Structure constant of twist-2 light-ray operators in the Regge limit

Ian Balitsky, Vladimir Kazakov, and Evgeny Sobko

I. INTRODUCTION

The problem of high-energy behavior of amplitudes has a long story [1,2]. One of the most popular approaches is to reduce the gauge theory at high energies to $2+1$ effective theory which can be solved exactly or by computer simulations. Unfortunately, despite the multitude of attempts, the Lagrangian for $2+1$ QCD is not written yet. In this context the idea to solve formally the former is now exactly and efficiently computable

In this work we calculate the three-point correlator of twist-2 operators $\mathcal{O}_j(x) = tr F_+^j F_+^{j+2} F_+^j$ + fermions + scalars in $\mathcal{N}=4$ SYM in the BFKL limit [5] when $\omega = j - 1 \to 0$, the 't Hooft coupling $g_s^2 \equiv N_c g_s^2 / 4 \pi \to 0$ and $\frac{1}{N_c}$

fixed, for arbitrary $N_c$. The symbol $+$ in the field-strength tensor $F_{+i}$ means contraction with light-ray vector $n_+$, and the summation over index $i$ goes over two-dimensional space orthogonal to $n_+$ and $n_-$. Since the contribution of fermions + scalars is subleading at this limit, including the internal loops, the result is valid for the pure Yang-Mills theory as well. The case of the two-point correlator was elaborated in our previous paper [6] where we defined the generalized operators with complex spin as special light-ray operators [7] (regularized as a narrow rectangular Wilson contour called a “frame”) and calculated their correlator using OPE over Wilson lines [8] with a rapidity cutoff and the BFKL evolution (see Fig. 1). Here we use the same light-ray operators: one along the $n_+$ direction and two along $n_-$. In this case we should use more general Balitsky-Kovchegov (BK) evolution [9,10], and the leading BFKL contribution comes from the BK vertex.

II. LIGHT-RAY OPERATORS AND THEIR RELATION TO LOCAL OPERATORS

The generalization of local operator $\mathcal{O}^j$ for the case of complex spin $j$ was constructed in Ref. [6]. It has a form of light-ray operator $\mathcal{S}^j$ stretched along the $n_+$ direction and realizing the principal series representation of $sl(2|4)$ with conformal spin $J = \frac{1}{2} + i\nu$ which is related to Lorentz spin $j$ as $J = j + 1$. The full regularized operator reads as follows,

\[
\mathcal{S}^{j+1}(x_{1\perp}) = \mathcal{S}^{j+1}_g(x_{1\perp}) + \frac{i}{2} (j - 1) \mathcal{S}^{j+1}_f(x_{1\perp}) - \frac{1}{2} (j - 1) \mathcal{S}^{j+1}_c(x_{1\perp}),
\]

where, for example, the regularized gluon operator is

\[
\mathcal{S}^{j+1}_g(x_{1\perp}) = \lim_{|x_{1\perp}| \to 0} |x_{1\perp}|^{-\gamma} \mathcal{S}^{j+1}(x_{1\perp}, x_{3\perp}).
\]

FIG. 1. Scheme of computation of the two-point correlator. In the lhs the long sides of regularizing rectangular Wilson frames are stretched along the light ray and the short sides in the orthogonal directions. In the rhs we use OPE of frames over color dipoles and compute their correlator; see Ref. [6] for details.
\[
S_{\ell}^{j} (x_{1\perp}, x_{3\perp}) = \int_{-\infty}^{\infty} \frac{dx_{1\perp}}{x_{1\perp}^2} \frac{dx_{3\perp}}{x_{3\perp}^2} \text{tr} F_{\ell}^4 (x_1) |1, 3\rangle |F_{\perp}^4 (x_3)\]
and \(x_1 = (x_{1\perp}, 0, x_{1\perp}), x_3 = (x_{3\perp}, 0, x_{3\perp})\). The anomalous dimension \(\gamma_j\) corresponds to operator \(\tilde{S}_{\ell}^{j+1} (x_{1\perp})\). Here we introduced the notation \([1, 3]\) for a rectangular Wilson contour with coordinates \(x_1, x_3\) of two diagonally opposite corners, as in Fig. 1. In the case of even integer Lorentz spin \(j\), it can be rewritten as an integral of local operator \(O^j (x)\) with dimension \(\Delta (j)\) along a light-ray direction \(n_{\perp}\):
\[
\tilde{S}_{\ell}^{j+1} (x_{1\perp}) |_{n_{\perp}} \sim \int_{-\infty}^{\infty} dx_{\perp} O^j (x). \tag{2}
\]

In this case the correlator of two light-ray operators stretched along \(n_{\perp}\) and \(-n_{\perp}\) vectors, normalized as \(\langle n_{\perp}, n_{\perp} \rangle = 1\), is just the double integral of two-point correlator of local operators with respect to light-ray directions \(n_{\perp}\):
\[
\langle \tilde{S}_{\ell}^{j+1} (x_{1\perp}) S_{\ell}^{j+1} (y_{1\perp}) \rangle = \frac{\delta (j_1 - j_2) b_{1j}}{(|x - y|^2)^{\Delta (j) - 1}}. \tag{3}
\]

In this work we calculate the correlator of three light-ray operators, restricting ourselves to a particular simple kinematics: one light-ray operator is stretched along the \(n_{\perp}\) light-ray direction, and two others are stretched along \(-n_{\perp}\). The correlator of three light-ray operators can be obtained by integrating the correlator of three local operators along these light rays. The tensor structures of such local correlators are known from general group-theoretical considerations [11], up to a few structure constants depending on the coupling and symmetry charges. The main problem which we are addressing here is the calculation of these nontrivial constants. Remarkably, if the coordinates of all three light-ray operators in the transverse space are restricted to the same line, all these structures collapse into a single one [12], with a single overall structure constant which we are going to compute. Note that after a conformal transformation the three points in the transverse space take arbitrary positions.

However, the configuration with two collinear light-ray operators is singular, so we first consider three different light-ray directions \(n_1, n_2, n_3\) and then take the limit \(n_2 \to n_3\). The result of integration along light rays is quite simple and contains only one unknown overall constant, \(C_{\langle n_{\perp} \rangle} |\{ \Delta_j \}, \{ j_i \} \rangle\):
\[
\langle \tilde{S}_{\ell}^{j_1+1} (x_{1\perp}) \tilde{S}_{\ell}^{j_2+1} (y_{1\perp}) \tilde{S}_{\ell}^{j_3+1} (z_{1\perp}) \rangle = C_{\langle n_{\perp} \rangle} |\{ \Delta_j \}, \{ j_i \} \rangle \times \frac{\langle n_1 n_2 \rangle |\{ j_{1,2,3} \}, \{ n_{1,2,3} \} \rangle |\{ j_{1,2,3} \}, \{ n_{1,2,3} \} \rangle}{(|x - y|^2)^{\Delta (j) - 1} (|x - z|^2)^{\Delta (j) - 1}}. \tag{4}
\]

In what follows, we assume the existence of a good analytic continuation for \(C_{\langle n_{\perp} \rangle} |\{ \Delta (j) \}, \{ j_i \} \rangle\) to noninteger \(\{ j_i \}\)’s. We take the limit \(n_1 = n_{\perp}, n_2 = n_{\perp}, n_3 \to n_2\) with the normalization \(\langle n_{\perp}, n_{\perp} \rangle = 1\). In the BFKL regime \(j_i = 1 + \omega_i \to 1\) we obtain
\[
\langle \tilde{S}_{\ell}^{j_1+1} (x_{1\perp}) \tilde{S}_{\ell}^{j_2+1} (y_{1\perp}) \tilde{S}_{\ell}^{j_3+1} (z_{1\perp}) \rangle \approx \frac{\langle n_2 n_3 \rangle |\{ n_{1,2,3} \} \rangle |\{ n_{1,2,3} \} \rangle}{(|x - y|^2)^{\Delta (j) - 1} (|x - z|^2)^{\Delta (j) - 1}}.
\]

Finally the structure constant is normalized using the corresponding two-point correlators:
\[
C_{\omega_1, \omega_2, \omega_3} = \frac{C_{\omega_1, \omega_2, \omega_3}}{b_{1+\omega_1} b_{1+\omega_2} b_{1+\omega_3}}. \tag{5}
\]

III. DECOMPOSITION OVER DIPOLES AND BK EVOLUTION

When calculating the two-point correlator [6], we used a point splitting regularization in the orthogonal direction, replacing light-ray operators by infinitely narrow Wilson frames with inserted fields in the corners (see Fig. 1). Now, for the sake of simplicity, we carry out our calculation for pure Wilson frames, related to our operators with zero \(R\)-charge in the following way:
\[
\partial_{x_{1\perp}} \cdot \partial_{x_{3\perp}} \int \frac{dx_{1\perp} dx_{3\perp}}{(x_3 - x_1)^{\gamma_1 + \gamma_3}} [x_1, x_3] \to \left| x_{1\perp}, x_{3\perp} \right| c (g_Y^2 N_c \omega_1) \tilde{S}_{\ell}^{j_1+1} (x_{1\perp}) \tilde{S}_{\ell}^{j_3+1} (x_{3\perp}) \to \left| x_{1\perp}, x_{3\perp} \right| N (1 - U^{\sigma_{\perp}} (x_{1\perp}, x_{3\perp})) \tag{7}
\]

The coefficient \(c (g_Y^2 N_c \omega_1)\) [denoted below as \(c (\omega_1)\)] depends on the local regularization procedure, and at weak coupling it behaves as \(c (\omega_1) \sim g_Y^2 \omega_1\), but its explicit form is irrelevant for us because we are going to calculate the normalized structure constant where it cancels. In general, there are a few types of leading twist-2 operators which appear in this decomposition, but in the BFKL limit a single one with the smallest anomalous dimension survives. In addition, in the \(\omega_i \to 0\) limit, only the term built out of gauge fields alone contributes [6].

Following the OPE method [8], the pure Wilson frames can be replaced by regularized color dipoles, \(x_{1\perp}, x_{3\perp} \to N (1 - U^{\sigma_{\perp}} (x_{1\perp}, x_{3\perp}))\).
\[
\mathbf{U}^{\sigma}(x_{1\perp},x_{3\perp}) = 1 - \frac{1}{N} \text{tr}(\mathbf{U}^{\sigma}_{x_{1\perp}} \mathbf{U}^{\sigma}_{x_{3\perp}}), \tag{9}
\]

\[
\mathbf{U}^{\sigma}_{x_{1\perp}} = P \exp \left[ ig_{\text{YM}} \int_{-\infty}^{\infty} dx_{1} A^{\sigma}_{+}(x) \right], \tag{10}
\]

\[
A^{\sigma}_{+}(x) = \int d^{4}k \theta(\sigma_{+} - |k_{+}|) e^{ikx} A_{\mu}(k) \tag{11}
\]

and \(\sigma_{+}\) is a longitudinal cutoff in the \(n_{+}\) direction. Now we can write

\[
\langle S^{2+\omega}(x_{1\perp},x_{3\perp})S^{2+\omega}(y_{1\perp},y_{3\perp})S^{2+\omega}(z_{1\perp},z_{3\perp}) \rangle
\]

\[
= -D_{\perp} \int_{-\infty}^{\infty} dx_{1\perp} \int_{y_{1\perp}}^{\infty} dy_{1\perp} \int_{y_{3\perp}}^{\infty} dz_{1\perp} \int_{y_{3\perp}}^{\infty} dz_{3\perp}
\]

\[
\times \langle \mathbf{U}^{\nu_{1\perp}}(x_{1\perp},x_{3\perp}) \mathbf{V}^{\nu_{2\perp}}(y_{1\perp},y_{3\perp}) \mathbf{W}^{\nu_{3\perp}}(z_{1\perp},z_{3\perp}) \rangle, \tag{12}
\]

where \(D_{\perp} = N^{4}(\partial_{\nu_{1\perp}}\partial_{\nu_{2\perp}}\partial_{\nu_{3\perp}})(\partial_{\nu_{1\perp}}\partial_{\nu_{2\perp}}\partial_{\nu_{3\perp}})\).

In our kinematics two dipoles \(V\) and \(W\) have zero \(n_{\perp}\) projection, and in the BFKL approximation they form a "pancake" field configuration in the reference frame related to \(U\). This means that the rapidity of \(U\) serves as the upper limit for integrations with respect to rapidities of \(V\) and \(W\) in our logarithmic approximation. Now we use the BK evolution equation \([9,10]\) to calculate the quantum average in (12). It gives the evolution of the dipole \(U\) with respect to rapidity \(Y = e^{\sigma}\), namely

\[
\sigma \frac{d}{d\sigma} \mathbf{U}(z_{1\perp},z_{2\perp}) = \mathbf{K}_{\text{BK}} \ast \mathbf{U}(z_{1\perp},z_{2\perp}), \tag{13}
\]

where \(\mathbf{K}_{\text{BK}}\) is an integral operator having the following form in the leading-order (LO) approximation:

\[
\mathbf{K}_{\text{LOBK}} \ast \mathbf{U}(z_{1\perp},z_{2\perp}) = \frac{2g^{2}}{\pi} \int d^{2}z_{3\perp} \frac{z_{3\perp}^{2}}{z_{13\perp}^{2}} \mathbf{U}(z_{1\perp},z_{3\perp})
\]

\[
+ \mathbf{U}(z_{3\perp},z_{2\perp}) - \mathbf{U}(z_{1\perp},z_{2\perp})
\]

\[
- \mathbf{U}(z_{1\perp},z_{3\perp}) \mathbf{U}(z_{2\perp},z_{2\perp}). \tag{14}
\]

The evolution of \(U^{Y_{1}}\) goes from \(Y_{1}\) to an intermediate \(Y_{0}\) with respect to the linear part of (13), and then the BK vertex acts at \(Y_{0}\) and generates two dipoles which can be contracted with \(V^{Y_{2}}\) and \(W^{Y_{3}}\). Schematically, it can be written as

\[
\int dY_{0} (\mathbf{U}^{Y_{1}} \rightarrow \mathbf{U}^{Y_{0}}) \otimes (\text{BK vertex at } Y_{0}) \otimes (\langle \mathbf{U}^{Y_{0}} \mathbf{V}^{Y_{2}} \rangle \otimes \langle \mathbf{U}^{Y_{0}} \mathbf{W}^{Y_{3}} \rangle).
\]

The linear BFKL evolution of \(U^{Y_{1}}\) from \(Y_{1}\) to \(Y_{0}\) gives
The usual delta-function $\delta(o_1-o_2-o_3)$ is a consequence of boost invariance as in the formula (5). $\gamma_{pl}$ represents the planar contribution of the BK vertex,

$$\gamma_{pl}(\nu_1,\nu_2,\nu_3;x_0,y_0,z_0) = \int \frac{d^2\nu \cdot d^2\gamma}{|\nu - \gamma|^4} E_{\nu_1}(\beta - x_0, \gamma - x_0) E_{\nu_2}(\beta - y_0, \gamma - y_0) E_{\nu_3}(\beta - z_0, \gamma - z_0)$$

$$= \frac{\Lambda(h_1, h_2, h_3)}{|x_0 - y_0|^4 |h_1, h_2, h_3|^2 |x_0 - z_0|^4 |h_1, h_2, h_3|^2 |y_0 - z_0|^4 |h_1, h_2, h_3|^2}, \quad (21)$$

where $h_1 = \frac{1}{2} + i\nu_1$, $h_2 = \frac{1}{2} + i\nu_2$, $h_3 = \frac{1}{2} + i\nu_3$ and the function $\Lambda(h_1, h_2, h_3)$ was presented in Ref. [15].

Remarkably we can also take into account the nonplanar contribution [15,16], thus providing the finite $N_c$ answer for the BFKL structure constant. It appears as a single extra term $\gamma_{npl}$,

$$\gamma_{npl}(\nu_1,\nu_2,\nu_3;x_0,y_0,z_0) = \int \frac{d^2\nu \cdot d^2\gamma}{|\nu - \gamma|^4} E_{\nu_1}(\beta - x_0, \gamma - x_0) E_{\nu_2}(\beta - y_0, \gamma - y_0) E_{\nu_3}(\beta - z_0, \gamma - z_0)$$

$$= -\frac{\psi(1) + \psi\left(\frac{1}{2} + i\nu_1\right) - \psi\left(\frac{1}{2} + i\nu_2\right) - \psi\left(\frac{1}{2} + i\nu_3\right)}{\pi N_c \gamma_{pl} |\nu|^2} [\psi(1) + \psi\left(\frac{1}{2} + i\nu\right) - \psi\left(\frac{1}{2} + i\nu_2\right) - \psi\left(\frac{1}{2} + i\nu_3\right)] \cdot \pi \Gamma^2(\frac{1}{2} + i\nu) \Gamma(-2i\nu) \Gamma(1 + 2i\nu), \quad (22)$$

The integrals over $x_0, y_0, z_0$ are easily computable, e.g.

$$\int d^2x_{\nu_1} E_{\nu_1}(\beta - x_0, \gamma - x_0) E_{\nu_1}(x_{10}, x_{30}) = (\tau^2)^{1+i\nu_1/2} F_1\left(\frac{1}{2} + i\nu, 1 + 1 + 2i\nu, \tau \right) F_1\left(\frac{1}{2} + i\nu, 1 + 1 + 2i\nu, \tau \right)$$

$$\times \frac{(1 + \nu^2)^2}{\nu^2} G(\nu) + (\nu_{\nu_{\nu_1}} - \nu), \quad (23)$$

where $\tau = \frac{|x_{10} - x_{30}|^{1/2}}{|x_{10} - x_{30}|^{1/2}}$. In the limit $x_{10}, x_{30} \to x$, we can replace $\frac{|x_{10} - x_{30}|^{1/2}}{|x_{10} - x_{30}|^{1/2}} \to 1$. For small $\tau$ we close the $\nu_1$ contour in the lower (upper) half-plane for first (second) term, respectively, both of them giving the same contribution. Integrators over $\alpha, \beta, \gamma$ in (19) can be reduced to $\gamma_{pl}$ represented in Ref. [15] in terms of hypergeometric and Meijer G functions and $\gamma_{npl}$ in terms of $\gamma$ functions. Integrals over $\nu_3$ can be done by picking up the BFKL poles $\nu_3 = \mathcal{N}(\nu_3)$.

Combining (19), (22) and (23), we come to the final expression for the three-point correlation function,
The structure of the three-point correlator.

\[ \Omega \left( z_{1\perp}, z_{2\perp}, z_{3\perp} \right) = \frac{1}{N_c^2} \left( \frac{z_{1\perp} \cdot z_{2\perp}}{z_{1\perp} \cdot z_{3\perp}} \right) \] (defined in [15]). The \( \alpha \beta \)-triangle in the first, planar term and \( \beta \gamma \)-link in the second, nonplanar term correspond to the triple pomeron vertex.

\[ \gamma_i = \gamma(1 + \omega_i) \] are anomalous dimensions and the coefficient \( \Psi(\nu_1, \nu_2, \nu_3) \) has the form

\[ \Psi(\nu_1, \nu_2, \nu_3) = \Omega(h_1^*, h_2^*, h_3^*) - \frac{2\pi}{N_c} \Lambda(h_1^*, h_2^*, h_3^*) \cdot \Re(\psi(1)) - \psi(h_1^*) - \psi(h_2^*) - \psi(h_3^*), \]

\[ h_i^* = \frac{1}{2} + i\nu_i^* = 1 + \gamma_i^* \frac{1}{2}. \]

The functions \( \Omega(h_1, h_2, h_3) \) and \( \Lambda(h_1, h_2, h_3) \) (defined in (20) and (21)) and calculated in [15].

Our final result for the normalized structure constant is

\[ C_{\omega_1, \omega_2, \omega_3} = -i\gamma^2 \frac{\sqrt{N_c^2 - 1}}{\sqrt{2\pi N_c} \frac{D}{\gamma_i} \frac{\gamma_i}{\omega_3}} (\omega_1^2 \omega_2 + \omega_3) \]

\[ + \omega_3^2 (\omega_1 + \omega_3) + \omega_2^2 (\omega_1 + \omega_2) + \omega_1 \omega_2 \omega_3 \underbrace{(1 + O(\gamma^2))}_{(30)}. \]

whereas the nonplanar one is \( O(g^6) \). It might seem strange that the planar contribution does not start from \( O(g^4) \) terms given by the leading Feynman graphs, e.g. with four gluon vertices. However, in the BFKL approximation, we should keep \( \frac{\gamma^2}{\omega_i} \gg \omega_i \). In addition when making the point-splitting regularization, we have to keep \( \gamma_i \ln(x_{31\perp}/(x - y))^2_i \gg 1 \). The limit \( |x_{31\perp}| \) has to be taken first, which makes the value \( g^2 \simeq 0 \) exceptional. This order of limits leads to the \( O(g^2) \) behavior of (30).

IV. DISCUSSION

Our result, Eq. (28), based on the BFKL approximation is a rare example of computation of a structure constant of three unprotected operators receiving contributions from all orders in a coupling constant, including infinitely many “wrapping” corrections. Moreover, our result is valid at any \( N_c \). Since in the LO BFKL, the contributions of all fields but gluons in \( N_c = 4 \) Supersymmetric Yang-Mills (SYM) disappear from both the definition of operators and internal loops, the result is applicable to pure Yang-Mills theory at any \( N_c \), including \( N_c = 3 \). It would be interesting to apply our structure constants to the OPE at hard scattering in real QCD and to work out the full “dictionary,” relating them to the OPE in the two-dimensional \( \mathrm{SL}(2, C) \) conformal field theory—the basis of our BFKL computation. It is also not hopeless, though challenging, to compute these structure constants in the next-to-leading-order approximation in \( N_c = 4 \) SYM. Our present result may serve as an important, all-wrappings test for the future computations of similar quantities in the integrability approaches to planar \( \mathrm{AdS}_5/\mathrm{CFT}_4 \), such as Ref. [4] and the BFKL limit of the quantum spectral curve [17].

ACKNOWLEDGMENTS

We thank J. Bartels, S. Caron-Huot, L. Lipatov, and V. Schomerus for discussions. Our special thanks to G. Korchemsky who participated in the initial stage of this work. The work of E.S. and V.K. was supported by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme FP7/2007-2013/
under REA Grant Agreement No. 317089 (GATIS). The work of V. K. has received funding from the European Research Council (Programme Ideas Grant No. ERC-2012-AdG 320769 AdS-CFT-solvable), from the ANR grant StrongInt (Grant No. BLANC- SIMI- 4-2011) and from ESF Grant No. HOLOGRAV-09- RNP- 092. The work of I. B. was supported by DOE Contract No. DE-AC05-06OR23177 and by Grant No. DE-FG02-97ER41028.

[12] The similar phenomenon was observed in Ref. [13].