Four-loop quark mass and field renormalization in the on-shell scheme and renormalons

Peter Marquard

DESY

in collaboration with

M. Beneke, P. Nason, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, D. Wellmann

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Outline

1. Introduction + Method
2. $\overline{\mathrm{MS}}$ – on-shell relation
3. Muon anomalous magnetic moment
4. wave function renormalization: Z_2^{OS}
5. Renormalons and the ultimate uncertainty of the top quark pole mass
6. Conclusions

Outline

Introduction

- Quark masses can be measured with high precision using different renormalization schemes
  $\Rightarrow$ have to be able to translate between them
- pole masses suffer from (infrared) renormalon problems
  $\Rightarrow$ try to quantify the effect
- Renormalization constants are on there own fundamental quantities of the given theory

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i\, Z_2}{\not{q} - Z_m m + \Sigma(q, m)}$$

with $\Sigma(q, m)$ the quark two-point function.
For the $\overline{\mathrm{MS}}$ scheme we require

$$S_F(q) \quad \text{finite}$$

and for the on-shell scheme we require a pole at the position of the mass

$$S_F(q) \stackrel{q^2 \to M^2}{\longrightarrow} \frac{-i}{\not{q} - M}$$

Setup of the calculation

expand the quark two-point function

$$\Sigma(q,M) \approx M\,\Sigma_1(M^2,M) + (\not{q} - M)\left(2M^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\right) + \dots$$

renormalization constants are then given by

$$Z_m^{\mathrm{OS}} = 1 + \Sigma_1(M^2,M)\,,$$

$$\left(Z_2^{\mathrm{OS}}\right)^{-1} = 1 + 2M^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\,.$$

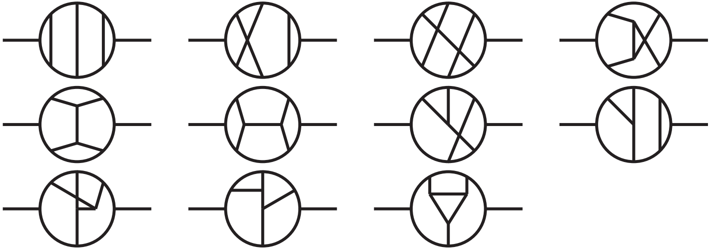
best to apply the projector

$$\mathrm{Tr}\left\{\frac{\not{Q} + M}{4M^2}\Sigma(q,M)\right\} = \Sigma_1(q^2,M) + t\Sigma_2(q^2,M)$$

$$= \Sigma_1(M^2,M) + \left(2M^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\right) t$$

Setup of the calculation cont'd

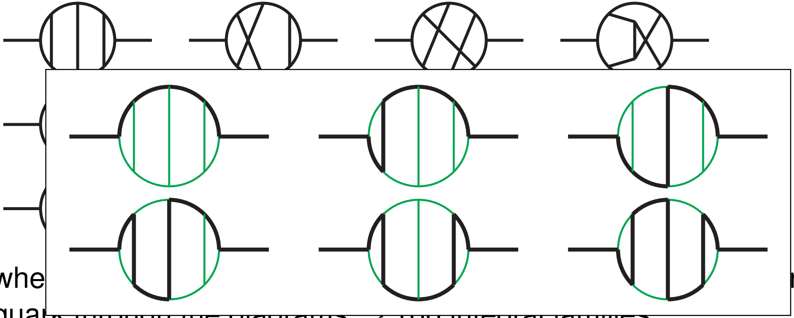
Need to calculate 4-loop on-shell diagrams of the form



where we have to consider all possible ways to route the external quark through the diagrams $\Rightarrow$ 100 integral families.

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form



whe … nal

quark through the diagrams ⇒ 100 integral families.

Setup of the calculation

Follow the *standard* procedure for multi-loop calculations

- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using `FIRE` [Smirnov] or `Crusher` [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)[`MB.m` [Czakon] `FIESTA` [Smirnov]]

Outline

Setup of the calculation

- Need to calculate mass renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\text{MS}}$-scheme $Z_m^{\overline{\text{MS}}}$

  [Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97; Baikov,Chetyrkin,Kühn '14]

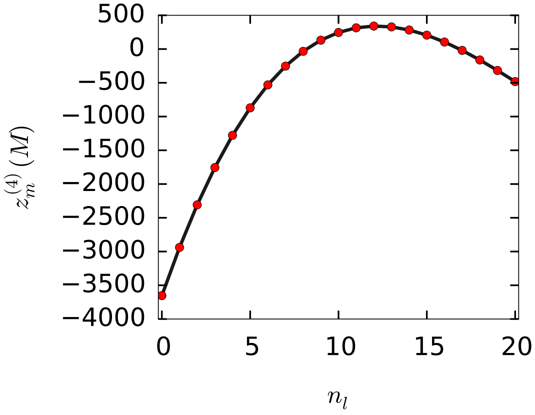
  we get for the relation between the two schemes

$$\left.\begin{aligned} m_{\text{bare}} &= Z_m^{\text{OS}} M \\ m_{\text{bare}} &= Z_m^{\overline{\text{MS}}} m \end{aligned}\right\} \Rightarrow m = M \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}}$$

  - 1-loop [Tarrach'81]
  - 2-loop [Gray,Broadhurst,Grafe,Schilcher'90]
  - 3-loop [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

$\overline{\text{MS}}$–on-shell relation at four-loop order

$$
\begin{aligned}
z_m^{(4)} &= -3654.15 \pm 1.64 + (756.942 \pm 0.040) n_l \\
&\quad -43.4824 n_l^2 + 0.678141 n_l^3 \, .
\end{aligned}
$$



$\overline{\mathrm{MS}}$–on-shell relation at four-loop order

$\overline{\mathrm{MS}} \to$ on-shell

$$
\begin{aligned}
m_t(m_t) &= M_t \big(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \\
&\quad -(8.949 \pm 0.018)\,\alpha_s^4\big) \\
&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \mathrm{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \to$ on-shell

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\begin{aligned}
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&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$$
\begin{aligned}
M_b &= m_b(m_b)\,\big(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 \\
&\quad +(12.685 \pm 0.025)\,\alpha_s^4\big) \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

Light quark mass dependence

- starts at 2 loops [Gray,Broadhurst,Grafe,Schilcher '90]
- available up to 3 loops [Bekavac,Grozin,Seidel,Steinhauser '07]
- for top quark
  - 11 MeV @ two loop
  - 16 MeV @ three loop
  - getting more important at higher orders due to renormalon enhancement

Threshold masses, e.g. PS mass

$$m^{\mathrm{PS}} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$

need static potential @ three loops

[Smirnov,Smirnov,Steinhauser '09; Anzai ,Kiyo,Sumino '09]

$$\begin{aligned} m_t^{\mathrm{PS}}(\mu_f = 80\text{ GeV}) &= 163.508 + (7.531 - 3.685) \\ &\quad +(1.607 - 0.989) + (0.495 - 0.403) \\ &\quad +(0.195 - 0.211 \pm 0.0004)\text{ GeV} \\ &= 163.508 + 3.847 + 0.618 + 0.092 \\ &\quad -(0.016 \pm 0.0004)\text{ GeV} \end{aligned}$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

Outline

Muon anomalous magnetic moment: Overview

$$a_\mu = \frac{g-2}{2}$$

has been measured and predicted with extreme accuracy

$$\begin{aligned} a_\mu|_{\text{exp}} &= 116592080(54)(33)[63] \cdot 10^{-11} \\ a_\mu|_{\text{theo}} &= 116591790(65) \cdot 10^{-11} \qquad \approx 3\sigma \text{ diff.} \end{aligned}$$

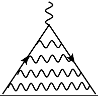
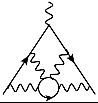
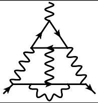
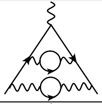
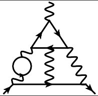
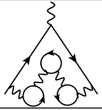
The four-loop contribution

$$a_\mu^{(8)} = (-1.912\,98 + 132.790\,3|_{e,\tau}) \left(\frac{\alpha}{\pi}\right)^4 \approx 381 \times 10^{-11} \,,$$

is of the same order as the difference

$\Rightarrow$ Need for independent calculations

Universal contributions to muon g-2

	-2.1755 ± 0.0020 -2.161 ± 0.065 $-2.176866027739540077443259355895893938670$
	0.05596 ± 0.0001 0.077 ± 0.031 $0.056110899897828364831469274418908842233$
	-0.3162 ± 0.0002 -0.3048 ± 0.021 $-0.316538390648940158843260382381513284828$
	-0.074665 ± 0.000006 -0.07461 ± 0.00008 $-0.074671184326105513860159965722793126809$
	0.598838 ± 0.000019 0.597204 ± 0.0012 $0.598842072031421820464649513201747727836$
	$0.000876865858889990697913748939713726165$ $0.000876865858889990697913748939713726165$ $0.000876865858889990697913748939713726165$

Muon g-2 @ four loops

$$
\begin{array}{lllll}
 & \text{universal} & e^- & \tau & e^- + \tau \\
a_\mu^{(8)} = & -1.912\,98(84) & +\,132.6852(60) & +\,0.04234(12) & +\,0.06272(4) \\
a_\mu^{(8)} = & -1.87(12) & +\,132.86(48) & +\,0.0424941(53) & +\,0.062722(10) \\
a_\mu^{(8)} = & -1.9122457649264\ldots & & &
\end{array}
$$

multiplying with $(\alpha/\pi)^4$ this becomes ($\times 10^{-11}$)

$$
\begin{array}{lllllll}
a_\mu^{(8)} = (& -5.56894(245) & +\,386.264(17) & +\,0.12326(35) & +\,0.18259(12) &) \\
a_\mu^{(8)} = (& -5.44(35) & +\,386.77(1.40) & +\,0.12371(15) & +\,0.182592(29) &) \\
a_\mu^{(8)} = (& -5.566798937\ldots & +\ldots & & &)
\end{array}
$$

comparing [Kinoshita et al] [Kurz et al] [Laporta]

- 3 sufficiently precise calculations of the universal part
- 2 for the remaining fermionic contributions

Outline

Overview

- final missing building block to complete the on-shell renormalization procedure of QCD at four loops.
- gauge dependence starts only at 3 loops

[Broadhurst,Gray,Schilcher]

[Melnikov,van Ritbergen;Marquard,Mihaila,Piclum,Steinhauser]

- computationally more involved due to additional derivative and gauge dependence
- N.B.: all results are preliminary

4-loop results, $N_c = 3$

ξ^0

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
n_l^0	-1.7728 ± 0.0034	-27.666 ± 0.026	33016.75 ± 0.17	4395746.8 ± 1.2	74527697.2 ± 8.3
n_l^1	0.46098 ± 0.00043	6.6913 ± 0.0029	74.655 ± 0.017	696.663 ± 0.084	6174.30 ± 0.41
n_l^2	-0.039931 ± 0	-0.51572 ± 0	-5.5055 ± 0	-48.777 ± 0	-418.93 ± 0
n_l^3	0.0011574 ± 0	0.012539 ± 0	0.12676 ± 0	1.0711 ± 0	8.916 ± 0

ξ^1

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
n_l^0	-0.01858 ± 0.00023	0.0344 ± 0.0016	-0.0574 ± 0.0090	5.226 ± 0.047	36.81 ± 0.22
n_l^1	0.0017361 ± 0	-0.0052085 ± 0	0.02242 ± 0.00000	-0.34864 ± 0.00003	-1.6111 ± 0.0002

ξ^2

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
n_l^0	0.0000 ± 0.0001	0.00190 ± 0.00044	-0.0302 ± 0.0023	-0.188 ± 0.012	-2.926 ± 0.052
n_l^1	0 ± 0	0 ± 0	0 ± 0	-0.00000 ± 0.00000	-0.00000 ± 0.00001

Z_2: General colour structure

$$
\begin{aligned}
\delta Z_2^{(4)} \quad = \quad & C_F^4\,\delta Z_2^{FFFF} + C_F^3 C_A\,\delta Z_2^{FFFA} + C_F^2 C_A^2\,\delta Z_2^{FFAA} + C_F C_A^3\,\delta Z_2^{FAAA} \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_c}\,\delta Z_2^{d_{FA}} + n_l \frac{d_F^{abcd} d_F^{abcd}}{N_c}\,\delta Z_2^{d_{FF}L} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c}\,\delta Z_2^{d_{FF}H} \\
& + C_F^3 T n_l\,\delta Z_2^{FFFL} + C_F^2 C_A T n_l\,\delta Z_2^{FFAL} + C_F C_A^2 T n_l\,\delta Z_2^{FAAL} \\
& + C_F^2 T^2 n_l^2\,\delta Z_2^{FFLL} + C_F C_A T^2 n_l^2\,\delta Z_2^{FALL} + C_F T^3 n_l^3\,\delta Z_2^{FLLL} \\
& + C_F^3 T n_h\,\delta Z_2^{FFFH} + C_F^2 C_A T n_h\,\delta Z_2^{FFAH} + C_F C_A^2 T n_h\,\delta Z_2^{FAAH} \\
& + C_F^2 T^2 n_h^2\,\delta Z_2^{FFHH} + C_F C_A T^2 n_h^2\,\delta Z_2^{FAHH} + C_F T^3 n_h^3\,\delta Z_2^{FHHH} \\
& + C_F^2 T^2 n_l n_h\,\delta Z_2^{FFLH} + C_F C_A T^2 n_l n_h\,\delta Z_2^{FALH} + C_F T^3 n_l^2 n_h\,\delta Z_2^{FLLH} \\
& + C_F T^3 n_l n_h^2\,\delta Z_2^{FLHH}
\end{aligned}
$$

$\Rightarrow$ 23 colour structures

Example for individual colour structures

hard 4-loop contribution

$$\begin{aligned}\delta Z_2^{FFFF,\mathrm{unren}} &= (-2.73194 \pm 0.00025)\epsilon^{-4} + (1.9450 \pm 0.0019)\epsilon^{-3} \\ &+(-22.265 \pm 0.017)\epsilon^{-2} + (93.41 \pm 0.16)\epsilon^{-1} \\ &+(167.9 \pm 1.5)\epsilon^{0}\end{aligned}$$

including mass counter term

$$\begin{aligned}\delta Z_2^{FFFF,\mathrm{ren}} &= (0.01317 \pm 0.00025)\epsilon^{-4} + (0.0836 \pm 0.0019)\epsilon^{-3} \\ &+(-0.084 \pm 0.017)\epsilon^{-2} + (-1.96 \pm 0.16)\epsilon^{-1} \\ &+(-4.1 \pm 1.5)\epsilon^{0}\end{aligned}$$

rather large cancellations

Outline

$\overline{\text{MS}}$–on-shell relation beyond 4 loops

$$m_P = m(\mu_m)\left(1 + \sum_{n=1}^{\infty} c_n(\mu, \mu_m, m(\mu))\, \alpha_s^n(\mu)\right)$$

for large n

$$c_n(\mu, \mu_m, m(\mu_m)) \underset{n\to\infty}{\longrightarrow} N c_n^{(\mathrm{as})}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)}\, \tilde{c}_n^{(\mathrm{as})}\,,$$

[Beneke,Braun '94;Beneke '94 '99]

where

$$\tilde{c}_{n+1}^{(\mathrm{as})} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right).$$

b_0, b, s_1, s_2: Combinations of coefficients of the β-function.

How well does the asypt. formula work?

Fit N to 1, . . . , 4-loop term and compare

n_l	$c_1/c_1^{(\mathrm{as})}$	$c_2/c_2^{(\mathrm{as})}$	$c_3/c_3^{(\mathrm{as})}$	$c_4/c_4^{(\mathrm{as})}$	Δ_{34}
-1000000	0.6953	0.9624	0.9349	0.9714	0.038
-10	0.4744	0.7152	0.6898	0.7005	0.015
0	0.4377	0.6357	0.6130	0.5977	0.025
3	0.3954	0.6150	0.5723	0.5370	0.064
4	0.3633	0.6120	0.5522	0.5056	0.088
5	0.3143	0.6119	0.5244	0.4616	0.127
6	0.2436	0.6089	0.4818	0.3942	0.200
7	0.1474	0.5378	0.4084	0.2786	0.378
8	0.0098	0.0379	0.2719	0.0564	1.312
10	0.2684	-0.0916	-0.1108	-1.7228	1.758

$$\Delta_{34} = 2\,\frac{|c_3/c_3^{(\mathrm{as})} - c_4/c_4^{(\mathrm{as})}|}{|c_3/c_3^{(\mathrm{as})} + c_4/c_4^{(\mathrm{as})}|}.$$

Beyond 4 loops

Fit N to 4-loop term and take higher orders from asymptotic formula

j	$\tilde{c}_j^{(\mathrm{as})}$	$\tilde{c}_j^{(\mathrm{as})}\alpha_s^j$
5	0.985499×10^2	0.001484
6	0.641788×10^3	0.001049
7	0.495994×10^4	0.000880
8	0.443735×10^5	0.000854
9	0.451072×10^6	0.000942
10	0.513535×10^7	0.001164

Asymptotic series $\Rightarrow$ converges only up to $\approx$ 8 loop

5+ loops and remaining ambiguity

An asymptotic series f can (sometimes) be summed by using the Borel transform $B[f]$

$$f(\alpha_s) = \sum_{n=0}^{\infty} c_n \alpha_s^n \qquad \Rightarrow \qquad B[f](t) = \sum_{n=0}^{\infty} c_{n+1} \frac{t^n}{n!}$$

the Borel integral

$$\int_0^{\infty} \mathrm{d}t \, e^{-t/\alpha_s} B[f](t)$$

has the same series expansion as $f(\alpha_s)$ and the same value.

5+ loops and remaining ambiguity cont'd

In our case we have

$$c_{n+1} = (2b_0)^n n! \qquad \Rightarrow \qquad \int_0^\infty \mathrm{d}t\, e^{-t/\alpha_s} \frac{1}{1-2b_0 t}$$

Not integrable due to pole at $1 - 2b_0 t = 0$
Possible prescription for the integral:
Take principle value and assign ambiguity Im/π

$$\begin{aligned} \delta^{(5+)} m_P \;&=\; 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \\ &\quad \pm 0.010\,(\alpha_s) \pm 0.071 \text{ (ambiguity) GeV} \end{aligned}$$

5+ loops and remaining ambiguity: light mass effects

- take 2-loop and 3-loop mass effects into account
- 4 loop: massless value using five-flavour theory
- 5 loop: massless value using four-flavour theory
- 6+ loops: massless value using three-flavour theory
- final estimate depends on $\Lambda_{\mathrm{QCD}}^{(3)}$

$$\begin{aligned}\delta^{(5+)} m_P &= 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \\ &\quad \pm 0.009\,(\alpha_s) \pm 0.108\ \text{(ambiguity) GeV}\end{aligned}$$

Cmp. analysis by [Hoang,Lepenik, Preisser '17] using different prescription

$$\frac{1}{2} \sum_{c_n < f c_n^{min}} c_n \to 250\mathrm{MeV} \qquad \text{with} \quad f = 5/4$$

Outline

Conclusions

Presented final results for

- $\overline{\mathrm{MS}}$–on-shell relation
- muon anomalous moment
- top quark pole mass ambiguity

and

- preliminary results for Z_2^{OS}