The coordinate representation of NLO BFKL and the dipole picture

Fadin V.S.
Institute of Nuclear Physics and Novosibirsk State University, 630090, Novosibirsk, Russia

Abstract
For scattering of colourless objects, the freedom in definition of the BFKL kernel resulting from invariance of scattering amplitudes under simultaneous transformations of the kernel and impact factors permits to present the kernel in the dipole form. This form is found in the next-to-leading order (NLO) by direct transfer from the momentum space to the coordinate one.

1 Introduction
The BFKL approach [1] gives the most common basis for theoretical description of small-x processes. Usually the approach is formulated in the momentum space. Scattering amplitudes \( A_{AB}^{A'B'} \) are presented in the form of convolution \( \Phi_{A'A} \otimes G \otimes \Phi_{B'B} \) of impact factors \( \Phi \) and Green’s function \( G \) for two interacting Reggeized gluons (see Fig.1). All dependence from properties of scattering particles is contained in the impact factors \( \Phi_{A'A} \) and \( \Phi_{B'B} \) describing transitions \( A \to A' \) and \( B \to B' \). The Green’s function \( G \) holds all energy dependence. The impact factors and the kernel of the BFKL equation for the Green function are defined in the transverse momentum space. The kernel is known now in the NLO for arbitrary momentum transfer \( t \) and all possible colour states in the \( t \) – channel [2–4]. The most important for phenomenological applications is the colour singlet state – Pomeron channel. In the following only this channel is considered. A distinctive feature of the colour singlet kernel is the cancellation of the infrared divergencies.

For scattering of colourless objects, in the leading order (LO), a remarkable property of the BFKL equation is its Möbius invariance [5]. In this case the BFKL kernel can be taken in a special representation, which we call dipole form. The Möbius invariance of the kernel in this representation can be made evident by transformation from the transverse momentum to the transverse coordinate space. Moreover, in the coordinate space the dipole form coincides with the kernel of the colour dipole approach [6]. It makes very interesting the finding of the dipole form of the BFKL kernel in the NLO. Of course, the conformal invariance is violated by renormalization. One may wonder, however, whether the renormalization is the only source of the violation. If so, one can expect the conformal invariance of the NLO BFKL kernel in supersymmetric extensions of QCD.
The dipole form of the BFKL kernel is very useful also for a better understanding of the relation between the BFKL and colour dipole approaches. It should help in further development of the theoretical description of small-x processes.

An additional reason is the complexity of the NLO BFKL kernel in the momentum space for \( t \neq 0 \). It is found in the form of intricate two-dimensional integrals. Its simplification is extremely desirable.

2 The dipole form in the LO

In the operator notations (see e.g. [7]) \( s \)-channel discontinuities of scattering amplitudes are presented in the form

\[
\delta(q_A - q_B)_{\text{discont}} A_{AB}^{A'B'} = \frac{i}{4(2\pi)^3} \langle A' \bar{A} | \hat{Y} \hat{K} \frac{1}{q_1 q_2} | B'B \rangle ,
\]

where \( \langle A' \bar{A} \rangle \) and \( | B'B \rangle \) are \( t \)-channel states related to the impact factors, \( Y = \ln(s/s_0) \), \( \hat{K} \) is the BFKL kernel, \( \hat{\Omega} = \hat{\Omega} + \hat{K}_r \), \( \hat{\Omega} = \omega(q_1) + \omega(q_2) \) is the “virtual” part, \( \langle q_1 | \omega_i | q_1' \rangle = \delta(q_1 - q_1') \omega(q_i) \), \( \omega(q) \) is the gluon Regge trajectory; \( \hat{K}_r \) is the “real” part:

\[
\hat{K}_r(q_1, q_1'; q_2, q_2') = \delta(q - q') \frac{1}{q_1 q_2} K_r(q_1, q_1'; q_2, q_2'),
\]

\( \hat{K}_r(q_1, q_1'; q_2, q_2') \) is the usually used expression for it. In the leading order at \( D = 4 + 2\epsilon \)

\[
\omega^{(1)}(q) = -\frac{g^2 N_c \Gamma(1 - \epsilon)}{(4\pi)^{2+\epsilon}} \epsilon(q), \quad K_r^{(1)}(q_1, q_2; q) = \frac{g^2 N_c}{(2\pi)^{D-1}} \left( \frac{q_1 q_2^2 + q_2 q_1'^2}{(q_1 - q_1')^2} - q^2 \right).
\]

The direct Fourier transform of the LO kernel gives [7]

\[
\langle \vec{r}_1 \vec{r}_2 | \hat{K} | \vec{r}_1' \vec{r}_2' \rangle = \langle \vec{r}_1 \vec{r}_2 | \hat{K}_{\text{dip}} | \vec{r}_1' \vec{r}_2' \rangle
\]

\[
= \frac{g^2 N_c \Gamma(1 + \epsilon)}{8\pi^{3+2\epsilon}} \left\{ \frac{\delta(\vec{r}_{11'})}{\vec{r}_{11'}^{2(1+\epsilon)}} + \frac{\delta(\vec{r}_{22'})}{\vec{r}_{22'}^{2(1+\epsilon)}} + \frac{2\delta(\vec{r}_{12'})}{\vec{r}_{11'}^{2(1+\epsilon)} \vec{r}_{22'}^{2(1+\epsilon)}} \right\},
\]

where

\[
\langle \vec{r}_1 \vec{r}_2 | \hat{K}_{\text{dip}} | \vec{r}_1' \vec{r}_2' \rangle = \frac{g^2 N_c \Gamma(1 + \epsilon)}{8\pi^{3+2\epsilon}} \int d^{2+\epsilon} \rho \left( \frac{\vec{r}_{1\mu}}{\vec{r}_{1\mu}^{2(1+\epsilon)}} - \frac{\vec{r}_{2\rho}}{\vec{r}_{2\rho}^{2(1+\epsilon)}} \right)^2 \times (\delta(\vec{r}_{11'}) \delta(\vec{r}_{2\rho}) + \delta(\vec{r}_{22'}) \delta(\vec{r}_{1\mu}) - \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}))
\]

is the dipole kernel in the \((D - 2)\)-dimensional space. Here and below \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \), \( \vec{r}_{ij'} = \vec{r}_i - \vec{r}_j', \vec{r}_{ij'} = \vec{r}_i - \vec{r}_j', \vec{r}_{i\mu} = \vec{r}_i - \vec{p} \). At \( D = 4 \) \( \langle \vec{r}_1 \vec{r}_2 | \hat{K}_{\text{dip}} | \vec{r}_1' \vec{r}_2' \rangle \) acquires well known form:

\[
\langle \vec{r}_1 \vec{r}_2 | \hat{K}_{\text{dip}} | \vec{r}_1' \vec{r}_2' \rangle = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \rho \frac{\vec{r}_{1\mu}^2}{\vec{r}_{1\mu}^{2(1+\epsilon)}} \left( \delta(\vec{r}_{11'}) \delta(\vec{r}_{2\rho}) + \delta(\vec{r}_{22'}) \delta(\vec{r}_{1\mu}) - \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \right).
\]
It is seen that the BFKL kernel is not equivalent to the dipole one. Actually the first is more general than the second. This is clear, because the BFKL kernel can be applied not only in the case of scattering of colourless objects.

However, when it is applied to the latter case, one can use the “dipole” and “gauge invariance” properties of targets and projectiles [5] and omit the terms in the kernel proportional to $\delta(\vec{r}_1\cdot\vec{r}_2)$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the “dipole” property.

Indeed, for colourless objects the impact factors in the representation (1) are “gauge invariant”:

$$\langle A'\bar{A}|\bar{g},0 \rangle = \langle A'\bar{A}|0,\bar{g} \rangle = 0.$$ 

Therefore $\langle A'\bar{A} | \Psi \rangle = 0$ if $\langle \vec{r}_1, \vec{r}_2 | \Psi \rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$. Since $\langle \vec{q}_1, \vec{q}_2 | \bar{\cal K}_r | \vec{q}_1, \vec{q}_2 \rangle$ vanishes at $\vec{q}_1 = 0$ or $\vec{q}_2 = 0$, $\langle A'\bar{A} | \bar{\cal K} \rangle$ is “gauge invariant” as well. It means that we can change $|In\rangle \equiv (\vec{q}_1^2 \vec{q}_2^2)^{-1} |\vec{B}'\vec{B}\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the “dipole” property $\langle \vec{r}, \vec{r} | In_d \rangle = 0$. After this the terms proportional to $\delta(\vec{r}_1\cdot\vec{r}_2)$ can be omitted, and independent either of $\vec{r}_1$ or of $\vec{r}_2$ terms can be changed so that $\bar{\cal K} \to \bar{\cal K}_{dip}$ with the property $\langle \vec{r}, \vec{r} | \bar{\cal K}_{dip} | \vec{r}', \vec{r}' \rangle = 0$. The coordinate representation of the kernel obtained in such a way is what we call the dipole form of the BFKL kernel.

3 A general structure of the dipole form of the NLO kernel and its ambiguity

In the NLO the dipole form can be written as

$$\langle \vec{r}_1, \vec{r}_2 | \bar{\cal K}^{NLO}_d | \vec{r}_1', \vec{r}_2' \rangle = \frac{\alpha_s^2(\mu) N_c^2}{4\pi^3} \left[ \delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \int d\vec{\rho} g^0(\vec{r}_1, \vec{r}_2; \vec{\rho}) + \delta(\vec{r}_{11'}) g(\vec{r}_1, \vec{r}_2; \vec{r}_{11'}) + \delta(\vec{r}_{22'}) g(\vec{r}_{22'}, \vec{r}_1; \vec{r}_{11'}) + \frac{1}{\pi} g(\vec{r}_1, \vec{r}_2; \vec{r}_{11'}') + \frac{1}{\pi} g(\vec{r}_{22'}, \vec{r}_1; \vec{r}_{11'}) \right]$$

(6)

with the functions $g$ turning into zero when their first two arguments coincide. The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the “dipole” property of the “target” impact factors. The coefficient of $\delta(\vec{r}_{11'}) \delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident. The term $g(\vec{r}_1, \vec{r}_2; \vec{r}_{11'}, \vec{r}_{22'})$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

The discontinuities (1) are invariant under the operator transformation

$$\bar{\cal K} \to \tilde{O}^{-1} \bar{\cal K} \tilde{O}, \quad \langle A'\bar{A} | \to \langle A'\bar{A} | \tilde{O}, \quad \frac{1}{\vec{q}_1^2 \vec{q}_2^2} |\vec{B}'\vec{B}\rangle \to \tilde{O}^{-1} \frac{1}{\vec{q}_1^2 \vec{q}_2^2} |\vec{B}'\vec{B}\rangle.$$ 

(7)

In the LO the kernel can be fixed by the requirement of the Möbius invariance of its dipole form. But even after this transformations with $\bar{\cal O} = 1 + \tilde{O}$, where $\tilde{O} \sim g^a$, are still possible. At that

$$\bar{\cal K} \to \bar{\cal K} - [\bar{\cal K}^B \tilde{O}].$$

(8)

These transformations rearrange NLO corrections to the kernel and impact factors. They can be used for simplification of the dipole form.
4 The quark contribution

The simplest piece of the NLO BFKL kernel is the “non-Abelian” (leading in $N_c$) part of the quark contribution. It is known at arbitrary $D$ [2]. Its dipole form is found [7] at arbitrary $D$ as well. However, it is rather complicated. In the physical space-time dimension $D = 4$ the dipole form can be obtained in a much easier way, starting from the renormalized BFKL kernel at $D = 4$ in a specific form [7]. It occurs that the dipole form of the original “non-Abelian” part [7] contains the term $g(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2')$ and is not very simple. But the operator transformation (8) with

$$\hat{O}_Q = \frac{\alpha_s(\mu)}{8\pi} \frac{2}{3} n_f \ln \left( \frac{\hat{q}_1^2}{\hat{q}_2^2} \right)$$  \hspace{1cm} (9)$$

removes this term and simplifies the dipole form considerably. After this transformation we remain with

$$g_Q(\vec{r}_1, \vec{r}_2; \vec{r}) = -g^0_Q(\vec{r}_1, \vec{r}_2; \vec{r}) = \frac{n_f}{3N_c} \left( \frac{r_{12}^2}{r_{1p}^2 r_{2p}^2} \ln \frac{\hat{r}_{1p}^2}{\hat{r}_{2p}^2} + \frac{r_{1p}^2 - r_{2p}^2}{r_{1p}^2 r_{2p}^2} \ln \frac{\hat{r}_{1p}^2}{\hat{r}_{2p}^2} \right),$$  \hspace{1cm} (10)$$

where

$$\ln \hat{r}_{12}^2 = \frac{5}{3} + 2\psi(1) - \ln \frac{\mu^2}{4}.$$  \hspace{1cm} (11)$$

The result agrees with the quark contribution to the small-x evolution of color dipoles [8].

The “Abelian” contribution was calculated in the momentum representation many years ago in the framework of QED [9] and is extremely complicated. It turns out, however, that the dipole form of the Abelian” part of the quark contribution is quite simple. This part contributes only to $g(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2')$:

$$g_Q(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2') = \frac{n_f}{N_c^3} \frac{1}{r_{12}^4} \left[ \left( \frac{r_{12}^2 r_{12'}^2 + r_{11'}^2 r_{22'}^2 - r_{12'}^2 r_{12'}^2}{2(r_{12'}^2 r_{12'}^2 - r_{11'}^2 r_{22'}^2)} \ln \frac{r_{12'}^2 r_{12'}^2}{r_{11'}^2 r_{22'}^2} - 1 \right) \right].$$  \hspace{1cm} (12)$$

It coincides with the corresponding part of the quark contribution to the dipole kernel [8]. Moreover, it is conformal invariant. It could be especially interesting for the QED Pomeron.

5 The gluon contribution

Note that the transformation of the kernel (8) must be accompanied by a corresponding transformation of the impact factors. The transformation (8) with $\hat{O} = \hat{O}_Q$ (9) removes quark parts of NLO corrections to impact factors, so that all related to quarks corrections turn out to be included in the kernel. The reason is the simplicity of the quark corrections. They are related to the charge renormalization only. For the gluon contribution transformations which include all corrections to the kernel are not known. It would be a miracle if they existed at all.

It seems reasonable to perform the transformation related to the charge renormalization, i.e. the transformation (8) with

$$\hat{O}_G = \frac{\alpha_s(\mu)}{8\pi} \left( -\frac{11}{3} N_c \right) \ln \left( \frac{\hat{q}_1^2}{\hat{q}_2^2} \right).$$  \hspace{1cm} (13)$$
With this transformation the result \([11]\) is the following.

\[
g_G^0(\vec{r}_1, \vec{r}_2; \rho) = \frac{3}{2} \frac{r_{12}^2}{r_{1p}^2 r_{2p}^2} \ln \left( \frac{r_{1p}^2}{r_{12}} \right) \ln \left( \frac{r_{2p}^2}{r_{12}} \right) - \frac{11}{12} \left[ \frac{r_{12}^2}{r_{1p}^2 r_{2p}^2} \ln \left( \frac{r_{1p}^2 r_{2p}^2}{r_G^2} \right) \right] + \left( \frac{1}{r_{2p}^2} - \frac{1}{r_{1p}^2} \right) \ln \left( \frac{r_{12}^2}{r_{2p}^2} \right),
\]

\[(14)\]

\[
g_G(\vec{r}_1, \vec{r}_2; \vec{r}_2') = \frac{11}{6} \frac{r_{12}^2}{r_{22'} r_{12}^2} \ln \left( \frac{r_{12}^2}{r_G^2} \right) + \frac{11}{6} \left( \frac{1}{r_{22'}^2} - \frac{1}{r_{12}^2} \right) \ln \left( \frac{r_{12}^2}{r_{22'}^2} \right) + \frac{1}{2r_{22'}^2} \ln \left( \frac{r_{12}^2}{r_{22'}^2} \right) - \frac{r_{12}^2}{2r_{22'}^2 r_{12}^2} \ln \left( \frac{r_{12}^2}{r_{22'}^2} \right) - \frac{r_{12}^2}{2r_{22'}^2 r_{12}^2} \ln \left( \frac{r_{12}^2}{r_{22'}^2} \right),
\]

\[(15)\]

where

\[
\ln r_G^2 = 2 \psi(1) - \ln \frac{\mu^2}{4} - \frac{3}{11} \left( \frac{67}{9} - 2 \zeta(2) \right).
\]

Both \(g_G^0(\vec{r}_1, \vec{r}_2; \rho)\) and \(g_G(\vec{r}_1, \vec{r}_2; \vec{r}_2)\) vanish at \(\vec{r}_1 = \vec{r}_2\). Then, these functions turn into zero for \(\vec{r}_1^2 \to \infty\) faster than \((\vec{r}_2^2)^{-1}\) to provide the infrared safety. The ultraviolet singularities of these functions at \(\vec{r} = \vec{r}_2\) and \(\vec{r} = \vec{r}_1\) cancel on account of the “dipole” property of the “target” impact factors.

The most complicated is the structure which is absent in the LO:

\[
g_G(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2') = \left[ \frac{\vec{r}_{12}^2}{r_{11'}^2 r_{22'}^2} - \frac{2}{r_{11'}^2 r_{12}^2} + \frac{2}{r_{22'}^2 r_{12}^2} \right] \ln \left( \frac{r_{12}^2}{r_{11'}^2 r_{12}^2} \right) + \frac{1}{2r_{11'}^2} \left[ \frac{\vec{r}_{11'}^2 \vec{r}_{22'}^2}{r_{11'}^2} + \frac{\vec{r}_{22'}^2}{r_{22'}^2} - \frac{2 \vec{r}_{11'} \vec{r}_{22'}}{r_{22'}^2} \right] \ln \left( \frac{r_{12}^2}{r_{22'}^2} \right) + \frac{1}{d} \left[ \frac{\vec{r}_{11'}^2 \vec{r}_{22'}^2}{d} - \frac{\vec{r}_{12}^2 \vec{r}_{22'}}{d} \right] \ln \left( \frac{r_{11'}^2}{r_{22'}^2} \right) - \frac{\vec{r}_{12}^2}{r_{11'}^2 r_{22'}^2} \ln \left( \frac{r_{11'}^2}{r_{22'}^2} \right) + (1 \leftrightarrow 2),
\]

\[(17)\]

where

\[
d = r_{12}^2 r_{21'}^2 - r_{11'}^2 r_{22'}^2.
\]

This term also vanishes at \(\vec{r}_1 = \vec{r}_2\), so that it possesses the “dipole” property. It has ultraviolet singularity only at \(\vec{r}_{12}^2 = 0\) and tends to zero at large \(\vec{r}_{11'}^2\) and \(\vec{r}_{22'}^2\) sufficiently quickly in order to provide the infrared safety.

The term \((17)\) violates conformal invariance, although it has no relation to the charge renormalization. Remind, however, the ambiguity of the NLO kernel discussed in Section (3). The transformation \((18)\) can change non-invariant contributions. It is not yet clear, if it is possible to find such transformation which makes \(g(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2')\) conformal invariant.
6 Summary

The colour singlet BFKL kernel is more general than the dipole one. However, in the case of scattering of colourless objects the BFKL kernel can be written in the dipole form (Möbius representation). In the NLO the dipole form is greatly simplified in comparison with the BFKL kernel in the momentum space. The quark contribution to the dipole form agrees with the corresponding contribution to the kernel of the colour dipole approach. It would be extremely interesting to compare corresponding gluon contributions. Unfortunately, such contribution to the kernel of the colour dipole approach is not yet known. The “Abelian” part of the quark contribution is conformal invariant. The ambiguity of the NLO kernel reserves a hope for conformal invariance of the dipole form of the colour singlet NLO BFKL kernel at $N = 4$ SUSY.

References


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