Leading-order hadronic contribution to the electron and muon $g − 2$

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Abstract

I present a new data driven update of the hadronic vacuum polarization effects for the muon and the electron $g − 2$. For the leading order contributions I find $a_μ^{\text{had}(1)} = (688.57 \pm 4.28) \times 10^{-10}$ based on $e^+e^−$ data [incl. $τ$ data], $a_μ^{\text{had}(2)} = (−9.92 \pm 0.10) \times 10^{-10}$ (NLO) and $a_μ^{\text{had}(3)} = (1.23 \pm 0.01) \times 10^{-10}$ (NNLO) for the muon, and $a_e^{\text{had}(1)} = (185.11 \pm 1.24) \times 10^{-14}$ (LO), $a_e^{\text{had}(2)} = (−22.15 \pm 0.16) \times 10^{-14}$ (NLO) and $a_e^{\text{had}(3)} = (2.80 \pm 0.02) \times 10^{-14}$ (NNLO) for the electron. A problem with vacuum polarization undressing of cross-sections (time-like region) is addressed. I also add a comment on properly including axial mesons in the hadronic light-by-light scattering contribution. My estimate here reads $a_μ[a_1, f_f', f_f] \sim (7.51 \pm 2.71) \times 10^{-11}$. With these updates $a_μ^{\text{exp}} - a_μ^{\text{the}} = (31.0 \pm 8.2) \times 10^{-10}$ a 3.8σ deviation, while $a_e^{\text{exp}} - a_e^{\text{the}} = (−1.14 \pm 0.82) \times 10^{-12}$ shows no significant deviation.

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Abstract. I present a new data driven update of the hadronic vacuum polarization effects for the muon and the electron $g - 2$. For the leading order contributions I find $a_{\mu}^{\text{had}(1)} = (688.91 \pm 3.52) \times 10^{-10}$ based on $e^+ e^-$ data [incl. $\tau$ data], $a_{\mu}^{\text{had}(2)} = (-10.11 \pm 5.70) \times 10^{-10}$ (NLO) and $a_{\mu}^{\text{had}(3)} = (1.23 \pm 0.01) \times 10^{-10}$ (NNLO) for the muon, and $a_{\mu}^{\text{had}(1)} = (185.11 \pm 1.24) \times 10^{-14}$ (LO), $a_{\mu}^{\text{had}(2)} = (-22.15 \pm 0.16) \times 10^{-14}$ (NLO) and $a_{\mu}^{\text{had}(3)} = (2.80 \pm 0.02) \times 10^{-14}$ (NNLO) for the electron. A problem with vacuum polarization undressing of cross-sections (time-like region) is addressed. I also add a comment on properly including axial mesons in the hadronic light-by-light scattering contribution. My estimate here reads $a_{\mu}[a_1, f_1, f_1] \sim (7.51 \pm 2.71) \times 10^{-11}$.

With these updates $a_{\mu}^{\text{had}} - a_{\mu}^{\text{exp}} = (31.0 \pm 8.2) \times 10^{-10}$ a 3.8 $\sigma$ deviation, while $a_{\mu}^{\text{had}} - a_{\mu}^{\text{LO}} = (-1.14 \pm 0.82) \times 10^{-12}$ shows no significant deviation.

1 Introduction: hadronic effects in $g - 2$.

A well known general problem in electroweak precision physics are the higher order contributions from hadrons (quark loops) at low energy scales. While leptons primarily exhibit the fairly weak electromagnetic interaction, which can be treated in perturbation theory, the quarks are strongly interacting via confined gluons where any perturbative treatment breaks down. Considering the lepton anomalous magnetic moments one distinguishes three types of non-perturbative corrections: (a) Hadronic Vacuum Polarization (HVP) of order $O(\alpha^2)$, $O(\alpha^3)$, $O(\alpha^4)$; (b) Hadronic Light-by-Light (HLbL) scattering at $O(\alpha^3)$; (c) hadronic effects at $O(\alpha G_f m^2)$ in 2-loop electroweak (EW) corrections, in all cases quark-loops appear as hadronic “blobs”. The hadronic contributions are limiting the precision of the predictions.

Evaluation of non-perturbative effects is possible by using experimental data in conjunction with Dispersion Relations (DR), by low energy effective modeling via a Resonance Lagrangian Approach (RLA) (Vector Meson Dominance (VMD) implemented in accord with chiral structure of QCD) [1–3], like the Hidden Local Symmetry (HLS) or the Extended Nambu Jona-Lasinio (ENJL) models, or by lattice QCD. Specifically: (a) HVP via a dispersion integral over $e^+ e^- \to \text{hadrons}$ data (1 independent amplitude to be determined by one specific data channel) [4]) as elaborated below, by the HLS effective Lagrangian approach [5–9], or by lattice QCD [10–15]; (b) HLbL via a RLA together with operator product expansion (OPE) methods [16–19], by a dispersive approach using $\gamma\gamma \to \text{hadrons}$ data (28 independent amplitudes to be determined by as many independent data sets in principle) [20–22] or by lattice QCD [23, 24]; (c) EW quark-triangle diagrams are well under control, because the possible large corrections are related to the Adler-Bell-Jackiw (ABJ) anomaly which is perturbative and non-perturbative at the same time. Since $VVV = 0$ by the Furry theorem, only VVA (of $\gamma\gamma Z$-vertex, $V$=vector, $A$=axialvector) contributes. In fact leading effects are of short distance type ($M_Z$ mass scale) and cancel against lepton-triangle loops (anomaly cancellation) [25, 26].

2 Leading-order $a_{\mu}^{\text{had}}$ via $\sigma(e^+ e^- \to \text{hadrons})$

The leading non-perturbative hadronic contribution to $a_{\mu}^{\text{had}}$, represented by the diagram figure 1, can be obtained in terms of undressed experimental cross-sections data

$$R_{\gamma}(s) \equiv \sigma^{(0)}(e^+ e^- \to \gamma \to \text{hadrons})/\frac{4\pi\alpha^2}{3s}, \quad (1)$$

$$s = E_{\text{cm}}^2, \; E_{\text{cm}} \text{ the center of mass energy, and the DR:}$$

$$a_{\mu}^{\text{had}} = \left(\frac{\alpha m_{\mu}}{3\pi}\right)^2 \left(\int_{4m^2_\text{lo}}^{E_{\text{cut}}^\mu} ds \frac{R_{\gamma}(s) \tilde{K}(s)}{s^2} + \int_{E_{\text{cut}}^{\text{LO}}}^{\infty} ds \frac{R_{\gamma}^{\text{QCD}}(s) \tilde{K}(s)}{s^2}\right). \quad (2)$$

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The pion form factor \( |F_\pi(s)|^2 = 4R_\pi/s^2 \) (\( \beta_\pi = \sqrt{1 - 4m^2_\pi/s} \)) dominated by the \( \rho \) resonance peak. Data include measurements from Novosibirsk (NSK) [27–29], Frascati (KLOE) [30–32], SLAC (BaBar) [33] and Beijing (BESIII) [34].

**Table 1.** Results for \( a_\mu^{\text{had}} \) (in units \( \times 10^{-10} \)).

<table>
<thead>
<tr>
<th>State</th>
<th>Range (GeV)</th>
<th>( a_\mu^{\text{had}} ) (stat + syst) [m]</th>
<th>rel[abs]%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>(0.28, 1.05)</td>
<td>507.55 (0.39, 2.88) [2.71]</td>
<td>0.3% [3.9%]</td>
</tr>
<tr>
<td>( \omega )</td>
<td>(0.42, 0.81)</td>
<td>352.23 (0.42, 0.95) [1.04]</td>
<td>3.0% [5.9%]</td>
</tr>
<tr>
<td>( \phi )</td>
<td>(1.00, 1.04)</td>
<td>341.48 (0.48, 0.79) [0.92]</td>
<td>2.7% [4.7%]</td>
</tr>
<tr>
<td>( T )</td>
<td>(0.11, 0.00)</td>
<td>8.94 (0.42, 0.41) [0.90]</td>
<td>66.1% [1.9%]</td>
</tr>
<tr>
<td>had (1.05, 2.00)</td>
<td>60.45 (0.21, 2.80) [2.80]</td>
<td>4.6% [42.9%]</td>
<td></td>
</tr>
<tr>
<td>had (2.00, 3.10)</td>
<td>21.63 (0.12, 0.92) [0.93]</td>
<td>4.3% [4.7%]</td>
<td></td>
</tr>
<tr>
<td>had (3.10, 3.60)</td>
<td>3.77 (0.03, 0.10) [0.10]</td>
<td>6.8% [0.0%]</td>
<td></td>
</tr>
<tr>
<td>had (3.60, 5.20)</td>
<td>7.50 (0.04, 0.01) [0.04]</td>
<td>0.3% [0.0%]</td>
<td></td>
</tr>
<tr>
<td>pQCD (5.20, 9.46)</td>
<td>6.27 (0.00, 0.01) [0.01]</td>
<td>0.0% [0.0%]</td>
<td></td>
</tr>
<tr>
<td>pQCD (9.46, 13.00)</td>
<td>1.28 (0.01, 0.07) [0.07]</td>
<td>5.4% [0.0%]</td>
<td></td>
</tr>
<tr>
<td>data (0.28,13.00)</td>
<td>680.77 (0.89, 4.19) [4.28]</td>
<td>0.6% [100%]</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>688.57 (0.89, 4.19) [4.28]</td>
<td>0.6% [100%]</td>
<td></td>
</tr>
</tbody>
</table>

The kernel \( \tilde{K}(s) \) is an analytically known monotonically increasing function, raising from about 0.64 at the two pion threshold 4\( m^2_\pi \) to 1 as \( s \to \infty \). This integral is well defined due to the asymptotic freedom of QCD, which allows for a perturbative QCD (pQCD) evaluation of the high energy contributions. Because of the \( 1/s^2 \) weight, the dominant contribution comes from the lowest lying hadronic resonance, the \( \rho \) meson (see figure 2). As low energy contributions are enhanced, about 75% come from the region \( 2m_\pi < \sqrt{s} < 1 \) GeV dominated by the \( \pi^+\pi^- \) channel. Experimental errors imply theoretical uncertainties, the main issue for the muon \( g - 2 \). Typically, results are collected from different resonances and regions as presented in table 2. Statistical errors (stat) are summed in quadrature, systematic (syst) ones are taken into account linearly (100% correlated) within the different contributions of the list, and summed quadratically from the different regions and resonances. From 5.2 GeV to 9.46 GeV and above 13 GeV pQCD is used. Relative (rel) and absolute (abs) errors are also shown. The distribution of contributions and errors are illustrated in the pie chart figure 3. As a result we find

\[
a_\mu^{\text{had}(1)} = (688.57 \pm 4.28) (688.91 \pm 3.52) \times 10^{-10} \tag{3}
\]

based on \( e^+e^- \) data [incl. \( \tau \)-decay spectra [35]]. In the last 15 years \( e^+e^- \) cross-section measurements have dramatically improved, from energy scans [27–29] at Novosibirsk (NSK) and later, using the radiative return mecha-
Flavour changing and conserving processes

input for HVP comes from BESIII [34]. Still the most precise ISR measurements from KLOE and BaBar are in conflict, but the new ISR data from BESIII steps to resolve this tension, as it lies in between the two. Other data recently collected, and published up to the end of 2014, include the $e^+e^- \rightarrow 3(\pi^+\pi^-)$ data from CMD–3 [46], the $e^+e^- \rightarrow \omega\eta^0 \rightarrow \pi^0\pi^0\gamma$ from SND [47] and several data sets collected by BaBar in the ISR mode [48–51]. These data samples highly increase the available statistics for the annihilation channels opened above 1 GeV and lead to significant improvements. Recent/preliminary results also included are $e^+e^- \rightarrow \pi^+\pi^-\mu^+\mu^-$ from Belle, $e^+e^- \rightarrow K^+K^-$ from CMD-3, $e^+e^- \rightarrow K^+K^-$ from SND. The resulting data sample is collected in figure 6, which has indicated the overall precision of the different ranges as well as the pQCD ranges, where data are replaced by pQCD results. Still one of the main issue in HVP is $R_\gamma(s)$ in the region 1.2 to 2.4 GeV, which actually has been improved dramatically by the exclusive channel measurements by BaBar in the last decade. The most important 20 out of more than 30 channels are measured, many known at the 10 to 15% level. The exclusive channel therefore has a much better quality than the very old inclusive data from Frascati (see figure 7).

2.2 NLO and NNLO HVP effects updated

The next-to-leading order (NLO) HVP is represented by diagrams in figure 8. With kernels from [53], the results of an updated evaluation are presented in table 2. The next-to-next leading order (NNLO) contributions have been calculated recently [54, 55]. Diagrams are shown in figure 8 and corresponding contributions evaluated with kernels from [54] are listed in table 3.

![Figure 7. $e^+e^-$ annihilation data in the 1.4 to 2.6 GeV region. Summed up exclusive (excl) channel data are shown together with old inclusive data (incl). Two-body channels represent a small fraction of $R_\gamma$ only. Above 2 GeV good quality inclusive BES-II data [52] provide a fairly well determined $R_\gamma(s)$.](image7)

![Figure 6. My 2015 compilation of $R_\gamma$ as a function of energy $E$.](image6)

![Figure 8. Feynman diagrams with hadronic insertions at NLO (top row) and NNLO.](image8)
1.0) \times 10^{-11} and \alpha^{\text{had}}_{\mu}[\text{NNLO VP}] = (12.4 \pm 0.1) \times 10^{-11} although relevant will be known well enough. These number also compare with the well established weak \alpha^{\text{EW}}_{\mu} = (154 \pm 1) \times 10^{-11} and the problematic HLB in estimated to contribute \alpha^{\text{had,1L}}_{\mu} = [(105 \pm 106) \pm (26 \pm 39)] \times 10^{-11}, which is representing a +0.90 \pm 0.28 ppm effect. Next generation experiments require a factor 4 reduction of the uncertainty optimistically feasible should be a factor 2 we hope.

3 Effective field theory: the Resonance Lagrangian Approach

As we know HVP is dominated by spin 1 resonance physics, therefore we need a low energy effective theory which includes \rho, \omega, \phi mesons. Principles to be implemented are the VMD mechanism, the chiral structure of QCD (chiral perturbation theory), and electromagnetic gauge invariance. A specific realization is the HLS effective Lagrangian [57] (see [8] for a brief account). In our context it has been first applied to HLB of muon g \nu \rightarrow 2 in [1], to HVP in [58]. Largely equivalent is the ENJL (just including the scalar QED [35] for the pion-photon interaction, with effective Lagrangian

\[ \mathcal{L} = \mathcal{L}_{\gamma p} + \mathcal{L}_x \]

(5)

where

\[ \mathcal{L}_{\gamma p} = -\frac{1}{4} F_{\mu
u} F^{\mu\nu} - \frac{1}{4} \rho_{\mu\nu} \rho^{\mu\nu} + \frac{M^2_{\rho}}{2} \rho_{\mu\nu} \rho^{\mu\nu} + \frac{e}{2 g_{\rho}} \rho_{\mu\nu} F^{\mu\nu} \]

\[ \mathcal{L}_x = D_{\mu} \rho^\dagger \rho \gamma - m_\rho^2 \rho^\dagger \rho \]

Photon and \rho self-energies are then pion-loops, which also implies non-trivial \gamma - \rho mixing. The clue is that the \rho^0 - \gamma interference is uniquely fixed by the electronic \rho-width \Gamma_{\text{JEC}}. Nothing unknown to be adjusted! Previous calculations à la Gounaris-Sakurai, considered the mixing term to be constant, i.e.

\[ -i \Pi^{(\text{eff})}_{\gamma \rho}(s)(q) = \frac{\Gamma_{\text{JEC}}}{s - \mu_{\rho}} \]

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\[ \Gamma_{\text{JEC}} \]

Figure 9. Irreducible self-energy contribution at one-loop.

}\end{verbatim}
which replaces the ρ contribution of the GS formula, which usually includes the ω−ρ mixing and higher ρ contributions ρ′ = ρ(1450) and ρ″ = ρ(1700). Properly normalized (VP subtraction: e^2(s) → e^2) we have

\[ F_\rho(s) = \left( e^2 D_\rho + e (g_{\rho\pi\pi} - g_{\rho\omega}) D_{\rho\omega} - g_{\rho\omega} g_{\rho\pi\pi} D_{\rho\pi\pi} \right) / \left( e^2 D_\gamma \right) \]  

Typical couplings are \( g_{\rho\pi\pi \text{ bare}} = 5.8935 \), \( g_{\rho\pi\pi \text{ ren}} = 6.1559 \), \( g_{\rho\omega} = 0.018149 \). \( x = g_{\rho\omega}/g_{\rho\pi\pi} = 1.15128 \) [35]. As a result a correction displayed in figure 11 is obtained, a ±5 to -10% correction! The proper relationship between \( e^+e^- \) and \( \tau \) spectral functions

\[ v_\tau(s) = \frac{\beta_\tau(s)}{12\pi} F_\tau^i(s)^2 \quad (i = 0, \pm) \]  

in terms of the pion form-factors \( F_\rho \) and the pion velocities \( \beta_\tau \), now reads

\[ v_\tau(s) = r_{\rho\tau}(s) R_{\tau\pi}(s) v_\rho(s) \]  

where \( R_{\tau\pi}(s) \) is the standard isospin breaking correction (see [35, 38–40]). The τ requires for correcting missing ρ−γ mixing as well, before being used as \( I = 1 \) \( e^+e^- \) data, because results obtained from \( e^+e^- \) data is what goes into the DR (2) (the photon coupled to \( \pi^+\pi^- \) not to \( \pi^0\pi^0 \)). The correction is large only for the ρ and affects narrower resonances only near very near resonance. The effect is part of the experimental data and as \( \omega \) and \( \phi \) have no charged partners there is nothing to be corrected in these cases. To include further mixing effects, like \( \omega - \rho^0 \) mixing, one has to extend the Lagrangian and including all possible fields and their possible interactions, which leads to the HLS Lagrangian or a related effective model. For details I refer to [9]. Nevertheless, let me add a few comments on the HLS approach:

Can global fits like our HLS implementation discriminate between incompatible data sets? The problem of inconsistent data is not a problem of whatever model, rather it is a matter of systematics of the measurements. Note that modeling is indispensable for interrelating different data channels. In the HLS global fit \( \tau \) data play a central role as they are simple, i.e. pure \( I = 1 \), no singlet contribution, no \( \gamma - \rho^0 \) mixing. In fact, \( \tau \) spectra supplemented with PDG isospin breaking, provide a good initial fit for most \( e^+e^- \)-data fits, which then are improved and optimized by iteration for a best simultaneous solution.

Why do we get slightly lower results for HVP and with reduced uncertainties? BaBar data according to [59] are in good accord with Belle \( \tau \)-data, before correcting τ-data for the substantial and quite unambiguous \( \gamma - \rho^0 \) mixing effects! i.e. for the BaBar data alone there seems to be no \( \tau \) vs. \( e^+e^- \) puzzle, while the puzzle exists for all other \( e^+e^- \) data sets. This is a problem for the BaBar data. They are disfavored by our global fit! BaBar data rise the HVP estimate quite substantially towards the uncorrected \( \tau \) data value. In contrast the NSK, KLOE10/12 and the new BESIII data are in very good agreement with the \( \tau + \text{PDG} \) prediction [7], so they dominate the fit and give somewhat lower HVP result\(^2\). Since, besides the \( e^+e^- \) data, additional data constrain the HLS Lagrangian and its parameters, we find a reduced uncertainty and hence an increased significance.

What are the model (using specifically HLS) errors of our estimates? This is hard to say. Best do a corresponding analysis based on different implementations of the results.

\(^2\) We are talking about a 1% shift, which is of the order of the size of the uncertainty.
onance Lagrangian approach. Try to include higher order corrections. However, the fit quality is surprisingly good and we do not expect that one has much flexibility. However, on can improve on photon radiation within a suitably extended HLS approach. Such processes have been implemented recently in the CARLOMAT Monte Carlo [62].

To conclude: our analysis is a first step in a direction which should allow for systematic improvements. A comparison of different estimates and leading uncertainties is shown in figure 12.

4 HVP for the electron anomaly

An up-to-date reevaluation of hadronic VP effects to the electron $g-2$ yields the results given in table 4. The present status is illustrated by the pie chart figure 13.

Table 4. 2015 update of HVP effects contributing to $a_e$

<table>
<thead>
<tr>
<th>Energy Range</th>
<th>$a_e^\text{had}(1)$</th>
<th>$a_e^\text{had}(2)$</th>
<th>$a_e^\text{had}(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 GeV</td>
<td>$185.11 \pm 1.24 \times 10^{-14}$ (LO)</td>
<td>$-22.15 \pm 0.16 \times 10^{-14}$ (NLO)</td>
<td>$2.80 \pm 0.02 \times 10^{-14}$ (NNLO)</td>
</tr>
</tbody>
</table>

Together with the hadronic and weak contribution we get the SM prediction (incl. $a_e^\text{had,LbL} = (3.7 \pm 0.5) \times 10^{-14}$)

$$a_e^\text{SM} = a_e^\text{QED} + 1.725(12) \times 10^{-12} \text{ (hadr & weak)}.$$ which can be used for extracting $\alpha_{QED}$ from $a_e$ at unprecedented precision. Matching the theory prediction with the very precise experimental result of Gabrielse et al. [64]

$$a_e^\text{exp} = 0.001 159 652 180 73(28),$$

one extracts

$$\alpha^{-1}(a_e) = 137.0359991685(342)(68)(46)(24)[353],$$

which is close [85 $\rightarrow$ 57] to the value

$$\alpha^{-1}(a_e) = 137.0359991657(342)[0.25 \text{ ppb}],$$

obtained by [63]. Note that the weak part has been reevaluated as

$$a_e^\text{weak} = 0.030 \times 10^{-12},$$

which is replacing the value $0.039 \times 10^{-12}$ which has been estimated in [65]. An inconsistency there has been noted by M. Passera [66].

The best test for new physics can be obtained by using $\alpha$ from atomic interferometry [67]. With $\alpha^{-1}(\text{Rb}11) = 137.035999037(91)[0.66 \text{ ppb}]$ as an input one finds

$$a_e^\text{the} = 0.001 159 652 181 87(77),$$

such that

$$a_e^\text{exp} - a_e^\text{the} = -1.14(0.82) \times 10^{-12},$$

in good agreement. We know that the sensitivity to new physics is reduced by $(\alpha_{QED}/\alpha_{QED})^2 \cdot \delta a_e^\text{exp}/\delta a_e^\text{the} \simeq 19$ relative to $a_e$. Nevertheless, one has to keep in mind that $a_e$ is suffering less form hadronic uncertainties and thus may provide a safer test. Presently, the $a_e$ prediction is limited by the, by a factor $\delta a(\text{Rb}11)/\delta a(a_e) = 5.3$ less precise, $\alpha$ available. Combining all uncertainties $a_e$ is about a factor 43 more sensitive to new physics at present.

5 HVP subtraction of $R_f(s)$: a problem of the DR method?

The full photon propagator is usually obtained by Dyson resummation of the 1pi part (blob) as illustrated by figure 14. As we know this is a geometric series

$$\gamma = \gamma^i \gamma^f = \ldots$$

$$\gamma^i D_q(q^2) = \frac{-i}{q^2} \left(1 + \frac{-i}{q^2} \Pi'_q(q^2) + \frac{-i}{q^2} \Pi'_q(q^2)^2 + \ldots\right)$$

$$= \frac{-i}{q^2} \left(1 + \frac{-i}{q^2} \Pi'_q(q^2) + \frac{-i}{q^2} \Pi'_q(q^2)^2 + \ldots\right)$$

$$= \frac{-i}{q^2} \left(1 + \frac{-i}{q^2} \Pi'_q(q^2) + \frac{-i}{q^2} \Pi'_q(q^2)^2 + \ldots\right)$$

Figure 14. The Dyson summation of the photon self-energy.
Usually, $\Delta a(s)$ is a correction i.e $\Delta a(s) \ll 1$ and the Dyson series converges well. Indeed for any type of perturbative effects no problem is encountered (besides possible Landau poles). For non-perturbative strong interaction physics there are exceptions. One would expect that, if there are problems, one would encounter them at low energy, but for the $q$, the $\omega$ and the $\phi$, this expectation is in sharp contradiction with the observed strong resonance enhancements. The electromagnetic contribution to the running charge is small relative to unity, as is the effect of suppression by the e.m. coupling $e^2$. The exception, surprisingly, we find at pretty high energies, like charge, i.e. $\mu$.

Locally, near OZI suppressed resonances the usual iterative procedure of getting $R^{\text{bare}}_y$ does not converge! The way out usually practiced is to use the smooth space-like charge, i.e. $R^{\text{phys}}_y = R^{\text{phys}}_y |1 + \Pi'_y(s)|^2$, expected to do the undressing “in average”. This actually does not look too wrong as we look in figure 16. Nevertheless, I see a problem her, not only for the interpretation of resonance data, where one would wish to be able to disentangle electromagnetic form strong interaction effects.

For what concerns the proper extraction of the hadronic effects contributing to the running of $\alpha_{\text{QED}}$ and to $\alpha_{\mu}^\text{had}$, I see no proof that this cannot produce non-negligible shifts!

Fortunately, experimental progress is in sight here: KLOE 2015 [68] has a first direct measurements of the time-like complex running $\alpha_{\text{QED}}(s)$! Similar measurements for the $J/\psi$ and other ultra-narrow resonances should be possible with BES III. It is a fundamental problem! An interesting possibility in this respect is a novel approach to determine $\alpha_{\mu}^\text{had}$ form a direct space-like measurement of $\alpha(-Q^2)$ as proposed in [69, 70], recently.

### 6 A comment on axial exchanges in HLbL

The Landau-Yang theorem says that the amplitude $\mathcal{A}(\text{axial meson } \gamma \gamma)_{\text{on-shell}} = 0$, e.g. $Z^0 \to \gamma \gamma$ is forbidden, while $Z^0 \to \gamma \pi^0$ is allowed as one of the photons is off-shell. For HLbL such type of contribution has been estimated in [17] to be rather large, which raised the question: Why $a_\mu(a_1, f_1) = 22 \times 10^{-11}$ is so large? From the data side we know, untagged $\gamma \gamma \to f_1$ shows no signal, while single-tag $\gamma \gamma \to f_1$ strongly peaked when $Q^2 \gg m_f^2$. The point: the contribution from axial mesons has been calculated assuming a symmetric form-factors under exchange of the two photon momenta. This violates the Landau-Yang theorem, which requires an antisymmetric form-factor. In fact antisymmetrizing the form-factor adopted in [17] reduces the contribution by a factor about 3, and the result agrees with previous findings [1, 2] and with the more recent result [20]. As a result one finds that the estimate $a_\mu^{\text{HLbL, LO}} = (116 \pm 39) \times 10^{-11}$ accepted in [65] must be replaced by

$$a_\mu^{\text{HLbL, LO}} = (102 \pm 39) \times 10^{-11}. \quad (16)$$

This also requires a modification of the result advocated in [71]. The evaluation of the axialvector mesons contribution, taking a Landau-Yang modified (i.e. antisymmetrized) Melnikov-Vainshtein form-factors yields [72]

$$a_\mu(a_1, f_1^*, f_1) \sim (5.51 \pm 1.89 + 5.04 + 0.58) \times 2.71 \times 10^{-11}, \quad (17)$$

where ideal mixing and nonet symmetry results have been averaged. In fact, the sum of the contributions from the $f_1$ and $f_1^*$ depends little on the mixing scheme. The result supersedes $a_\mu(a_1, f_1^*, f_1) \sim 22(5) \times 10^{-11}$ we included in [65].

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**Figure 15.** OZI suppressed strong decays let e.m. interaction look to be almost of equal strength.

**Figure 16.** Time-like vs. space-like effective finestructure constant $\alpha$ as a function of the energy $E$: $\alpha(s)$ in the mean follows $\alpha(t = -s)$ ($s = E^2$). Note that the smooth space-like effective charge agrees rather well with the non-resonant “background” above the $\phi$ (kind of duality).
7 Theory vs. experiment: do we see New Physics?

Table 5. A list of small shifts in theory [in units $10^{-11}$]. The error from new entries in the list reduces from the old 5 to 3.6.

<table>
<thead>
<tr>
<th>New contribution</th>
<th>$a_\mu$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old axial exchange HLbL</td>
<td>$22\pm5$</td>
<td>[17]</td>
</tr>
<tr>
<td>New axial exchange HLbL</td>
<td>$7.51\pm2.71$</td>
<td>[20, 72]</td>
</tr>
<tr>
<td>NNLO HVP</td>
<td>$12.4\pm0.1$</td>
<td>[54]</td>
</tr>
<tr>
<td>NLO HLbL</td>
<td>$3\pm2$</td>
<td>[73]</td>
</tr>
<tr>
<td>Tensor exchange HLbL</td>
<td>$1.1\pm0.1$</td>
<td>[20]</td>
</tr>
</tbody>
</table>

Total change $+2.0\pm3.4$

Here I briefly summarize what is new and where we are. Some new results/evaluations are collected in table 5. We finally compare the SM prediction for $a_\mu$ with its experimental value [77] in table 6, which also summarizes the present status of the different contributions to $a_\mu$. A deviation between 3 and 5 $\sigma$ is persisting and was slightly increasing. Resonance Lagrangian models, like the HLS model, provide clear evidence that there is no $\tau$ version HVP which differs from the $e^+e^-$ data result. This consolidates a larger deviation $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{the}}$. Also the decrease of the axial HLbL contribution goes in this direction, it is compensated however by the new NNLO HVP result. What represents the 4 $\sigma$ deviation: new physics? Is it a statistical fluctuation? Are we underestimating uncertainties (experimental, theoretical)? Do experiments measure what theoreticians calculate? I refer to [78] for possible interpretations and conclusions.

8 Outlook

Although progress is slow, there is evident progress in reducing the hadronic uncertainties, most directly by progress in measuring the relevant hadronic cross-sections. Near future progress we expect from BINF Novosibirsk/Russia and from IHEP Beijing/China. Energy scan as well as ISR measurement of cross-sections in the region from 1.4 to 2.5 GeV are most important to reduce the errors to a level competitive with the factor 4 improvement achievable by the upcoming new muon $g-2$ experiments at Fermilab/USA and at J-PARC/Japan [56]. Also BaBar data are still being analyzed and are important for improving the results. Promising is that lattice QCD evaluations come closer to be competitive [15].

Acknowledgements

I thank Z. Zhang for helpful discussions on the isospin breaking corrections and M. Benayoun for close collaboration on HLS driven estimates of $g_\mu^{\text{had}}$.

References


Table 6. Standard model theory and experiment comparison [in units $10^{-11}$].

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value</th>
<th>Error</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED at 5-loops</td>
<td>11 658 471 . 8851</td>
<td>0 . 036</td>
<td>[55, 74, 75]</td>
</tr>
<tr>
<td>LO HVP</td>
<td>688 . 91</td>
<td>3 . 52</td>
<td>(3)</td>
</tr>
<tr>
<td>NLO HVP</td>
<td>-9 . 917</td>
<td>0 . 100</td>
<td>table 2</td>
</tr>
<tr>
<td>NNLO HVP</td>
<td>1 . 225</td>
<td>0 . 012</td>
<td>[54], table 3</td>
</tr>
<tr>
<td>HLbL</td>
<td>10 . 6</td>
<td>3 . 9</td>
<td>[65, 72], table 5</td>
</tr>
<tr>
<td>EW at 2-loops</td>
<td>15 . 40</td>
<td>0 . 10</td>
<td>[25, 26, 76]</td>
</tr>
<tr>
<td>Theory</td>
<td>11 659 178 . 10</td>
<td>5 . 26</td>
<td>–</td>
</tr>
<tr>
<td>Experiment</td>
<td>11 659 209 . 1</td>
<td>6 . 3</td>
<td>[77] (updated)</td>
</tr>
<tr>
<td>Exp-The</td>
<td>3 . 8 $\sigma$</td>
<td>31 . 0</td>
<td>8 . 2</td>
</tr>
</tbody>
</table>


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