SIMPLE EMERGENT POWER SPECTRA FROM COMPLEX INFLATIONARY PHYSICS

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1502.03125, 1607.xxxxx, Jonathan Frazer, David Seery and David Mulryne
WHAT CAN WE LEARN?

No running, isocurvature, non-Gaussianity, tensor spectrum...
WHAT CAN WE LEARN?

The power spectrum of the primordial curvature perturbation can be parameterised by two numbers:
Amplitude $A_s$ and tilt $n_s$.

$$P_\zeta(k) = A_s \left( \frac{k}{k_*} \right)^{n_s-1}$$

$$n_s|_{k_0} \equiv \left. \frac{d \log P_\zeta}{d \log k} \right|_{k_0} = 0.968 \pm 0.006$$

$$\alpha_s|_{k_0} \equiv \left. \frac{d n_s}{d \log k} \right|_{k_0} = -0.003 \pm 0.007$$

Planck 2015 results. XX. arXiv:1502.02114
Simplest inflation models fit data

\[ \epsilon \propto \left( \frac{V_{,\phi}}{V} \right)^2 \ll 1 \]
\[ |\eta| \propto \left| \frac{V_{,\phi\phi}}{V} \right| \ll 1 \]

power spectrum:
\[ \langle \zeta(k_1)\zeta(k_2) \rangle = \frac{2\pi^2}{k^3} \delta(k_1 + k_2) P_\zeta \]

\[ P_\zeta = N_{,\phi}^2 \Sigma_{*}^{\phi\phi} = \frac{1}{2\epsilon_*} \left( \frac{H_*}{2\pi} \right)^2 \]

\( \langle \delta \phi \delta \phi \rangle \) at horizon crossing

Power spectrum conserved on superhorizon scales
\[ n_s(k) - 1 = -6\epsilon + 2\eta \]
\[ r = 16\epsilon \]

**SIMPLEST INFLATION MODELS FIT DATA**

Planck Collaboration: Constraints on inflation

Fig. 11. Marginalized joint 68\% and 95\% CL regions for \((\epsilon_1, \epsilon_2, \epsilon_3)\) (top panels) and \((\epsilon_1, \epsilon_2, \epsilon_3)\) (bottom panels) for Planck TT+lowP (red contours), Planck TT,TE,EE+lowP (blue contours), and compared with the Planck 2013 results (grey contours).

Fig. 12. Marginalized joint 68\% and 95\% CL regions for \(n_s, r\) from Planck in combination with other data sets, compared to the theoretical predictions of selected inflationary models.

Fundamental Physics

Occam’s razor should be handled with care; UV sensitivity

Phenomenology: Single Field Slow Roll Inflation?
FUNDAMENTAL PHYSICS PICTURE OF INFLATION

single field inflation

multifield inflation
FUNDAMENTAL PHYSICS PICTURE OF INFLATION

Complex inflationary physics  EMERGENT SIMPLICITY ?
Fundamental Physics

Emergence

Phenomenology
Emergent simple power spectrum from complex physics?
STUDYING COMPLICATED MODELS IS COMPLICATED

Very little is known about inflation with many fields. Two challenges:
STUDYING COMPLICATED MODELS IS COMPLICATED

Very little is known about inflation with many fields. Two challenges:

1. Constructing the model: scaling problem e.g.

\[ V = \Lambda_v^4 \sum_{k_{\text{min}}}^{k_{\text{max}}} \left[ a_{\vec{k}} \cos (\vec{k} \cdot \vec{\phi}) + b_{\vec{k}} \sin (\vec{k} \cdot \vec{\phi}) \right] \]

No. of terms \( \sim (k_{\text{max}}/k_{\text{min}})^{N_f} \)

\[ \tilde{\phi}^a \equiv \phi^a / \Lambda_h \]
STUDYING COMPLICATED MODELS IS COMPLICATED

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Two challenges:

1. Constructing the model: scaling problem \( e.g. \)

\[
V = \Lambda_v^4 \sum_{k_{\text{min}}}^{k_{\text{max}}} \left[ a_{\vec{k}} \cos (\vec{k}.\vec{\phi}) + b_{\vec{k}} \sin (\vec{k}.\vec{\phi}) \right] \]

No. of terms \( \sim (k_{\text{max}}/k_{\text{min}})^{N_f} \)

2. Computing observables: another scaling problem \( i.e. \)

\[
\frac{d\Sigma^{\alpha\beta}}{dN} = u^{\alpha}_{\gamma} \Sigma^{\gamma\beta} + u^\beta_{\gamma} \Sigma^{\alpha\gamma} \]

No. of coupled ODEs \( \sim N_f^2 \)
AN INTERLUDE

COMPUTING OBSERVABLES IN MULTIFIELD INFLATION

OR

THE TRANSPORT METHOD
Computing observables in multifield inflation

Curvature perturbations: $\zeta = \delta N$

Constant $\rho$

Flat

In single field, remains unchanged

Time of evaluation

Horizon exit

$V(\phi) \sim \rho$

Computing observables in multifield inflation

Curvature perturbations: $\zeta = \delta N$

In multifield, superhorizon evolution

Time of evaluation

Horizon exit

Computing observables in multifield inflation

Not trivial in the presence of heavy fields and turns in the trajectory. Some analytical attempts: QSFI (Chen et al), gelaton (Tolley and Wyman), sharp features (Achucarro et al and Langlois et al).
COMPUTING OBSERVABLES IN MULTIFIELD INFLATION

Oscillating mode functions:
Numerical advantages in working with non-oscillatory correlation functions: transport method.

Compute Hubble-scale correlators:
Not trivial in the presence of heavy fields and turns in the trajectory. Some analytical attempts: QSFI (Chen et al), gelaton (Tolley and Wyman), sharp features (Achucarro et al and Langlois et al).
THE TRANSPORT METHOD

Differential form of usual In-In calculation:

\[
\frac{d\langle \hat{X} \rangle}{dt} = \left\langle \frac{d\hat{X}}{dt} \right\rangle = \left\langle -i \left[ \hat{X}, \hat{H} \right] \right\rangle + \left\langle \frac{\partial \hat{X}}{\partial t} \right\rangle
\]

with Hamiltonian:

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}
\]

\[
\frac{d\delta \varphi_\alpha}{dN} = u_{\alpha\beta} \delta \varphi_\beta + \frac{1}{2} u_{\alpha\beta\gamma} \delta \varphi_\beta \delta \varphi_\gamma + \cdots
\]

\[
\left[ \delta \varphi_\alpha, \hat{H}_0 \right] = iu_{\alpha\beta} \delta \varphi_\beta \quad \quad \left[ \delta \varphi_\alpha, \hat{H}_{\text{int}} \right] = iu_{\alpha\beta\gamma} \delta \varphi_\beta \delta \varphi_\gamma
\]

Dias, Frazer, Seery: 1502.03125; Mulryne astro-ph/1302.3842
THE TRANSPORT METHOD

\begin{align*}
\frac{d\delta \phi^a}{dN} &= u^a_b \delta \phi^b \\
\langle \delta \phi^a(k_1) \delta \phi^b(k_2) \rangle &= (2\pi)^3 \delta(k_1 + k_2) \frac{\Sigma^{ab}}{k^3} \\
\frac{d\Sigma^{ab}}{dN} &= u^c_b \Sigma^{cb} + u^a_c \Sigma^{ac}
\end{align*}

\[ P_\zeta(N) = N_a N_b \Sigma^{cd}(N_f) \]
THE TRANSPORT METHOD

\[ \frac{d\delta \phi^a}{dN} = u^a_b \delta \phi^b \]

\[ \langle \delta \phi^a(k_1)\delta \phi^b(k_2) \rangle = (2\pi)^3 \delta(k_1 + k_2) \frac{\Sigma^{ab}}{k^3} \]

\[ \frac{d\Sigma^{ab}}{dN} = u^c_b \Sigma^{cb} + u^a_c \Sigma^{ac} \]

\[ \delta \phi(N)^a = \Gamma^a_b \delta \phi^b(N_0) \]

\[ \Gamma^{ab}(N, N_0) = \mathcal{P} \exp \int_{N_0}^{N} u^{ab}(N')dN' \]

\[ \frac{d\Gamma^{ab}}{dN} = u^a_c \Gamma^{cd} \]

\[ P_\zeta(N) = N_a N_b \Sigma^{cd}(N) \]

\[ P_\zeta(N) = N_a N_b \Gamma^a_c \Gamma^b_d \Sigma^{cd}(N_0) \]
THE TRANSPORT METHOD

Example: bispectrum at fixed $k_T$

$$V = \frac{\lambda}{4} \phi^4 + \Lambda^4 (1 - \cos (2\pi \chi))$$
End Of Interlude

Studying multifield models with RMT
**Basics of Random Matrix Theory**

Any large real symmetric matrix with entries $M_{ab}$ drawn from a random distribution (GOE)

$$p(\lambda_1, \ldots, \lambda_{N_f}) = Ce^{-\frac{1}{2}W}$$

$$W = \frac{1}{\sigma^2} \sum_{a=1}^{N_f} \lambda_a^2 - \sum_{a \neq b} \ln |\lambda_a - \lambda_b|$$

**Eigenvector repulsion**

Eigenvalues behave like a gas of electrons confined to a line and subject to a quadratic potential

**Wigner Semi-Circle**
BASICS OF RANDOM MATRIX THEORY

Atypical distribution of a GOE

\[ P(\lambda_{\text{min}} > 0) \propto e^{-N^2} \]
BASICS OF RANDOM MATRIX THEORY

DYSON BROWNIAN MOTION

\[ \delta M_{ab} = \delta A_{ab} - M_{ab} \frac{\delta s}{\Lambda_h} \]

stochastic piece \quad restoring force

Dyson 1962: “A Brownian-Motion Model for the Eigenvalues of a Random Matrix”
CONSTRUCTING THE POTENTIAL WITH RMT

A LOCAL APPROACH:

\[ V \bigg|_{p_0} = \Lambda^4_v \sqrt{N_f} \left( v_0|_{p_0} + v_a|_{p_0} \tilde{\phi}^a + \frac{1}{2} v_{ab}|_{p_0} \tilde{\phi}^a \tilde{\phi}^b \right) \]

\[ v_0|_{p_1} = v_0|_{p_0} + v_a|_{p_0} \delta s^a \]

\[ v_a|_{p_1} = v_a|_{p_0} + v_{ab}|_{p_0} \delta s^b \]

\[ v_{ab}|_{p_1} = v_{ab}|_{p_0} + \delta v_{ab}|_{p_0 \to p_1} \]

\[ \tilde{\phi}^a \equiv \phi^a / \Lambda_h \]
Constructing the potential with RMT

Rare, fluctuated spectrum, suitable for inflation

Typical configuration, not suitable for inflation

We will present a new way of defining random functions locally around a path in field space: for a given path \( p \) in field space, we first specify the values of the potential \( V \), gradient \( V_0 \), and Hessian matrix \( \nabla^2 V \) at a point \( p_0 \). The values of the potential and the gradient vector at a nearby point \( p_1 \) that is separated from \( p_0 \) by a small path length \( s \) may then be obtained to leading order in Taylor expansion from the (known) values of the potential and its first and second derivatives at \( p_0 \). The key element of our proposal is to specify the Hessian matrix at \( p_1 \) by adding a random matrix to the Hessian at \( p_0 \),

\[
H(p_1) = H(p_0) + \mathbf{R}
\]

(1.1)

where we have yet to define the statistical distribution of the random symmetric matrix \( \mathbf{R} \). By repeating this process along the entire path, we obtain a random function defined in the vicinity of \( p \). In the limit, one obtains a continuous description of the evolution of the Hessian.

We will discuss the restrictions on \( s \) in §3.1: in particular, it must not self-intersect.

Rare, fluctuated spectrum, suitable for inflation

Typical configuration, not suitable for inflation
One could construct many different classes of potentials by modifying the rule (3.5) governing the evolution of the Hessian.

Figure 4. The random potentials presented in this paper exhibit non-trivial structure on scales larger than a few $\star h$, as is illustrated above for $N_f = 2$ and a path length of 4 $\star h$. For illustration purposes we have exaggerated the separation between subsequent charts, though a smaller separation will be used in §4 to ensure a good approximation to the smooth evolution of the eigenvalues of the Hessian.

Because our method defines the potential in a semi-local as opposed to global fashion, it has some obvious drawbacks. First of all, as the random potential is defined as a sequence of quadratic approximations in a string of coordinate patches, the global structure of the potential far from the generating path is not readily available with this method. Constraints on the structure of the potential from e.g. Morse theory are therefore not immediately applicable to these potentials.

Furthermore, the path length along a curve does not always give a good measure of distance in field space. In particular, self-intersecting trajectories will generally not give rise to single-valued potentials. For trajectories that are nearly self-intersecting — which is not uncommon for low-dimensional field spaces, but is extremely rare at large $N_f$ — a more careful analysis is required.

$N_f = 2$
THE TRANSPORT METHOD

\[ \frac{d\phi^a}{dN} = u^a_b \phi^b \]

\[ \langle \phi^a(k_1)\phi^b(k_2) \rangle = (2\pi)^3 \delta(k_1 + k_2) \frac{\Sigma^{ab}}{k^3} \]

\[ \frac{d\Sigma^{ab}}{dN} = u^c_b \Sigma^{cb} + u^a_c \Sigma^{ac} \]

\[ P_\zeta(N) = N_a N_b \Sigma^{cd}(N) \]

\[ \delta\phi(N)^a = \Gamma^a_b \delta\phi^b(N_0) \]

\[ \Gamma^{ab}(N, N_0) = \mathcal{P} \exp \int_{N_0}^{N} u^{ab}(N') dN' \]

\[ \frac{d\Gamma^{ab}}{dN} = u^a_c \Gamma^{cd} \]

\[ P_\zeta(N) = N_a N_b \Gamma^a_c \Gamma^b_d \Sigma^{cd}(N_0) \]

Dias, Frazer, Seery: 1502.03125; Mulryne astro-ph/1302.3842
Computing Observables with RMT

Each patch can be rotated to a sum-separable basis

\[ [O^T \text{diag}(\lambda)O]^{ab} = v^{ab} \quad \xrightarrow{\text{V}} \quad V|_{p_i} = \sum_{a=1}^{N_f} V_a(\phi^a) \]

Assuming SR

\[ \Gamma^{ab}(p_i, p_{i-1}) = \frac{(v^a)_p_i}{(v^b)_p_{i-1}} \left( \delta_{ab} + \frac{(v^b_0)_p_{i-1} - (v^b_0)_p_i}{(v_0)_p_i} \right) \]

\[ \delta \vec{\phi}|_{pf} = O^T_{pf} \Gamma(p_f, p_{f-1}) O_{pf} \ldots O^T_{p1} \Gamma(p_1, p_0) O_{p1} \delta \vec{\phi}|_{p_0} \]

No need to solve any more ODEs!

\[ V_a = \Lambda^4 v \sqrt{N} \left((v_0)_a + v_a \phi_a + \frac{1}{2} \lambda_a \phi_a^2\right) \]

Dias, Frazer, Marsh: arXiv:1604.05970
$P_\zeta(N) = N_\alpha N_\beta \Gamma^\alpha_c \Gamma^b_d \Sigma^{cd}(N_0)$

$N_f = 50$
RESULTS

$N_f = 2$
RESULTS

$N_f = 50$

SMOOTHER AND MORE PREDICTIVE SPECTRA

Dias, Frazer, Marsh: arXiv:1604.05970
RESULTS

SMOOTHER AND MORE PREDICTIVE SPECTRA
Significant superhorizon evolution of the primordial curvature perturbation, implying the presence of many active fields.
**Random Matrix Theory Interpretation**

**More Predictive Spectra**: Analyse the Variance of the Tilt

\[
\begin{align*}
    n_s - 1 &= \frac{d \ln P_\zeta}{d \ln k} = \frac{1}{P_\zeta} N_a N_b \Gamma^a_c \Gamma^b_d n^{cd} \\
    n_{*}^{ab} &\equiv \left. \frac{d \Sigma}{d \ln k} \right|_* = (-\epsilon \delta^{ab} - u^{ab})_* H_*^2 \\
    n_s - 1 &\approx 2 e_a e_b \left( \frac{\nu^{ab}}{v_0 \Lambda^{2}_h} \right)_*
\end{align*}
\]

Variance of Smallest Eigenvalue of \( \nu^{ab} \)
SMOOTHER SPECTRA: STUDY THE BEHAVIOUR OF THE RUNNING

\[
\alpha_s = \frac{dn_s}{d \ln k} = \frac{1}{P_\zeta} N_a N_b \Gamma^a_c \Gamma^b_d \alpha^c_d - (n_s - 1)^2
\]

\[
\alpha_{*}^{ab} \equiv \left. \frac{dn^{ab}}{d \ln k} \right|_* = \left( 2\epsilon^2 - \epsilon' \right) \delta^{ab} - \left( u^{ab} + 2\epsilon u^{ab} \right)_* H_*^2 - 2 \left[ u^a_c n^{cb} \right]_*
\]

\[
\alpha_s \approx 4 e_a e_b \left. \frac{v^a_c v^{cb}}{v_0^2 \Lambda_h^4} \right|_* - 4 \left( \frac{e_a e_b v^{ab}_*}{v_0^* \Lambda_h^2} \right)^2 + 2 e_a e_b \left. \frac{v^{ab}_*}{v_0 \Lambda_h^2} \right|_*
\]

CHANGES IN THE SMALLEST EIGENVALUE OF \( v^{ab} \)
RANDOM MATRIX THEORY INTERPRETATION

EIGENVALUE REPULSION
Random Matrix Theory Interpretation

Tracy-Widom distribution

\[ N_f^{-1/6} \]

when \( \sigma^2 = 2/N_f \)

\[ c_{b'} = |\lambda_1^0 - \lambda_{b'}^0|^{-1} \propto N_f^2 \]

D. S. Dean and S. N. Majumdar; cond-mat/0609651
C.A. Tracy and H. Widom; CMP 159, 151 (1994)
RANDOM MATRIX THEORY INTERPRETATION

2nd order perturbation theory:

\[ \lambda_1^k = \nu_{11}^k - \sum_{b'=2}^{N_f} c_{b'} |\nu_{1b'}^k|^2 \]

\[ \langle \lambda_1(s) \rangle = e^{-s} \lambda_1^0 - s \sigma^2 \sum_{b'=2}^{N_f} c_{b'} \sim N_f \]

more negative, so redder tilt

\[ \text{Var} \,(\lambda_1(s)) = 2s \sigma^2 \left( 1 + s \sigma^2 \sum_{b'=2}^{N_f} c_{b'}^2 \right) \sim \frac{1}{N_f} \]

shrinking variance, so more predictivity
OPEN QUESTIONS?

- **Presence of isocurvature: Evolution of curvature perturbation past end of inflation?**

- **Reheating?**

- **Transport for the 3PF: Interesting bispectrum arising around horizon exit?**
Overview and Prospect

- **Emergent simplicity from complex systems can help bridge the gap between fundamental physics and data.**

- **The study of complex systems requires numerical tools for observables: Transport Method.**

- **Simple power spectra can emerge from complex inflationary physics:**
  - As $N_f$ increases, spectra become both more *predictive* and *smoother*, both can be understood through RMT.

- **Open questions:** Isocurvature, Bispectrum