = M_{+} M_{-}\right) \quad \hookrightarrow \quad \left(\Upsilon, G_{\Upsilon} = M_{+} \text{ or } M_{-}\right)$$

$$M_{+} = \begin{pmatrix} 1 & A(\lambda)\left(\dfrac{\lambda - q}{\lambda + q}\right)^{-2\nu(\lambda)} e^{ix\lambda} \\ 0 & 1 \end{pmatrix} \qquad M_{-} = \begin{pmatrix} 1 & 0 \\ B(\lambda)\left(\dfrac{\lambda - q}{\lambda + q}\right)^{-2\nu(\lambda)} e^{-ix\lambda} & 1 \end{pmatrix}$$

The Parametrix

New variables $\zeta = x(\lambda + q)$

- $\mathcal{P}$ analytic outside of $\Gamma_+ \cup \Gamma_- \cup \mathbb{R}_+$ and $\mathcal{P}(\zeta) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \zeta^{-\nu(-q)\,\sigma_3}$
- $\mathcal{P}(\zeta) = (I_2 + \mathrm{o}(1))\, \mathrm{e}^{i\sigma_3 \frac{\zeta}{2}} \zeta^{\sigma_3 \nu(\zeta/x)}$, uniformly for $\zeta \in \partial D_{-q,x\delta}$,
- $\begin{cases} \mathcal{P}_+(\zeta)\begin{pmatrix} 1 & A(\nu(\zeta/x)) \\ 0 & 1 \end{pmatrix} = \mathcal{P}_-(\lambda) & \text{for } \zeta \in \Gamma_+ \\ \mathcal{P}_+(\zeta)\begin{pmatrix} 1 & 0 \\ B(\nu(\zeta/x)) & 1 \end{pmatrix} = \mathcal{P}_-(\lambda) & \text{for } \zeta \in \Gamma_- \end{cases}$ $, \mathcal{P}_+(\zeta) = \mathcal{P}_-(\zeta)\, \mathrm{e}^{2i\pi\nu(\zeta/x)\sigma_3}$ for $\zeta \in \mathbb{R}_+$.

Constant Case

Solution by differential equation method (Its) for constant jump matrices $\nu(\lambda) \equiv \nu_0 = Const$.

$\dfrac{\mathrm{d}\mathcal{P}}{\mathrm{d}\zeta} \times \mathcal{P}^{-1}$ has no cuts $\oplus$ behavior at $\zeta \to \infty$ $\oplus$ pole at $\zeta = 0$

⋙ *Fuchsian differential equation* $\Longrightarrow$ Solution $\mathcal{P}(\nu_0; \zeta)$

General Case

$\Longrightarrow$ Add the ζ dependence on ν by hand $\mathcal{P}(\nu(\zeta/x); \zeta)$

Conclusion

Review of the results

- ✔ New approach to Asymptotics of certain multiple integrals ;
- ✔ Conjecture on structure of corrections in asymptotics of Fredholm determinants ;
- ✔ Construction of a new type of parametrix .

Next possible targets

- ✓ Prove the conjecture ;
- ✓ Differential equations for GSK Fredholm determinant ;
- ✓ Analyse other multiple integrals to other cases (Scalar products in SoV) .