Time scales of thermalization of a strongly interacting non-Abelian plasma

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We use gauge/gravity duality to investigate the thermalization of a strongly interacting plasma, taken out-of-equilibrium by boundary quenching. In particular, we show how information can be obtained from local and nonlocal observables, in particular concerning the time scales of thermalization.

1 Introduction

There are experimental evidences that the plasma produced in ultrarelativistic heavy ion collisions at RHIC and LHC is a realization of a strongly coupled deconfined phase of QCD. It behaves as a perfect fluid, with a small viscosity to entropy density ratio $\eta/s \sim 0(0.1)$, at odds with perturbative QCD which predicts larger values for $\eta/s$. The out-of-equilibrium configuration lasts for a short time, $\mathcal{O}(1 \text{ fm/c})$, followed by a hydrodynamic evolution [1]. The description of the early-time out-of-equilibrium dynamics is a challenging task, which can be faced by holographic methods. Indeed, using gauge/gravity duality [2], the equilibration problem of the 4d strongly interacting non-Abelian system can be mapped into a gravity problem in a 5d space-time [3]. \footnote{Ref. [4] contains a review of applications of holographic methods to the heavy ion collision phenomenology.}

In the gauge/gravity duality framework, thermalization in the boundary gauge theory occurs in correspondence to the formation of a black brane in the higher dimensional space. The equilibration can be monitored through the boundary stress-energy tensor $T_{\mu \nu}$. For a plasma behaving like a perfect fluid the $T_{\mu \nu}$ components obey the Bjorken relations [5]. Within the AdS/CFT correspondence, the dual of the boundary stress-energy tensor is the bulk metric. In particular, the dual of $T_{\mu \nu}$ fulfilling Bjorken hydrodynamics is identified as a black brane metric with time-dependent horizon [6]. In general cases, to determine $T_{\mu \nu}$ one has to solve the bulk Einstein equation, subject to appropriate initial and boundary conditions, and the components of the boundary stress-energy tensor are obtained from of the coefficients of the near-boundary expansion of the gravity metric, by the holographic renormalization procedure [7]. This approach has been applied within different contexts [8].

In Refs. [9] it has been proposed to study thermalization of a strongly coupled plasma after a distortion of the boundary metric (a “quench”), to mimic a perturbation driving the system out-of-equilibrium. The response depends on the profile of the quench, which is expected also considering the nonlinear nature of the dual gravity problem: this motivated a dedicated
investigation of different quenches [10], to assess universal features in the time dependence of
the components of $T_{\mu\nu}$. Moreover, it was possible to establish a hierarchy of the thermalization
times at different length scales in the system [11], studying nonlocal observables and their time
evolution, as proposed in [12]. We shall describe the results from two-point correlation function
of operators defined in the boundary theory, and from the expectation values of Wilson loops
defined on the boundary. Nonlocal probes have also been scrutinized in other contexts [13].

2 Thermalization by boundary sourcing: results from local probes

To study the thermalization of a boost-invariant non-Abelian plasma one can make use of
boundary sourcing, to mimic the way the system is driven out-of-equilibrium [9]. In [10] differ-
ent kinds of quenches in the boundary metric where scrutinized, monitoring the stress-energy
tensor of the boundary theory. The tensor is written as $T_{\mu\nu} = \frac{N^2}{2\pi^2} \text{diag}(-\epsilon, p_\perp, p_\perp, p_\parallel)$, with
components the system energy density $\epsilon$, the pressure $p_\perp$ along one of the two transverse direc-
tions (with respect, e.g., to the heavy ion collision axis) and the pressure $p_\parallel$ in the longitudinal direction.\footnote{Along the paper, energy density and pressures are referred to without considering the factor $N^2/2\pi^2$.} The 4d boundary coordinates are denoted as $x^\mu = (x^0, x^1, x^2, x^3)$, and the $x^3 = x_\parallel$ direction is identified with the collision axis along which the plasma is supposed to expand. Boost-invariance along this axis, as well as translational and $O(2)$ rotational invariances in the transverse plane $x_\perp = \{x^1, x^2\}$ are assumed. In terms of the proper time $\tau$ and the spacetime rapidity $y$, defined through the relations $x^0 = \tau \cosh y$ and $x_\parallel = \tau \sinh y$, the 4d boundary line element reads:

$$ds_4^2 = -d\tau^2 + e^{\gamma(\tau)} dx_\perp^2 + \tau^2 e^{-2\gamma(\tau)} dy^2 ;$$

(1)

this leaves the spatial three-volume unchanged, and respects the translational and $O(2)$ sym-
metries in the transverse plane.

The gravity dual is defined on a 5d spacetime, described adopting the Eddington-Finkelstein
coordinates with a fifth radial coordinate $r$ and having the metric (1) as a boundary. The 5d
line element can be written as

$$ds^2 = -A(r, \tau)d\tau^2 + \Sigma(r, \tau)^2 e^{B(r, \tau)} dx_\perp^2 + \Sigma(r, \tau)^2 e^{-2B(r, \tau)} dy^2 + 2d\tau dr ,$$

(2)

with the boundary obtained for $\tau \to \infty$, and the metric functions $A$, $\Sigma$ and $B$ depending only
on $r$ and $\tau$, due to the symmetries imposed to the system. Such metric function are determined
solving 5d Einstein equations with negative cosmological constant, written in the form [9]:

\[
\Sigma(\dot{\Sigma})' + 2\Sigma'\dot{\Sigma} - 2\Sigma^2 = 0
\]

\[
\Sigma(\dot{B})' + \frac{3}{2} \left( \Sigma'\dot{B} + B'\dot{\Sigma} \right) = 0
\]

\[
A'' + 3B'\dot{B} - 12\frac{\Sigma'\dot{\Sigma}}{\Sigma^2} + 4 = 0
\]

\[
\ddot{\Sigma} + \frac{1}{2} \left( B^2\dot{\Sigma} - A'\dot{\Sigma} \right) = 0
\]

\[
\Sigma'' + \frac{1}{2} B^2\Sigma = 0.
\]

In (3), for a generic function \( \xi(r, \tau) \), the directional derivatives along the infalling radial null geodesics and the outgoing radial null geodesics are denoted by \( \xi' = \partial_r \xi \) and \( \dot{\xi} = \partial_t \xi + \frac{1}{2} A \partial_r \xi \), respectively.

Two boundary conditions need to be imposed. The first one consists in the requirement that the metric (2) reproduces the 4d metric Eq. (1) for \( r \to \infty \). The second condition is that, at the initial time slice \( \tau = \tau_i \) when the quench is switched on, one has the AdS\(_5\) bulk metric:

\[
ds^2 = r^2 \left[ -dr^2 + dx^2_\perp + \left( \tau + \frac{1}{r} \right)^2 dy^2 \right] + 2drd\tau.
\]

In [10] several quench profiles were considered, to understand whether and how thermalization depends on the specific choice of sourcing. Each profile is characterized by a function \( \gamma(\tau) \), representing quenches with different number, structures and intensities, parametrized as

\[
\gamma(\tau) = w \left[ \tanh \left( \frac{\tau - \tau_0}{\eta} \right) \right]^7 + \sum_{j=1}^N \gamma_j(\tau, \tau_{0,j})
\]

with

\[
\gamma_j(\tau, \tau_{0,j}) = c_j f_j(\tau, \tau_{0,j})^6 e^{-1/f_j(\tau, \tau_{0,j})} \Theta \left( 1 - \frac{(\tau - \tau_{0,j})^2}{\Delta_j^2} \right)
\]

and

\[
f_j(\tau, \tau_{0,j}) = 1 - \frac{(\tau - \tau_{0,j})^2}{\Delta_j^2}.
\]

The set of parameters \( w, \eta, \tau_0, \tau_{0,j}, c_j \) and \( \Delta_j \) defines the various quench models: among these, we focus on two models, denoted by \( A(2) \) and \( B \) in Ref.[10]. Model \( A(2) \) has two short pulses in the boundary metric, obtained using the parameters \( w = 0, N = 2, c_1 = 1, \Delta_{1,2} = 1, \tau_{0,1} = \frac{5}{4}\Delta_1, c_2 = 2, \tau_{0,2} = \frac{3}{4}\Delta_2 \). The quench ends at \( \tau_{f,A}^4 = 3.25 \). Model \( B \) represents the superimposition of a short pulse to a slow deformation, and is obtained when the parameters are set to \( w = \frac{2}{5}, \eta = 1.2, \tau_0 = 0.25, N = 1, c_1 = 1, \Delta_1 = 1, \tau_{0,1} = 4\Delta_1 \). The pulse ends at \( \tau_{f,B}^2 = 5 \), even if the slow distortion continues with \( \tau \) and approaches a constant value. In both cases, the quench is switched off at \( \tau_i = 0.25 \). The two profiles \( \gamma(\tau) \) are depicted in Fig. 6.

The metric functions \( A(r, \tau), \Sigma(r, \tau) \) and \( B(r, \tau) \) in Eq. (2) are calculated by solving the Einstein equations (3) with the boundary conditions at the initial time slice and at \( r \to \infty \).
[10]. Using the solutions, several quantities of interest are determined, in particular the thermalization time, comparing the boundary stress-energy tensor to the same quantity computed in the viscous hydrodynamic regime. Notice that, due to homogeneity, boost-invariance and invariance under rotations in the transverse plane, the various components of $T_{\mu}^\nu$ depend only on the proper time $\tau$ [5]. Moreover, imposing that $T_{\mu}^\nu$ is conserved and traceless, its components depend on a single function $f(\tau)$, so that

$$T_{\mu}^\nu = \text{diag} \left( -f(\tau), f(\tau) + \frac{1}{2} \tau f'(\tau), f(\tau) + \frac{1}{2} \tau f'(\tau), -f(\tau) - \tau f'(\tau) \right). \quad (8)$$

For a perfect fluid one has $\epsilon = 3p$ and the relation $p = p_\parallel = p_\perp$: the $\tau$ dependence is hence fixed, $\epsilon(\tau) = \frac{\text{const}}{\tau^{4/3}}$. This dependence includes subleading terms in $1/\tau$ when viscous effects are taken into account [6]. An effective temperature $T_{\text{eff}}(\tau)$ can be defined through the relation $\epsilon(\tau) = \frac{3}{4} \pi^2 T_{\text{eff}}(\tau)^4$, and for $T_{\text{eff}}(\tau)$ the subleading terms in the large-$\tau$ expansion can be computed in $N = 4$ SYM [14]:

$$T_{\text{eff}}(\tau) = \frac{\Lambda}{(\Lambda \tau)^{1/3}} \left[ 1 - \frac{1}{6\pi(\Lambda \tau)^{2/3}} + \frac{1 + \log 2}{36\pi^2(\Lambda \tau)^{4/3}} \left( -21 + 2\pi^2 + 51 \log 2 - 24(\log 2)^2 \right) \right. \right.$$  

$$\left. + \left( \frac{1}{(\Lambda \tau)^{2/3}} \right) \right], \quad (9)$$

with $\Lambda$ a parameter. This expression corresponds to the large $\tau$ dependence for the energy density $\epsilon$, the longitudinal $p_\parallel$ and transverse $p_\perp$ pressures:

$$\epsilon = \frac{3\pi^4 \Lambda^4}{4(\Lambda \tau)^{4/3}} \left[ 1 - \frac{2c_1}{(\Lambda \tau)^{2/3}} + \frac{c_2}{(\Lambda \tau)^{4/3}} + \mathcal{O} \left( \frac{1}{(\Lambda \tau)^2} \right) \right], \quad (10)$$

$$p_\parallel(\tau) = \frac{\pi^4 \Lambda^4}{4(\Lambda \tau)^{4/3}} \left[ 1 - \frac{6c_1}{(\Lambda \tau)^{2/3}} + \frac{5c_2}{(\Lambda \tau)^{4/3}} + \mathcal{O} \left( \frac{1}{(\Lambda \tau)^2} \right) \right], \quad (11)$$

$$p_\perp(\tau) = \frac{\pi^4 \Lambda^4}{4(\Lambda \tau)^{4/3}} \left[ 1 - \frac{c_2}{(\Lambda \tau)^{4/3}} + \mathcal{O} \left( \frac{1}{(\Lambda \tau)^2} \right) \right], \quad (12)$$

with $c_1 = \frac{1}{3\pi}$ and $c_2 = \frac{1 + 2 \log 2}{18\pi^2}$. The large $\tau$ dependence of the pressure ratio $\frac{p_\parallel}{p_\perp}$ and anisotropy $\frac{\Delta p}{\epsilon} = \frac{p_\perp - p_\parallel}{\epsilon}$ can also be worked out. The differences among different quench models are encoded in $\Lambda$. Hence, the components of $T_{\mu}^\nu$, computed from the bulk geometry, can be compared with the asymptotic expressions (10), (11) and (12).

We summarize the results for the quench models $\mathcal{A}(2)$ and $\mathcal{B}$, although all models scrutinized in [10] share common features concerning the horizon formation in the bulk and the evolution towards the hydrodynamic regime of temperature and pressures. However, the equilibration occurs with different features, depending on the observable. The effective temperature and the energy density start to follow their hydrodynamic expressions immediately after the quench is switched off. On other contrary, the two pressures take longer to reach the hydrodynamic forms. This effect can be described introducing a "thermalization time" $\tau_p$, by the condition $\left| \frac{d}{d\tau} \left( \frac{p_\parallel(\tau_p)}{\tau_p} - \frac{p_\parallel(\tau_p)}{\tau_p} \right) H \right| = 0.05$, with $(p_\parallel(\tau_p)/\tau_p) H$ obtained from (11),(12).
In Fig. 1 the metric function $A(r,\tau)/r^2$ is displayed for the two models; the apparent and event horizons are depicted. Fig. 2 shows the time dependence of the temperature, the area of the apparent horizon per unit of rapidity, $\Sigma(r_h,\tau)^3$, and the components of $T^\mu_{\nu}$. Considering the temperature, the relaxation of the system, right after the end of the perturbation follows the hydrodynamic expression. The same conclusion can be inferred looking at the area of the apparent horizon per unit of rapidity, $\Sigma(r_h,\tau)^3$, that reaches a constant value soon after the pulses. As for the components of $T^\mu_{\nu}$, their behaviour after the end of the quench is displayed in Fig. 3, which shows that the time $\tau^*$, at which $\epsilon(\tau^*)$ differs from the corresponding hydrodynamic value $\epsilon_H(\tau^*)$ in (10) by less than 5%, essentially coincides with the end of the perturbation. On the other hand, the thermalization time $\tau_p$ is larger. The difference $\tau_p - \tau^*$ is the time elapsed after the end of the quench to restore the hydrodynamic regime. It can be expressed in physical units introducing a scale in the system, namely imposing $T_{\text{eff}}(\tau^*) = 500$ MeV, with the results collected in Table 1 for various models [10]. The difference $\tau_p - \tau^*$ is always found to be order (or less than) 1 fm/c, a result comparable to the values inferred from the analyses of heavy ion collision phenomenology.

<table>
<thead>
<tr>
<th>model</th>
<th>$\tau^*$</th>
<th>$\tau_p$</th>
<th>$\Delta\tau = \tau_p - \tau^*$ (fm/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (1)</td>
<td>5.25</td>
<td>6.8</td>
<td>2.25</td>
</tr>
<tr>
<td>A (2)</td>
<td>3.25</td>
<td>6.0</td>
<td>1.73</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>6.74</td>
<td>1.12</td>
</tr>
<tr>
<td>C</td>
<td>9.45</td>
<td>10.24</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Table 1: Results from different quench models [10].
Figure 2: For two different quench profiles, models A(2) and B, the panels show (from top to bottom) the profile $\gamma(\tau)$, the temperature $T_{\text{eff}}(\tau)$, the horizon area $\Sigma^3(r_h, \tau)$ per unit of rapidity, and the three components $\epsilon(\tau)$, $p_{\perp}(\tau)$, and $p_{\parallel}(\tau)$ of the stress-energy tensor $T^\mu_\nu$.

3 Nonlocal probes of thermalization

Several nonlocal probes can be computed using the calculated metric functions $A(r, \tau)$, $\Sigma(r, \tau)$ and $B(r, \tau)$ in the geometry (2). The properties of the same observables in the hydrodynamic setup are useful for the comparison. The 5d metric reproducing, through holographic renormalization, the results in (10)-(12) was derived for the Fefferman-Graham [6, 14, 15] and Eddington-Finkelstein coordinates [16]. To connect it with the stress-energy tensor components, the 5d metric dual to viscous hydrodynamics can be written as [11]

$$ds^2 = -A^H(r, \tau)d\tau^2 + [\Sigma^H(r, \tau)]^2e^{B^H(r, \tau)}d\mathbf{x}_\perp^2 + [\Sigma^H(r, \tau)]^2e^{-2B^H(r, \tau)}dy^2 + 2drd\tau , \quad (13)$$
Figure 3: For models $A(2)$ and $B$ the panels show (from top to bottom) the temperature $T_{\text{eff}}(\tau)$, the components $\epsilon(\tau)$, $p_\perp(\tau)$, and $p_\parallel(\tau)$ of the stress-energy tensor, pressure anisotropy $\Delta p/\epsilon = (p_\perp - p_\parallel)/\epsilon$ and ratio $p_\parallel/p_\perp$, computed for $\tau > \tau_{A(2)}^f$. The short and long dashed lines correspond to the hydrodynamic result and to the NNLO result in the $1/\tau$ expansion.

with the metric functions given in terms of the energy density and the pressures:

$$A^H(r, \tau) = r^2 \left( 1 - \frac{4}{3r^4} \epsilon(\tau) \right), \quad \Sigma^H(r, \tau) = r \left( \tau + \frac{1}{r} \right)^{1/3},$$

$$B^H(r, \tau) = \frac{1}{3r^4} (p_\perp(\tau) - p_\parallel(\tau)) - \frac{2}{3} \log \left( \tau + \frac{1}{r} \right).$$

(14)
Among various nonlocal observables, we first consider the equal-time two-point correlation functions, computed in geodesic approximation. The AdS/CFT correspondence establishes a connection between a boundary scalar operator $O(t, \mathbf{x})$ of conformal dimension $\Delta$ in $d$ dimensions and its dual bulk field $\phi(t, \mathbf{x}, r)$ with mass $m$ in $(d + 1)$ dimensions, with $\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$. In the strong coupling regime of the boundary theory, the equal-time two-point function $\langle O(t, \mathbf{x})O(t, \mathbf{x}') \rangle$ can be computed starting from the on-shell 5$d$ supergravity action. In the geometric optic limit, such two-point correlation function is given in terms of the length $L$ of the space-like geodesics that connect the two points on the boundary,

$$\langle O(t, \mathbf{x})O(t, \mathbf{x}') \rangle \simeq \sum_{\text{geodesics}} e^{-\Delta L}, \quad (20)$$

an approximation holding for large $\Delta$. $L$ is computed by extremizing the length of the curves connecting the two points, given by

$$L = \int_{P}^{Q} d\lambda \sqrt{\pm g_{MN} \dot{x}^{M} \dot{x}^{N}}, \quad (16)$$

where $g_{MN}$ is the metric, $P$ and $Q$ are boundary points and the coordinates $x^{M}(\lambda)$ ($M = 1, \ldots, d + 1$) are functions of a parameter $\lambda$, and the derivative $\dot{x}^{M} \equiv dx^{M}/d\lambda$ is defined. The integrand in (16) can be treated as a Lagrangian, from which Euler-Lagrange equations stem. The solutions of such equations provide the geodesic that extremizes $L$.

Wilson loops can be treated analogously. In the boundary theory, the loop is defined as

$$W_{c}[A] = \frac{1}{N_{c}} \text{Tr} \left( P e^{-ig \oint_{c} \text{d}x^{a} A_{a}^{T} T^{a}} \right), \quad (17)$$

with $c$ a closed contour. In the strong coupling limit, the holographic expression for the expectation value of (17) is [17]:

$$\langle W_{c} \rangle \sim e^{-S_{NG}}, \quad (18)$$

where $S_{NG}$ is the Nambu-Goto action, giving the area of the string worldsheet bounded by $c$:

$$S_{NG} = \frac{1}{2\pi \alpha'} \int d^{2}\xi \sqrt{det \left[ g_{MN} \partial_{\alpha}X^{M}\partial_{\beta}X^{N} \right]}. \quad (19)$$

$\xi^{\alpha}$ ($\alpha, \beta = 1, 2$) are the worldsheet coordinates, and $X^{M}(\xi^{\alpha})$ the embedding functions. The details of the calculation of the two-point correlation function and of the vacuum expectation values of different Wilson loops can be found in [11]. Here we only mention that the space-like paths connecting the boundary points $P = (t_{0}, -\ell/2, x_{2}, y)$ and $Q = (t_{0}, \ell/2, x_{2}, y)$, and extending in the bulk at fixed $(x_{2}, y)$ are described by two functions $\tau(x)$ and $r(x)$ which depend on the coordinate $x_{1} \equiv x$. In the middle point $x = 0$ they assume the values $\tau(0) = \tau_{*}$, $r(0) = r_{*}$ and have a minimum: $\tau'(0) = r'(0) = 0$, with the prime standing for the derivative with respect to $x$. The conditions $\tau(-\ell/2) = \tau(\ell/2) = t_{0}$, $r(-\ell/2) = r(\ell/2) = r_{0}$ are satisfied, with $r_{0}$ the maximum radial coordinate used in the calculation. The geodesic length

$$L = \int_{-\ell/2}^{\ell/2} dx \frac{\tilde{\Sigma}(r, \tau)}{\sqrt{\tilde{\Sigma}(r_{*}, \tau_{*})}} \quad (20)$$
is given in terms of the metric functions, with \( \widetilde{\Sigma}(r, \tau) = \Sigma(r, \tau)^2 e^{B(r, \tau)} \). Its expression requires a regularization, which we implement subtracting from the length (20) the same quantity computed in pure AdS5. The distance \( \ell \) is provided by the relation \( r(\ell/2) = r_0 \). Typically resulting geodesics are depicted in Fig. 4, and the thermalization of the boundary theory is studied computing their lengths as time proceeds. The hydrodynamic expression of the geodesic lengths \( L_H \) are determined in the geometry (13)-(14) with the same regularization prescription. This allows to identify the difference \( \Delta L = L - L_H \) as an observable.

In the case of Wilson loops as nonlocal probes of the boundary theory thermalization, we consider two shapes, circles and strips. For a Wilson loop along a circumference of radius \( R = \ell/2 \) on the plane \( x_\perp \equiv (x_1, x_2) \) at the boundary, the space-like worldsheet of minimal area based on the circular path and extending in the bulk at fixed \( y \) must be computed. Such a surface has an azimuthal symmetry and a tip at \((\tau, x_\perp, r) = (\tau, 0, r_\ast)\) with \((r_\ast, \tau_\ast)\) input values in the calculation. The transverse section at fixed \( \tau \) and \( r \) is a circumference. For each section the worldsheet can be parameterized in polar coordinates \( \xi^\alpha = (\rho, \varphi) \), so that \( r = r(\rho) \), \( x_1 = \rho \cos \varphi \), \( x_2 = \rho \sin \varphi \), \( r = r(\rho) \), with \( y \) fixed. The area of the worldsheet is obtained from the Nambu-Goto action

\[
A_C = \frac{1}{\alpha} \int_0^{\ell/2} d\rho \rho \left( \widetilde{\Sigma}(r, \tau) \left[ -A(r, \tau) r'(\rho)^2 + \tilde{\Sigma}(r, \tau) + 2\tau'(\rho) r'(\rho) \right] \right)^{1/2},
\]

with the prime in the functions \( \tau \) and \( r \) denoting a derivative with respect to \( \rho \), and the angle \( \varphi \) integrated out. The area of the extremal surface is regularized using the same scheme adopted for the geodesic lengths. The corresponding quantity in the hydrodynamic geometry is obtained using the metric functions (14), and thermalization is probed using the difference \( \Delta A_C = A_C - A_{C,H} \). Examples of extremal surfaces of circular Wilson loops computed using the bulk geometry (2) are shown in Fig. 4.

A less symmetric Wilson loop is an infinite rectangular strip, with one side length \( q \) is taken...
to infinity on the other one set to $\ell$. The extremal area has the expression
\begin{equation}
A_R = \frac{q}{2\pi\alpha'} \int_{-\ell/2}^{\ell/2} dx \frac{\tilde{\Sigma}(r,\tau)}{\tilde{\Sigma}(r_*,\tau_*)},
\end{equation}
to be regularized. This is done computing $A_{RH}$ in the geometry (14), and considering the difference $\Delta A_R = (A_R - A_{RH})/q$ at various $\tau_0$ and for different $\ell$ defines an observable to study thermalization of the boundary theory. An example of rectangular Wilson loops computed in the geometry (2) is shown in Fig. 4.

Let us now describe the results for geodesics in the quench model $B$. Depending on $r_*$ and $\tau_*$, two sets of geodesics $r(x)$ are found: those reaching the AdS boundary, the class we are interested in, and those falling into the bulk. After the quench, at a fixed $\tau_*$, a critical value $r_{**}$ separates the two classes of solutions, and corresponds to the position of the black brane event horizon. The solutions at large $\ell$ approach and follow the horizon, as shown in Fig. 5. On the other hand, during the quench, when large time gradients are present, solutions starting from the boundary and crossing the apparent horizon are also found, a remarkable phenomenon observed for nonlocal observables in rapidly changing time-dependent setups.

The regularized geodesic length $L(t_0, \ell)$, the regularized area of extremal surfaces for the rectangular Wilson loop $A_R(t_0, \ell)$ (divided by $q$) and the circular Wilson loop $A_C(t_0, \ell)$ in model $B$ are shown in Fig. 6 for several values of the distance $\ell$ in the correlation function, of the side (again denoted by $\ell$) of the rectangular Wilson loop, and of the diameter $\ell$ of the circular Wilson loop ($\alpha'$ is set to 1). The curves start at different values of the initial time $t_0$, all corresponding to $\tau_* = 0.25$. Fig. 6 shows that the observables follow the quench profile, with a delay that increases for increasing sizes of the probes.

Let us focus on the time region that follows the end of the spike in the quench, when the profile $\gamma(\tau)$ is nearly constant. We are interested in understanding if the nonlocal observables follow the hydrodynamic behavior and, in that case, how fast such a regime is reached after the end of the quench, in comparison with the thermalization time determined through local observables (in particular the pressures). In Fig. 7 we display the differences $\Delta L$, $\Delta A_R$ (divided by $q$) and $\Delta A_C$. Each observable thermalizes at different times, where all differences vanish. The thermalization times are different for different sizes of the probes. This result indicates how nonlocal observables recover the hydrodynamic regime after the end of the quench in comparison...
Figure 6: Quench models $A(2)$ (left) and $B$ (right). From top down: profile of the quench $\gamma$, geodesics regularized lengths, regularized areas of the extremal surfaces for rectangular (divided by $q$) and circular Wilson loops versus $t_0$, for the sizes of the probes specified in the legendae. The regularization scheme consists in subtracting from each observable the corresponding quantity computed in pure $AdS_5$.

with the local observables: $\Delta L$, $\Delta A_R$ and $\Delta A_C$ are smaller for low values of $\ell$: the system is seen to thermalize faster using observables remaining as local as possible.
To provide a quantitative measure of thermalization for the nonlocal probes, several criteria can be used to determine the value of the size $\ell$ above which the observables are not thermalized, at a fixed value $t_0$ which corresponds to the restoration of pressure isotropy. For example, we define $t_{1/2}(\ell)$ as the value of $t_0$ at which $|\Delta L|$ is reduced by a half with respect to the end of the quench at fixed $\ell$, and similarly for $|\Delta A_R|$ and $|\Delta A_C|$. The results in Fig. 8 show that $t_{1/2}(\ell)$ exceeds the thermalization time obtained using local observables for size $\ell \simeq 1$. The rectangular Wilson loop takes more time to thermalize. Another feature emerging for $t_{1/2}(\ell)$ is the linear increase against the size $\ell$. The hierarchy found between the thermalization times of the energy density, the pressures and the large size probes indicates that the onset of thermalization starts at short distances.

As for the quench model $A(2)$, the results in Fig. 6 show how the nonlocal observables follow the quench in the boundary, and how thermalization is reached with the curves approaching the hydrodynamic behavior, as understood inspecting Fig. 7. The half thermalization time $t_{1/2}(\ell)$ is depicted in Fig. 8, and its behavior is linear for large sizes. As in model $B$, the rectangular Wilson loop thermalizes more slowly than the other two observables.

The conclusion, for both the quench models, is the emergence of time scales related to the onset of hydrodynamics, depending on the size of the probes. The observation of a hierarchy in thermalization among the different sizes and distances is connected to the use of space-like...
probes. Analyses based on time correlators, or on horizon-to-boundary propagators in the same dynamical framework, are useful to show a hierarchy in thermalization among different frequencies and modes of the boundary field theory [18].

4 Conclusions

Equilibration of a strongly interacting non-Abelian plasma can be studied in a fully dynamical holographic 5d setup with boundary sourcing, using local and nonlocal observables that provide indications on the hierarchy of thermalization times. The energy density follows the hydrodynamic viscous behavior immediately after the end of the quenches that drive the system out-of-equilibrium, while there is a time delay for the pressures to reach the viscous time dependence and the isotropy condition $p_\perp = p_\parallel$. For nonlocal observables, the thermalization time changes with the size of the observable. For larger sizes, the thermalization time increases linearly with the size of the probe. The hierarchy among the thermalization times of the energy density, pressures and large probes indicates that short distances thermalize first.

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References


Figure 8: Quench model $A_\parallel(2)$ (left) and $B$ (right): time $t_{1/2}$ versus the size $\ell$ for the three nonlocal observables. The horizontal dashed line indicates the thermalization time obtained from the pressure anisotropy.
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