The $\mathcal{N} = 2$ superconformal bootstrap

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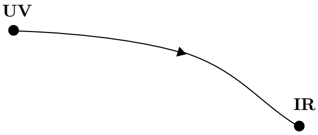
The $\mathcal{N} = 2$ bootstrappers:
C. Beem, M. Lemos, W. Peelaers, I. Ramirez, L. Rastelli, J. Seo, B. van Rees.

Outline

1. Motivation: Why the bootstrap?
2. (Super)conformal bootstrap review
3. $2d$ chiral algebras
4. Analytic bounds

Conformal Field Theories



- Renormalization Group fixed points: Quantum Field Theories flow to conformal fixed points in the extreme ultraviolet and infrared energy regimes.
- In Statistical Mechanics: critical behavior near a phase transition.
- In String Theory: worldsheet description, and also spacetime physics thanks to AdS/CFT.
- Potential Phenomenological applications, Superconformal Field Theory, ...

The bootstrap philosophy

- Lagrangians are not enough!
- The bootstrap idea attempts to constrain theories using basic principles such as unitarity and crossing symmetry.
- It is an algebraic viewpoint based on the operator algebra.
- Well suited for $\mathcal{N} = 2$ SCFTs were many non-Lagrangian examples are known.

A vast zoo of $\mathcal{N} = 2$ SCFTs

$\mathcal{N} = 2$ superconformal dynamics is extremely rich. We have an ample catalog of theories:

- Lagrangian theories built with vectors and hypers such that $\beta(g) = 0$.
- Solutions to the Seiberg-Witten curve: some strongly interacting with no known Lagrangian.
- Low energy limit of string theory on D3-branes.
- Gaiotto theories obtained by compactifying $6d$ $(2,0)$ theories on Riemann surfaces.

Is there an underlying structure?

Having such a vast zoo, instead of solving theories one by one, maybe we should try to find the underlying principles, and attempt to classify $\mathcal{N} = 2$ SCFTs.
Work in this direction includes

- Classification of Lagrangian theories. (Bhardwaj,Tachikawa)
- Classification of solutions to the Seiberg-Witten curve. (Argyres, Lotito, Lu, Martone, Xie, Yau)
- Classification of Gaiotto theories. (Chacaltana, Distler, Tachikawa, Trimm)
- <u>The Superconformal Bootstrap.</u> (The $\mathcal{N} = 2$ bootstrappers)

The Conformal Group

The conformal group puts tight restrictions on correlation functions

$$\begin{aligned}
\langle \phi_1(x_1)\phi_2(x_2)\rangle &= \begin{cases} \frac{1}{|x_1-x_2|^{2\Delta_\phi}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases}, \\
\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &= \frac{\lambda_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} .
\end{aligned}$$

The collection $\{\lambda, \Delta\}$ is the CFT data.
The four-point function is not completely fixed, for identical fields,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} ,$$

Operator Product Expansion

The product of two primary fields can be written as a sum of local operators,

$$\phi(x)\phi(0) \sim \sum_{\mathcal{O}} \lambda_{\mathcal{O}} C(x,\partial)\mathcal{O}(0)\,.$$

Taking the OPE in the four-point function

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}(1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 g_{\mathcal{O}}(u,v))$$

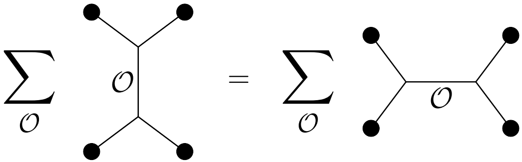
where the "conformal block" $g_{\mathcal{O}}(u,v)$ collects the contributions of the operator $\mathcal{O}$ and all its descendants. They can be written in terms of hypergeometric functions (Dolan-Osborn).

Crossing Symmetry

Invariance under the interchange of $x_1 \leftrightarrow x_3$ gives *crossing symmetry*:

$$v^{\Delta_\phi}(1 + \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} g_{\mathcal{O}}(u, v)) = u^{\Delta_\phi}(1 + \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} g_{\mathcal{O}}(v, u))$$

It can be represented pictorially,



Overconstrained system for the CFT data, but what to do with it?

Bootstrap techniques

There are several approaches to study crossing:

- **Virasoro symmetry.** In $2d$ the enhancement to Virasoro allows for an exact solution: the celebrated minimal models.
  (Belavin, Polyakov, Zamolodchikov)
- **The numerical bootstrap.** Powerful numerical techniques that constrain the low-lying spectrum.
  (Poland, Rattazi, Rychkov, Simmons-Duffin, Tonni, Vichi)
- **The analytical bootstrap.** Analytic constraints for operators with high spin.
  (Alday, Poland, Kaplan, Komargodski, Fitzpatrick, Simmons-Duffin, Zhiboedov)
- **Solvable truncation.** In supersymmetric theories there is a solvable truncation of the crossing equations.
  (C. Beem, M. Lemos, PL, W. Peelaers, L. Rastelli, B. van Rees.)

The $\mathcal{N} = 2$ superconformal bootstrap

The $\mathcal{N} = 2$ superconformal bootstrap relies only on the algebra and not on a Lagrangian! It can be formulated as a two-step process:

- **$2d$ chiral algebras**

  Any $4d$ $\mathcal{N} = 2$ SCFT contains a subset of operators described by a $2d$ chiral algebra.

  $$4d \text{ SCFT} \qquad \rightarrow \qquad 2d \text{ Chiral Algebra}$$

  The $2d$ chiral algebra only describes *protected* operators.

- **Numerics**

  For operators outside the $2d$ chiral algebra, we have less analytic control and we have to resort to numerics. In general *unprotected* quantities.

The $\mathcal{N} = 2$ Superconformal algebra

Let's start reviewing the $SU(2,2|2)$ algebra.

- Conformal generators : $\{\mathcal{P}_{\alpha\dot{\alpha}}, \mathcal{K}^{\dot{\alpha}\alpha}, \mathcal{M}_{\alpha}^{\ \beta}, \bar{\mathcal{M}}^{\dot{\alpha}}_{\ \dot{\beta}}, \mathcal{D}\}$

  Includes translations, special conformal, Lorentz and the dilatation operator.

- R-symmetry generators : $\{\mathcal{R}_i^{\ j}, r\}$

  In $\mathcal{N} = 2$ theories the R-symmetry is $SU(2)_R \times U(1)_r$

- Supercharges : $\{\mathcal{Q}_{\alpha}^{i}, \bar{\mathcal{Q}}_{\dot{\alpha},i}, \mathcal{S}_i^{\alpha}, \bar{\mathcal{S}}^{\dot{\alpha}i}\}$

A solvable truncation

Let's go back to crossing

$$1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 G_{\mathcal{O}}(z, \bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} (1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 G_{\mathcal{O}}(1-z, 1-\bar{z}))$$

where we have defined $z\bar{z} = u$, $(1-z)(1-\bar{z}) = v$.

In $\mathcal{N} = 2$ SCFTs there is a *solvable truncation*.

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 h_{\mathcal{O}}(z) \sim \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 h_{\mathcal{O}}(1-z)$$

The correlators in the subsector are meromorphic:

$$\langle \mathcal{O}(0)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}(\infty)\rangle \sim f(z)$$

They can be thought of as correlators describing a *2d chiral algebra*.

Intermezzo: The $\mathcal{N} = 1$ chiral ring

In $\mathcal{N} = 1$ theories there are chiral operators that satisfy

$$[\mathcal{Q}_\alpha, \phi(x)] = 0$$

The translation generators are Q-exact:

$$\mathcal{P}_{\alpha\dot{\alpha}} = \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\}$$

Derivatives of ϕ are also $\mathcal{Q}$-exact:

$$\partial_{\alpha\dot{\alpha}}\phi(x) = [\mathcal{P}_{\alpha\dot{\alpha}}, \phi(x)] = \{\mathcal{Q}_\alpha, X_{\dot{\alpha}}\}$$

One can then prove

$$\partial\langle\phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle = 0$$

The correlator doesn't care about $\mathcal{Q}$-exact terms.

Meromorphy in $\mathcal{N} = 2$ SCFTs

First, we fix a plane $\mathbb{R}^2 \in \mathbb{R}^4$.

We want a subsector with meromorphic correlators.

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = f(z_i)$$

We consider operators killed by a particular combination of supercharges:

$$[\mathbb{Q}, \mathcal{O}(z, \bar{z})] = 0 \qquad \mathbb{Q} = \mathcal{Q} + \mathcal{S}$$

If the $\bar{z}$-dependence is $\mathbb{Q}$-exact, the $\mathcal{N} = 1$ chiral argument goes through.

Is the cohomology of $\mathbb{Q}$ non-empty?

Cohomology: Schur operators

Translations in z commute with $\mathbb{Q}$: We only need to look operators at the origin of $\mathbb{R}^2$.

Operators in the cohomology satisfy

$$\frac{\Delta - \ell}{2} = R$$

Δ is the conformal dimension, ℓ is the spin in $\mathbb{R}^2$, and R is the Cartan of $SU(2)_R$.

Only short operators are described by the $2d$ chiral algebra: Schur operators.

2d chiral algebras in 4d SCFTs

The construction is as follows:

- Restrict an operator $\mathcal{O}_{Schur}$ to a plane $\mathbb{R}^2 \subset \mathbb{R}^4$ with coordinates $z, \bar{z}$.
- Set Lorentz indices to $+$ and $\dot{+}$ and contract $SU(2)_R$ indices with $u_i = (1, \bar{z})$.

$$\mathcal{O}(z, \bar{z}) = u_i u_j \mathcal{O}^{ij}_{+\dot{+}}(z, \bar{z})$$

- Correlators of this operator are meromorphic:

$$\langle \mathcal{O}(0)\mathcal{O}(1)\mathcal{O}(z, \bar{z})\mathcal{O}(\infty)\rangle = f(z)$$

This construction only works for $\mathcal{N} \geq 2$ because we need $SU(2)_R$.

The R-symmetry current

The R-symmetry current is a Schur operator:

$$J^j_{i\mu}(z,\bar{z}) \to T(z)$$

The $4d$ OPE implies

$$T(z)T(0) \sim -\frac{6\,c_{4d}}{z^4} + 2\frac{T(0)}{z^2} + \frac{\partial T(0)}{z} + \dots .$$

we have Virasoro symmetry with central charge

$$c_{2d} = -12\,c_{4d}\,.$$

The R-symmetry current is related to the $4d$ stress tensor $T_{\mu\nu} \sim (\mathcal{Q}\bar{\mathcal{Q}}J)_{\mu\nu}$.

The moment map operator

The moment map operator M^{ij} is the highest weight of a 1/2 BPS multiplet and is a generator in the Higgs branch chiral ring.

$$M^{ij}(z,\bar{z}) \to J(z)$$

The $4d$ OPE implies

$$J^a(z)J^b(0) \sim \frac{k_{2d}}{z^2}\delta^{ab} + if^{abc}\frac{J^c(0)}{z} + \dots .$$

we have an AKM algebra at level

$$k_{2d} = -\frac{1}{2}\,k_{4d}\,.$$

The moment map is related to the flavor current $J_{F\,\mu} \sim (\mathcal{Q}\bar{\mathcal{Q}}M)_\mu$.

The four-point function of $T(z)$ is in textbooks!

$$\langle T(0)T(z)T(1)T(\infty)\rangle \sim 1 + \frac{z^4}{(1-z)^4} + \frac{8}{2c_{2d}}z^2 + \dots$$

A truncation of the full correlator $\langle \mathcal{J}_R \mathcal{J}_R \mathcal{J}_R \mathcal{J}_R \rangle$ is solvable.
We can expand it in $4d$ conformal blocks

$$\sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} G_{\mathcal{O}}(z,\bar{z})$$

and solve for an infinite number of OPE coefficients.
Assuming absence of higher-spin currents (Maldacena, Zhiboedov)

$$\lambda^2_{\mathcal{O}_0} = \left(2 - \frac{11}{15c_{4d}}\right)$$

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Unitarity implies (PL, I. Ramirez, J. Seo)

$$c_{4d} \geqslant \frac{11}{30}$$

What did we assume?

Let's recap what were the basic assumptions that lead to the bound

- $\mathcal{N} = 2$ superconformal symmetry.
- A stress-tensor or locality.
- That the theory is interacting.

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G	A_0	A_1	A_2	D_4	E_6	E_7	E_8

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G	A_0	A_1	A_2	D_4	E_6	E_7	E_8
c	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{13}{6}$	$\frac{19}{6}$	$\frac{31}{6}$
k	—	$\frac{8}{3}$	3	4	6	8	12

Table: Central charges for canonical $\mathcal{N} = 2$ theories. (Tachikawa, Aharony)

These are theories of rank one: only one Coulmob branch generator.

Let's add flavor

We can repeat a similar analysis for the correlator

$$\langle J(0)J(z)J(1)J(\infty)\rangle$$

It implies (Beem, Lemos, PL, Rastelli, Peelaers, van Rees),

$$\frac{1}{k} \leqslant \frac{12c + \dim_G}{24ch^\vee} \qquad k \geqslant k_{min}(G)$$

We can also look at the mixed system

$$\langle T(0)T(z)T(1)T(\infty)\rangle \; \langle J(0)J(z)J(1)J(\infty)\rangle \; \langle T(0)T(z)J(1)J(\infty)\rangle$$

It implies (Lemos, PL),

$$k\left(-180c^2 + 66c + 3\dim_G\right) + 60c^2h^\vee - 22ch^\vee \leqslant 0$$

Combining all the bounds

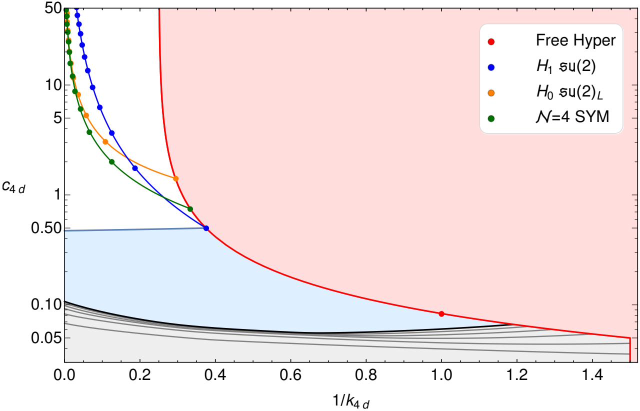

Figure: The landscape of $\mathcal{N} = 2$ SCFTs with flavor group $SU(2)$.

The corner is at $(c, k) = (1/2, 8/3)$.

The E_6 landscape

Combining all the bounds

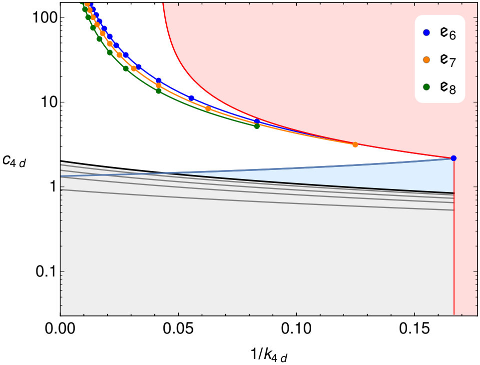

Figure: The landscape of $\mathcal{N} = 2$ SCFTs with flavor group E_6.

The corner is at $(c, k) = (13/6, 6)$.

Summary

- Every $\mathcal{N} = 2$ SCFT contains a subsector of operators with meromorphic correlators described by a $2d$ chiral algebra.
- Studying the interplay between $4d$ and $2d$ drescriptions we obtained general analytic bounds valid for any $\mathcal{N} = 2$ SCFT with the given flavor group.
- The canonical rank one $\mathcal{N} = 2$ superconformal field theories live in special corners of the parameter space. Their c and k central charges are uniquely fixed.
- To do: Numerical bootstrap, develop block technology, study analytic limits.

Thank you.