The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer

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Abstract

We calculate the massive flavor non-singlet Wilson coefficient for the heavy flavor contributions to the polarized structure function $g_1(x, Q^2)$ in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in Quantum Chromodynamics at general values of the Mellin variable $N$ and the momentum fraction $x$, and derive heavy flavor corrections to the Bjorken sum-rule. Numerical results are presented for the charm quark contribution. Results on the structure function $g_2(x, Q^2)$ in the twist-2 approximation are also given.
1 Introduction

Massless and massive contributions to the unpolarized and polarized structure functions in deep-inelastic scattering exhibit different scaling violations. For a precise determination of the QCD scale $\Lambda_{\text{QCD}}$ or the strong coupling constant $\alpha_s(M_Z^2)$ their precise knowledge is therefore of importance [1]. In the case of the polarized structure function $g_1(x, Q^2)$ the heavy quark mass. The $O(\alpha_s^2)$ corrections in the polarized case were calculated in Refs. [6, 7]. In the case of the structure function $g_1(x, Q^2)$, the 1-loop heavy flavor corrections have been accounted for at next-to-leading order (NLO) QCD analysis [8]. The corresponding flavor non-singlet corrections in the unpolarized case were calculated for pure photon exchange to $O(\alpha_s^2)$ in [5, 9] and in Ref. [10] to $O(\alpha_s^3)$.

In the present paper we calculate the $O(\alpha_s^2)$ massive flavor non-singlet Wilson coefficient for the inclusive structure function $g_1(x, Q^2)$ in the asymptotic region $Q^2 \gg m^2$, and also present the corresponding $O(\alpha_s^2)$ result, extending Refs. [6, 7].

The differential cross section for polarized deep-inelastic scattering [11–13] is given by

$$\frac{d^2\sigma_B}{dx dy} = \frac{2\alpha e^2}{Q^4} \lambda_N^p f^p S \left[ S_1^p(x, y) g_1(x, Q^2) + S_2^p(x, y) g_2(x, Q^2) \right], \quad (1.1)$$

with

$$f^L = 1, \quad f^T = \cos(\beta - \varphi) \frac{d\varphi}{2\pi} \sqrt{\frac{4M^2x}{Sy}} \left[ 1 - y - M^2xy \frac{S}{S} \right],$$

$$S_1^L = 2xy \left[ (2 - y) - 2 \frac{M^2}{S} xy \right], \quad S_1^T = 2xy^2,$$

$$S_2^L = -8x^2y \frac{M^2}{S}, \quad S_2^T = 4xy. \quad (1.2)$$

Here $\alpha = e^2/(4\pi)$ denotes the fine structure constant, $M$ is the nucleon mass, $S = (p + l)^2$ is the center of mass energy of the lepton-nucleon system, with $p$ and $l$ the nucleon and lepton 4-momenta, respectively, $q = l - l'$ is the 4-momentum transfer and $Q^2 = -q^2$. $x = Q^2/(2p.q)$ and $y = p.q/p.l$ are the Bjorken variables. $\lambda_N^p$ denotes the degree of the nucleon polarization. The spin 4-vectors in the longitudinal and transverse cases are given by

$$S_L = M(0, 0, 0; 1) \quad (1.3)$$

and $\varphi$ denotes the angle between the vectors of the spin and the outgoing lepton. It contributes in a non-trivial way in the case of transverse polarization.

The polarized structure functions are denoted by $g_1(x, Q^2)$ and $g_2(x, Q^2)$. In the leading twist approximation, the heavy flavor contributions to the structure function $g_1(x, Q^2)$ is given by, cf. [14],

$$g_1(x, Q^2) = \frac{1}{2} \sum_{k=1}^{N_F} \epsilon_i^k \left\{ L_{q,g_1}^{NS} \left[ x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right] \otimes \left[ \Delta f_k(x, \mu^2, N_F) + \Delta f_k(x, \mu^2, N_F) \right] \right\}$$

1For an implementation in Mellin space, see [4].
The actual flavor non-singlet distribution is defined by

\[ \Delta f(x, Q^2) = \sum_{k=1}^{N_f} e_k^2 \left[ \Delta f_k(x, \mu^2, N_f) + \Delta f_{\bar{k}}(x, \mu^2, N_f) - \frac{1}{N_f} \Delta \Sigma(x, \mu^2, N_f) \right] . \]  

However, according to the representation (1.5), we will consider its whole first term, depending on \( L_{q,1}^{PS} \) as the non-singlet contribution in what follows. The structure function \( g_2(x, Q^2) \) can be obtained from \( g_1(x, Q^2) \) using the Wandzura-Wilson relation [15].

The paper is organized as follows. In Section 2 we calculate the heavy flavor contributions to the non-singlet Wilson coefficient in the asymptotic region \( Q^2 \gg m^2 \) to the structure function \( g_1(x, Q^2) \) to 3-loop order in the strong coupling constant. We present the results both in Mellin \( N \) and \( x \)-space. Numerical results are given in Section 3. Consequences for the polarized Bjorken \( x \) sum rule are discussed in Section 4, and Section 5 contains the conclusions.

## 2 The Wilson Coefficient

The heavy flavor non-singlet Wilson coefficient contributing to the structure function \( g_1(x, Q^2) \) in the asymptotic region \( Q^2 \gg m^2 \) receives its first contributions at \( O(\alpha_s^2) \). In previous analyses [6,7] the tagged flavor case at \( O(\alpha_s^3) \) has been considered. In what follows we will refer to the inclusive case, i.e. the complete contribution to the structure function \( g_1(x, Q^2) \), and consider the terms due a single heavy quark.

The non-singlet heavy flavor Wilson coefficient contributing to the structure function \( g_1(x, Q^2) \) in the asymptotic region \( Q^2 \gg m^2 \) is given by [16]

\[ L_{q,1}^{NS}(N_F + 1) = a_s^2 \left[ A_{q,q,Q}^{(2),NS}(N_F + 1) + C_{q,g_1}^{(2),NS}(N_F) \right] + a_s^3 \left[ A_{q,q,Q}^{(3),NS}(N_F + 1) + A_{q,g_1}^{(2),NS}(N_F + 1)C_{q,g_1}^{(1),NS}(N_F + 1) + C_{q,g_1}^{(3),NS}(N_F) \right] . \]
Here $A_{qq,Q}^{\text{NS}}$ is the massive non-singlet operator matrix element (OME) and the label ‘$N_F + 1$’ symbolically denotes that the OME is calculated at $N_F$ massless and one massive flavor, $\alpha_s = \alpha_s/(4\pi) \equiv g_s^2/(4\pi)^2$ parameterizes the strong coupling constant, and we use the convention
\[
\hat{f}(N_F) = f(N_F + 1) - f(N_F) .
\]

The calculation of the different contributions to the Wilson coefficient is performed in $D = 4 + \varepsilon$ dimensions to regulate the Feynman integrals. In the present polarized case the treatment of $\gamma_5$ has to be considered. In the flavor non-singlet case both for the massive OMEs and the massless Wilson coefficients $\gamma_5$, always appears in traces along one massless line and there is a Ward-Takahashi identity which implies the use of anti-commuting $\gamma_5$.

The inclusive massive OME $A_{qq,Q}^{\text{NS}}$ to 3-loop order for even and odd moments $N$ has been calculated in Ref. [10]. The corresponding diagrams have been reduced using integration-by-parts relations [17] applying an extension of the package Reduze 2 [18].\(^2\) The master integrals have been calculated using hypergeometric, Mellin-Barnes and differential equation techniques, mapping them to recurrences, which have then been solved by modern summation technologies using extensively the packages Sigma [21, 22], EvaluateMultiSums, SumProduction [23], $\rho$sum [24], and HarmonicSums [25].

The massless Wilson coefficients $C_{q,q_1}(x, Q^2)$ from 1- to 3-loop order were calculated in Refs. [26–29]. At 3-loop order those of the structure function $g_1$ are obtained by that of $F_3$ [29], setting the $d_{abc}$ terms in $\hat{C}_{q,q_1}^{(3),\text{NS}}(N_F)$ to zero, cf. also [30, 31]. The non-singlet OMEs $A_{qq,Q}^{(k),\text{NS}}$ at 2- and 3-loop order were calculated in [5, 9] and [10], respectively.

For comparison, the massless flavor non-singlet Wilson coefficient in Mellin space is given by [28, 29]
\[
L_{q,q_1}^{\text{NS}}(N_F) = 1 + \sum_{k=1}^3 \alpha_s^k C_{q,q_1}^{(k),\text{NS}}(N_F) .
\]

In Mellin $N$ space the Wilson coefficient can be expressed by nested harmonic sums $S_\vec{a}(N)$ [32] which are defined by
\[
S_{b,\vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{b}|\vec{a}|} S_{\vec{a}}(k), \quad S_\vec{a} = 1, \ b, a_i \in \mathbb{Z}, b, a_i \neq 0, N > 0, N \in \mathbb{N}.
\]

In the following, we drop the argument $N$ of the harmonic sums and use the short-hand notation $S_\vec{a}(N) \equiv S_{\vec{a}}$. The Wilson coefficients depend on the logarithms
\[
L_Q = \ln \left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad L_M = \ln \left(\frac{m^2}{\mu^2}\right),
\]
where the renormalization scale has been set equal to the factorization scale $\mu = \mu_R = \mu_F$.

As a short-hand notation we define the leading order splitting function $\Delta\gamma_{qq}^{(0)}$ up to its color factor
\[
\Delta\gamma_{qq}^{(0)} = 4 \left[ 2S_1 - \frac{3N^2 + 3N + 2}{2N(N + 1)} \right] .
\]

The massive Wilson coefficient for the structure function $g_1(x, Q^2)$ in the asymptotic region in Mellin space in the on-shell scheme is given by
\[
L_{q,q_1}^{b,\text{NS}}(N) = \frac{1}{2} \left[ 1 - (-1)^N \right] \left\{ a_s^2 C_F T_F \left\{ -\frac{1}{3} [L_M^3 + L_M^2] \Delta\gamma_{qq}^{(0)} + L_M \right\} - \frac{2P_1}{9N^2(N + 1)^2} - \frac{80}{9} S_1 \right\}
\]

\(^2\)The package Reduze 2 uses the packages Fermat [19] and Ginac [20].
\[\begin{align*}
\frac{16}{3}S_2 + L_Q \left[-\frac{2P_6}{9N^2(N+1)^2} + \frac{4(29N^2 + 29N - 6)}{9N(N+1)}S_1 + \frac{8}{3}S_1^2 - 8S_2 \right] + \frac{16}{3}S_{2,1} \\
+ \frac{P_{34}}{27N^3(N+1)^3} + \left(-\frac{2P_{11}}{27N^2(N+1)^2} + \frac{8}{3}S_2 \right) S_1 - \frac{2(29N^2 + 29N - 6)}{9N(N+1)}S_1^2 \\
- \frac{8}{9}S_3^3 + \frac{2(35N^2 + 35N - 2)}{3N(N+1)}S_2 - \frac{112}{9}S_3 \right)\right] \\
+ a_3^3 \left[ C_B^2 T_F + \frac{L^3}{6} [L_Q + L_M^2 L_Q] \Delta \gamma_{qq}^{(0)} + L_M^2 \left[-\frac{2P_{28}}{3N^3(N+1)^3} - \frac{16}{3}S_1 \right] \\
+ \frac{2P_6}{3N^2(N+1)^2} S_1 - \frac{4(N - 1)(N + 2)}{N(N+1)}S_2^2 + \frac{64}{3}S_3 + \frac{64}{3}S_{-3} - \frac{128}{3}S_{-2,1} \\
+ \left(-\frac{64}{3N(N+1)} + \frac{128}{3}S_1 \right) S_{-2} + \frac{10}{3} \Delta \gamma_{qq}^{(0)} S_2 \right] + L_Q \left[-\frac{2P_{30}}{9N^3(N+1)^3} \right] \\
+ \frac{2P_{21}}{9N^2(N+1)^2} S_1 - \frac{4(107N^2 + 107N - 54)}{9N(N+1)}S_1^2 - 16S_3^3 + \frac{64}{3}\left[S_3 + S_{-3}\right] \\
+ \left(-\frac{64}{3N(N+1)} + \frac{128}{3}S_1 \right) S_{-2} - \frac{128}{3}S_{-2,1} + \frac{22}{3} \Delta \gamma_{qq}^{(0)} S_2 \right] \\
+ L_M L_Q \Delta \gamma_{qq}^{(0)} \left[ \frac{P_1}{9N^2(N+1)^2} + \frac{40}{9}S_1 - \frac{8}{3}S_2 \right] + L_M \left[ \frac{P_{40}}{9N^4(N+1)^4} \right] \\
+ \left(\frac{2P_{31}}{9N^3(N+1)^3} + \frac{16(59N^2 + 59N - 6)}{9N(N+1)}S_2 - \frac{256}{3}S_3 - \frac{256}{3}S_{-2,1}\right) S_1 \\
+ \left(-\frac{4P_3}{3N^2(N+1)^2} + \frac{32}{3}S_2 \right) S_2^2 - \frac{160}{9}S_3^3 - \frac{4P_8}{9N^2(N+1)^2}S_2 - 32S_2^2 \\
+ \frac{32(29N^2 + 29N + 12)}{9N(N+1)} S_3 - \frac{256}{3}S_4 + \left(-\frac{64(16N^2 + 10N - 3)}{9N^2(N+1)^2} - \frac{128}{3}S_2 \right) \\
+ \frac{1280}{9}S_1 \right) S_{-2} + \left(\frac{64(10N^2 + 10N + 3)}{9N(N+1)} - \frac{128}{3}S_1 \right) S_{-3} - \frac{128}{3}S_{-4} \\
+ \frac{128}{3}S_{-3} + \frac{128}{3}(10N^2 + 10N - 3)S_{-2,1} + \frac{128}{3}S_{-2,2} + \frac{512}{3}S_{-2,1,1} \\
+ 8 \Delta \gamma_{qq}^{(0)} \zeta_4 \right] + L_Q \left[ \frac{4P_{48}}{27N^4(N+1)^4(N+2)^3} + \left(-\frac{4P_{36}}{27N^3(N+1)^3} + \frac{640}{9}S_3 \right) \\
- \frac{32(67N^2 + 67N - 21)}{9N(N+1)} S_2 + \frac{64}{3}S_{2,1} + \frac{512}{3}S_{-2,1} \right) S_1 + \left(-\frac{2P_{15}}{27N^2(N+1)^2} \right) \\
- \frac{224}{3}S_2 \right) S_1^2 + \frac{32(4N - 1)(4N + 5)}{9N(N+1)}S_2^3 + \frac{80}{9}S_1^4 + \left(-\frac{2P_{27}}{9N(N-1)N^2(N+1)^2(N+2)} \right) S_2^2 + 48S_2^2 \\
- \frac{32(53N^2 + 53N + 16)}{9N(N+1)} S_3 + \frac{352}{3}S_4 + \left(-\frac{64P_{27}}{9(N-1)N^2(N+1)^2(N+2)} \right) \\
+ \frac{128(10N^2 + 10N - 3)}{9N(N+1)} S_1 - \frac{256}{3}S_2 + \frac{256}{3}S_2^2 \right) S_{-2} + 64S_2^2 + \frac{448}{3}S_{-4} \\
+ \left(-\frac{64(10N^2 + 10N + 9)}{9N(N+1)} + \frac{256}{3}S_1 \right) S_{-3} + \frac{16(9N^2 + 9N - 2)}{3N(N+1)}S_{2,1}
\end{align*}\]
\[+64S_{3,1} + \frac{128(10N^2 + 10N - 3)}{9N(N + 1)}S_{-2,1} - \frac{256}{3}S_{-3,1} - 64S_{2,1,1} - \frac{512}{3}S_{-2,1,1} + \left(-\frac{16(9N^2 + 9N - 2)}{N(N + 1)} + 64S_1\right)\zeta_3 + \frac{P_{46}}{162N^5(N + 1)^5} - \frac{128(112N^3 + 112N^2 - 39N + 18)}{81N^2(N + 1)}S_{-2,1} + \left(\frac{P_{45}}{162N^4(N + 1)^4} - \frac{64}{9}S_2\right)
\]
\[+ \frac{8P_{16}}{81N^2(N + 1)^2}S_2 - \frac{8(347N^2 + 347N + 54)}{27N(N + 1)}S_3 + \frac{128}{9N(N + 1)}S_{2,1} - \frac{256}{9}S_{-2,2}S_1 + \left(\frac{P_{25}}{9N^3(N + 1)^3} + \frac{16(5N^2 + 5N - 4)}{9N(N + 1)}\right)S_2 - \frac{128}{9}S_{2,1}
\]
\[+ 16S_3 - \frac{256}{9}S_{-2,1}S_1^2 + \left(-\frac{16P_4}{27N^2(N + 1)^2} + \frac{128}{27}S_2\right)S_3^2 + \left(\frac{400}{27}S_3\right)
\]
\[+ \frac{P_{34}}{81N^3(N + 1)^3} + \frac{256}{3}S_{-2,1}S_2 - \frac{32(23N^2 + 23N - 3)}{27N(N + 1)}S_2^2 + \frac{512}{9}S_5
\]
\[+ \frac{8P_7}{81N^3(N + 1)^2}S_3 - \frac{176(17N^2 + 17N + 6)}{27N(N + 1)}S_4 + \left(-\frac{64P_9}{81N^3(N + 1)^3}\right)
\]
\[+ \frac{128P_4}{81N^2(N + 1)^2}S_1 - \frac{128}{9N(N + 1)}S_2^2 + \frac{256}{27}S_1S_3^2 - \frac{1280}{27}S_2 + \frac{512}{27}S_3
\]
\[- \frac{512}{9}S_{2,1}S_{-2} + \left(\frac{64(112N^3 + 224N^2 + 169N + 39)}{81N(N + 1)^2} + \frac{128}{9}S_1^2\right)
\]
\[+ \frac{128}{9}S_2 - \frac{128(10N^2 + 10N + 3)}{27N(N + 1)}S_1S_{-3} + \left(-\frac{128(10N^2 + 10N + 3)}{27N(N + 1)}\right)
\]
\[+ \frac{256}{9}S_1S_{-4} + \frac{256}{9}S_{-5} + \frac{16P_2}{9N^2(N + 1)^2}S_{2,1} + \frac{256}{9}S_{2,3} - \frac{512}{9}S_{2,-3}
\]
\[+ \frac{16(89N^2 + 89N + 30)}{27N(N + 1)}S_{3,1} - \frac{512}{9}S_{4,1} - \frac{128(10N^2 + 10N - 3)}{27N(N + 1)}S_{-2,2}
\]
\[+ \frac{512}{9}S_{-2,3} + \frac{512}{9}S_{2,1,-2} + \frac{256}{9}S_{3,1,1} + \frac{512(10N^2 + 10N - 3)}{27N(N + 1)}S_{-2,1,1}
\]
\[+ \frac{512}{9}S_{-2,2,1} - \frac{2048}{9}S_{-2,1,1,1} + \frac{16(2N^3 + 2N^2 + 2N + 1)}{3N^3(N + 1)^3}\zeta_2 + \left(-\frac{64}{3}S_2\right)
\]
\[+ \frac{16P_{13}}{9N^2(N + 1)^2} - \frac{1208}{9}S_1\zeta_3 + \left(\frac{8}{3}S_{2,1,1} - \frac{8}{3}B_4 + 12\zeta_4\right)\Delta\gamma^{(0)}
\]
\[+(-1)^N\left(-\frac{L^2}{3(N + 1)^3} - \frac{L^2}{3(N + 1)^3} + L_M\right)\left[-\frac{256(4N + 1)}{9(N + 1)^4}\right]
\]
\[+ \frac{128}{3(N + 1)^3}S_1 + \frac{64}{9N^3(N + 1)^3}S_1 + \frac{64(2N^2 + 2N + 1)}{9N^3(N + 1)^3}S_2
\]
\[
+ \frac{16(2N^3 + 2N^2 + 2N + 1)}{3N^3(N + 1)^3} \zeta_2 \right) + C_A C_F T_F \left[ L_M^3 \frac{22}{27} \Delta_{\gamma qq}^{(0)} + L_Q^3 \frac{44}{27} \Delta_{\gamma qq}^{(0)} \right] \\
+ L_M^2 \left[ \frac{2P_{21}}{9N^3(N + 1)^2} - \frac{184}{9} S_1 + \left( \frac{32}{3N(N + 1)} - \frac{64}{3} S_1 \right) S_2 - \frac{32}{3} [S_3 + S_{-3}] \right] \\
+ \frac{64}{3} S_{-2,1} + L_Q^2 \left[ \frac{2P_{23}}{27N^3(N + 1)^2} - \frac{16(194N^2 + 194N - 33)}{27N(N + 1)} S_1 - \frac{176}{9} S_1^2 \right] \\
+ \frac{176}{3} S_2 - \frac{32}{3} S_3 + \left( \frac{32}{3N(N + 1)} - \frac{64}{3} S_1 \right) S_2 - \frac{32}{3} S_{-3} + \frac{64}{3} S_{-2,1} \right] \\
+ L_M \left[ \frac{P_{38}}{81N^4(N + 1)^3} + \left( \frac{8P_{29}}{81N^3(N + 1)^3} + \frac{32S_3 + 128}{3} S_{-2,1} \right) S_1 \right] \\
+ \frac{1792}{27} S_2 - \frac{16(31N^2 + 31N + 9)}{9N(N + 1)} S_3 + \frac{160}{3} S_4 + \left( \frac{32(16N^2 + 10N - 3)}{9N^2(N + 1)^2} \right) S_{-2,1} \\
- \frac{640}{9} S_1 + \frac{64}{3} S_2 \right) S_{-2} + \left( \frac{-32(10N^2 + 10N + 3)}{9N(N + 1)} + \frac{64}{3} S_1 \right) S_{-3} + \frac{64}{3} S_{-4} \\
- \frac{128}{3} S_{3,1} + \frac{64(10N^2 + 10N - 3)}{9N(N + 1)} S_{-2,1} + \frac{64}{3} S_{-2,2} - \frac{256}{3} S_{-2,1,1} - \frac{8\Delta_{\gamma qq}^{(0)} \zeta_3}{\zeta_3} \right] \\
+ L_Q \left[ \frac{16(230N^3 + 460N^2 + 213N - 11)}{9N(N + 1)^2} S_2 - \frac{4P_{49}}{81N^4(N + 1)^4(N + 2)^3} \right] \\
\left( \frac{4P_{37}}{81N^3(N + 1)^3} - \frac{32(11N^2 + 11N + 3)}{9N(N + 1)} S_2 - \frac{128}{3} S_{2,1} - \frac{256}{3} S_{-2,1} \right) S_1 \right] + \frac{32S_3}{27N(N + 1)} S_2 + \left( \frac{16(194N^2 + 194N - 33)}{27N(N + 1)} \right) S_{-2,1} + \frac{128}{3} S_{2,1} \\
\left( \frac{32S_3}{27N(N + 1)} S_3 - \frac{224}{3} S_4 + \left( \frac{32P_{27}}{9(N - 1)N^2(N + 1)^2(N + 2)} \right) S_{-2,1} \right) S_1 \right] \\
+ \frac{64(10N^2 + 10N - 3)}{9N(N + 1)} S_1 + \frac{128}{3} S_2 - \frac{128}{3} S_2 \left( S_{-2} - 32S_2 \right) \right] + \left( \frac{-128}{3} S_1 \right) S_{-2,1} \\
\left( \frac{32(10N^2 + 10N + 9)}{9N(N + 1)} \right) S_{-3} + \frac{224}{3} S_{4} - \frac{64(11N^2 + 11N - 3)}{9N(N + 1)} S_{2,1} \right] \\
- \frac{64}{3} S_{3,1} - \frac{64(10N^2 + 10N - 3)}{9N(N + 1)} S_{-2,1} + \frac{128}{3} S_{-3,1} + \frac{64S_{2,1,1}}{3} + \frac{256}{3} S_{-2,1,1} \right] \\
\left( 96 - 64S_1 \right) \zeta_3 \right] + \frac{64(112N^3 + 112N^2 - 39N + 18)}{81N^2(N + 1)} S_{-2,1} \right] \\
+ \frac{P_{37}}{729N^3(N + 1)^5} \left( \frac{-16(N - 1)(2N^3 - N^2 - N - 2)}{9N^2(N + 1)^2} S_2 + \frac{112}{9} S_2 \right) \right] \\
- \frac{4P_{44}}{729N^3(N + 1)^4} + \frac{80(2N + 1)^2}{9N(N + 1)} S_2 - \frac{208}{9} S_4 - \frac{8(9N^2 + 9N + 16)}{9N(N + 1)} \right] S_{2,1} \\
\left( \frac{64}{3} S_{3,1} + \frac{128(10N^2 + 10N - 3)}{27N(N + 1)} S_{-2,1} + \frac{128}{9} S_{-2,2} - \frac{512}{9} S_{-2,1,1} \right) S_1 \right] \\
\left( \frac{4P_{18}}{9N^3(N + 1)^3} + \frac{32}{9N(N + 1)} S_2 - \frac{80}{9} S_3 + \frac{128}{9} S_{2,1} + \frac{128}{9} S_{-2,1} \right) S_1^2 \right]
\]
\[ + \left( \frac{4P_{35}}{81N^3(N+1)^3} + \frac{496}{27}S_3 - \frac{64}{3}S_{2,1} - \frac{128}{3}S_{-2,1} \right) S_2 - \frac{64}{27}S_1^3S_2 \\
- \frac{4(15N^2 + 15N + 14)}{9N(N+1)}S_2^2 - \frac{8P_{20}}{81N^2(N+1)^2}S_3 + \frac{4(443N^2 + 443N + 78)}{27N(N+1)}S_4 \\
- \frac{224}{9}S_5 + \left( \frac{32P_9}{81N^3(N+1)} - \frac{64P_7}{81N^2(N+1)^2} \right) S_1 + 128 \frac{1}{27}S_1^3 - \frac{128}{27}S_3^3 \\
+ \frac{640}{27}S_2 - \frac{256}{27}S_3 + \frac{256}{9}S_{2,1} \right) S_2 + \left( \frac{32(112N^3 + 224N^2 + 169N + 39)}{81N(N+1)^2} \right) S_3 - \frac{64(10N^2 + 10N + 3)}{27N(N+1)} \\
- \frac{128}{9}S_{-1} - \frac{256}{9}S_{-2} - \frac{64}{9}S_{1,1} \right) S_{-2} + \frac{64}{9}S_{1,1} - \frac{64}{9}S_2 \right) S_{-3} + \left( \frac{64(10N^2 + 10N + 3)}{27N(N+1)} \right) \\
- \frac{128}{9}S_{-1} - \frac{128}{9}S_{-2} - \frac{8P_{19}}{9N^2(N+1)^2}S_{2,1} - \frac{8(13N + 4)(13N + 9)}{27N(N+1)}S_{3,1} \\
+ \frac{256}{9} \left[ S_{-2,3} + S_{1,1} - S_{-2,3} - S_{1,1} - S_{-2,1,1} \right] - \frac{128}{9}S_{1,1} \\
+ \frac{64}{3}S_{2,1,1} + \frac{256}{27}S_{2,1,1} - \frac{8(2N^3 + 2N^2 + 2N + 1)}{3N^3(N+1)^3} \] \]
Here the color factors are given by
\[ C_A = N_c, \quad C_F = (N_c^2 - 1)/(2N_c), \quad T_F = 1/2 \text{ in } SU(N_c), \quad \text{and} \quad N_c = 3 \text{ in the case of Quantum Chromodynamics.} \]
\[ C_{q,g}^{NS,(3)}(N_F) \text{ denotes the massless Wilson coefficient at 3-loop order, cf. (2.2), and the polynomials } P_i \text{ are given by} \]

\[ P_1 = -3N^4 - 6N^3 - 47N^2 - 20N + 12 \]
\[ P_2 = 7N^4 + 14N^3 + 3N^2 - 4N - 4 \]
\[ P_3 = 19N^4 + 38N^3 - 9N^2 - 20N + 4 \]
\[ P_4 = 28N^4 + 56N^3 + 28N^2 + 2N + 1 \]
\[ P_5 = 33N^4 + 54N^3 + 9N^2 - 52N - 28 \]
\[ P_6 = 57N^4 + 96N^3 + 65N^2 - 10N - 24 \]
\[ P_7 = 112N^4 + 224N^3 + 121N^2 + 9N + 9 \]
\[ P_8 = 141N^4 + 246N^3 + 241N^2 - 8N - 84 \]
\[ P_9 = 181N^4 + 266N^3 + 82N^2 - 3N + 18 \]
\[ P_{10} = 235N^4 + 524N^3 + 211N^2 + 30N + 72 \]
\[ P_{11} = 359N^4 + 772N^3 + 335N^2 + 30N + 72 \]
\[ P_{12} = 501N^4 + 894N^3 + 541N^2 - 116N - 204 \]
\[ P_{13} = 561N^4 + 1122N^3 + 767N^2 + 302N + 48 \]
\[ P_{14} = 1131N^4 + 2118N^3 + 1307N^2 + 32N - 276 \]
\[ P_{15} = 1139N^4 + 2710N^3 + 635N^2 + 216N + 828 \]
\[ P_{16} = 1199N^4 + 2398N^3 + 1181N^2 + 18N + 90 \]
\[ P_{17} = 1220N^4 + 2359N^3 + 1934N^2 + 357N - 138 \]
\[ P_{18} = 3N^5 + 11N^4 + 10N^3 + 3N^2 + 7N + 8 \]
\[ P_{19} = 12N^5 + 16N^4 + 18N^3 - 15N^2 - 5N - 8 \]
\[ P_{20} = 27N^5 + 863N^4 + 1573N^3 + 1151N^2 + 144N - 36 \]
\[ P_{21} = 51N^5 + 102N^4 + 121N^3 + 118N^2 + 48N + 48 \]
\[ P_{22} = 648N^5 - 2103N^4 - 4278N^3 - 3505N^2 - 682N - 432 \]
\[ P_{23} = 1407N^5 + 2418N^4 + 1793N^3 + 134N^2 - 384N + 144 \]
\[ P_{24} = -11145N^6 - 32355N^5 - 37523N^4 - 14329N^3 + 2392N^2 + 120N - 1512 \]
\[ P_{25} = -151N^6 + 469N^5 - 181N^4 + 305N^3 + 208N^2 + 40N + 8 \]
\[ P_{26} = 3N^6 + 9N^5 + 70N^4 + 77N^3 + 39N^2 - 10N - 12 \]
\[ P_{27} = 6N^6 + 18N^5 - N^4 - 20N^3 + 46N^2 + 29N - 6 \]
\[ P_{28} = 15N^6 + 36N^5 + 30N^4 + 8N^3 + 3N^2 + 16N + 20 \]  
(2.35)

\[ P_{29} = 155N^6 + 465N^5 + 465N^4 + 371N^3 + 108N^2 + 108N + 54 \]  
(2.36)

\[ P_{30} = 216N^6 + 567N^5 + 687N^4 + 381N^3 + 37N^2 - 44N + 12 \]  
(2.37)

\[ P_{31} = 309N^6 + 807N^5 + 693N^4 - 271N^3 - 638N^2 + 68N + 216 \]  
(2.38)

\[ P_{32} = 525N^6 + 1575N^5 + 1535N^4 + 973N^3 + 536N^2 + 48N - 72 \]  
(2.39)

\[ P_{33} = 609N^6 + 1485N^5 + 1393N^4 + 83N^3 - 422N^2 + 156N + 216 \]  
(2.40)

\[ P_{34} = 795N^6 + 2043N^5 + 2075N^4 + 517N^3 - 298N^2 + 156N + 216 \]  
(2.41)

\[ P_{35} = 868N^6 + 2469N^5 + 2487N^4 + 940N^3 + 27N^2 + 63N + 72 \]  
(2.42)

\[ P_{36} = 1770N^6 + 4671N^5 + 4765N^4 + 1205N^3 - 227N^2 + 1044N + 756 \]  
(2.43)

\[ P_{37} = 7531N^6 + 23673N^5 + 23055N^4 + 7375N^3 + 1614N^2 + 936N - 324 \]  
(2.44)

\[ P_{38} = -4785N^7 - 14355N^6 - 4399N^5 + 10327N^4 + 3548N^3 + 3000N^2 + 1080N - 1728 \]  
(2.45)

\[ P_{39} = 25N^7 + 138N^6 + 311N^5 + 464N^4 + 672N^3 + 670N^2 + 264N + 48 \]  
(2.46)

\[ P_{40} = -45N^8 - 162N^7 - 858N^6 - 1960N^5 - 1885N^4 - 1094N^3 - 804N^2 - 40N + 192 \]  
(2.47)

\[ P_{41} = 39N^8 + 138N^7 + 847N^6 + 1371N^5 + 1283N^4 + 485N^3 + 101N^2 + 132N + 72 \]  
(2.48)

\[ P_{42} = 3549N^8 + 14196N^7 + 23870N^6 + 25380N^5 + 15165N^4 + 1712N^3 - 2016N^2 + 144N + 432 \]  
(2.49)

\[ P_{43} = 5487N^8 + 21948N^7 + 36370N^6 + 28836N^5 + 11943N^4 + 4312N^3 + 2016N^2 - 144N - 432 \]  
(2.50)

\[ P_{44} = 10807N^8 + 43228N^7 + 62898N^6 + 39178N^5 + 7027N^4 + 702N^3 + 3240N^2 + 3456N + 1620 \]  
(2.51)

\[ P_{45} = 42591N^8 + 166764N^7 + 245088N^6 + 128254N^5 - 26735N^4 - 40762N^3 - 3928N^2 - 1272N - 2160 \]  
(2.52)

\[ P_{46} = -18351N^{10} - 89784N^9 - 208773N^8 - 267222N^7 - 192265N^6 - 46700N^5 + 14565N^4 + 7730N^3 + 1240N^2 + 1464N + 144 \]  
(2.53)

\[ P_{47} = 165N^{10} + 825N^9 + 106856N^8 + 321746N^7 + 396657N^6 + 247433N^5 + 126914N^4 + 51804N^3 + 6336N^2 + 4752N + 5184 \]  
(2.54)

\[ P_{48} = 828N^{11} + 7632N^{10} + 29217N^9 + 59592N^8 + 66844N^7 + 35738N^6 + 7405N^5 + 16688N^4 + 27880N^3 + 11552N^2 - 3312N - 2304 \]  
(2.55)

\[ P_{49} = 8274N^{11} + 78519N^{10} + 313841N^9 + 686295N^8 + 881001N^7 + 638778N^6 + 204948N^5 + 7992N^4 + 32296N^3 + 26544N^2 - 10656N - 8640 \]  
(2.56)

We would like to note that we disagree with the \( O(a_s^2 \ln(Q^2/\mu^2)) \) terms given in [28], but agree with the representation in [29,51].

One obtains the analytic continuation of the harmonic sums to complex values of \( N \) by performing their asymptotic expansion analytically, cf. [33,34]. Furthermore, the nested harmonic sums obey the shift relations

\[ S_{b,a}(N) = S_{b,a}(N-1) + \frac{\text{sign}(b)N}{N|b|} S_{a}(N) \]  
(2.57)

\[^{3}\text{These expansions can now be obtained automatically using the package HarmonicSums [25].}\]
through which any regular point in the complex plane can be reached using the analytic asymptotic representation as input. The poles of the nested harmonic sums $S_q(N)$ are located at the non-positive integers. In data analyses, one may thus encode the QCD evolution [35] together with the Wilson coefficient for complex values of $N$ analytically and finally perform one numerical contour integral around the singularities of the problem.\footnote{For precise numerical implementations of the analytic continuation of harmonic sums see [36].}

In $x$-space the Wilson coefficient is represented in terms of harmonic polylogarithms [37] over the alphabet \{f_0, f_1, f_{-1}\}, which were again reduced applying the shuffle relations [38]. They are defined by

$$H_{b,a}(x) = \int_0^x dy f_b(y) H_a(y), \quad H_{0,\ldots,0}(x) = \frac{1}{k!} \ln^{k} x, \quad H_0 = 1,$$  

(2.58)

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1 - x}, \quad f_{-1}(x) = \frac{1}{1 + x}.$$  

(2.59)

The Wilson coefficient is represented by three contributions, the \((\ldots)_{+}\)-function term, the $\delta(1-x)$-term, and the regular term. Here the $\delta$-distribution is defined by

$$\int_0^1 dy \left[ F(y)\right]_+ g(y) = \int_0^1 dy F(y) \left[ g(y) - g(1) \right].$$  

(2.60)

One obtains

$$L_{q_{NS}}^{NS}(x) = \alpha_s^2 \left\{ \left[ \frac{1}{1-x} C_F T_F \left[ \frac{8}{3} L_Q^2 + L_M^2 \right] + L_M \left[ \frac{80}{9} + \frac{16}{3} H_0 \right] + L_Q \left[ -\frac{116}{9} - \frac{32}{3} H_0 \right] - \frac{16}{3} H_1 \right] + \frac{718}{27} H_Q + \frac{268}{9} H_0 + \frac{32}{3} \zeta_2 \right\} + \delta(1-x) \left( C_F T_F \left[ 2 \left[ L_M^2 + L_Q^2 \right] + L_M \frac{2}{3} + L_Q \frac{8}{3} + \frac{265}{9} \right] \right)$$

$$+ C_F T_F \left[ \frac{4}{3} \left[ L_Q^2 + L_M^2 \right] + L_M \left[ -\frac{8}{9} \left( 11x - 1 \right) - \frac{8}{3} \left( x + 1 \right) H_0 \right] + \frac{8}{9} \left( 14x + 5 \right) + \frac{16}{3} \left( x + 1 \right) H_0 + \frac{8}{3} \left( x + 1 \right) H_1 \right] - \frac{4}{27} (218x + 47)$$

$$- \frac{8}{9} (28x + 13) H_0 - 4 \left( x + 1 \right) H_0^2 + \left[ -\frac{8}{9} (14x + 5) - \frac{8}{3} \left( x + 1 \right) H_0 \right] H_1$$

$$- \frac{4}{3} \left( x + 1 \right) H_1^2 - \frac{8}{3} \left( x + 1 \right) H_{0,1} + \frac{16}{3} \left( x + 1 \right) \zeta_2 \right\}$$

$$+ \alpha_s^3 \left\{ \left[ \frac{1}{(1-x)^2} C_A C_F T_F \left[ -\frac{4}{81} (800x - 773) \right] H_0^2 + \frac{32}{81} (94x - 121) \zeta_2 \right] + \frac{32}{9} \left( x + 2 \right) H_{0,1} \right\} + \frac{1}{1-x} \left[ C_A C_F T_F \left[ -L_M^3 \frac{176}{27} - L_M^2 \frac{352}{27} + L_M^2 \frac{184}{9} \right] + \frac{16}{3} \left( H_0^2 - \frac{32}{3} \zeta_2 \right) + L_Q \left[ 3104 \frac{27}{27} + 704 \frac{9}{9} H_0 + \frac{16}{3} H_0^2 + 352 \frac{9}{9} H_1 - \frac{32}{3} \zeta_2 \right] \right\}.$$
\begin{align*}
+ & L_M \left[ \frac{1240}{81} + \frac{1792}{27}H_0 + \frac{248}{9}H_0^3 + \frac{32}{9}H_0^5 - 16H_0^2H_1 + 32H_0H_{0,1} - \frac{64}{3}H_{0,0,1} \right] \\
+ & \left( -\frac{320}{9} - \frac{64}{3}H_0 \right) \zeta_2 + 96\zeta_3 \\
+ & L_Q \left[ -\frac{80}{9}H_0^3 - \frac{30124}{81} - \frac{14144}{27}H_0 - \frac{1216}{9}H_0^2 \right] \\
+ & \left( -\frac{6208}{27} - \frac{704}{9}H_0 - \frac{16}{3}H_0^2 \right) H_1 + \left( -\frac{352}{9} + \frac{32}{3}H_0 \right) H_1^2 - 64H_0H_{0,0,1} \\
+ & \left( -\frac{704}{9} + \frac{32}{3}H_0 - \frac{128}{3}H_1 \right) H_{0,0,1} - \frac{128}{3}H_{0,0,1} + 128H_{0,0,1} - 64H_{0,1,1} \\
+ & \left( 192 + \frac{128}{3}H_0 + 64H_1 \right) \zeta_2 - \frac{256}{3} \zeta_3 \\
+ & \frac{16}{3}H_0^2 + \left( -8 - \frac{128}{9}H_0 \right) H_1 + \frac{128}{9}H_1^2 \right) H_{0,0,1} - \frac{32}{9}H_{0,0,1} + \left( -\frac{1072}{27} + \frac{32}{9}H_0 \right) \\
+ & \left( \frac{320}{9}H_1 \right) H_{0,0,1} + \frac{224}{9} \left[ H_{0,1,1,1} - H_{0,0,0,1} \right] + \left( \frac{160}{9}H_0 - 32H_1 + 24 \right) H_{0,1,1} \\
+ & \left( \frac{496}{27}H_0 - \frac{112}{9}H_0^2 + \left( 8 - \frac{160}{9}H_0 \right) H_1 - \frac{128}{9}H_1^2 + \frac{32}{9}H_{0,1} \right) \zeta_2 + \left( \frac{1196}{27} \right. \\
+ & \frac{160}{9}H_0 - \frac{32}{9}H_1 \right) \zeta_3 - \frac{296}{3} \zeta_4 - \frac{32}{3}B_4 \right] + C_F^2 T_F \left[ \frac{L_3^3}{L_Q} \left[ 16 - \frac{32}{3}H_0 - \frac{64}{3}H_1 \right] \\
+ & L_3^2 L_Q \left[ 16 - \frac{32}{3}H_0 - \frac{64}{3}H_1 \right] + L_M \left[ -22 - 16H_0 + \left( 8 + \frac{128}{3}H_0 \right) H_1 \right] \\
+ & \frac{16}{3}H_0^2 + 16H_1^2 - \frac{64}{3}H_2 \right] + L_Q \left[ -\frac{334}{3} + \frac{32}{9}H_0 + \left( \frac{856}{9} + \frac{320}{3}H_0 \right) H_1 \right] \\
+ & \frac{80}{3}H_0^2 + 48H_1^2 - \frac{256}{3} \zeta_2 \right] + L_M \left[ -\frac{206}{3} - \frac{112}{3}H_0 + \frac{88}{9}H_0^2 + \left( \frac{160}{3} + \frac{32}{3}H_0 \right) H_1 \right] \\
+ & \frac{32}{3}H_0^3 + \left( \frac{152}{3} + \frac{1424}{9}H_0 + \frac{160}{3}H_0^2 \right) H_1 - \frac{64}{3}H_0H_{0,1} - \frac{784}{9} - \frac{128}{3}H_0 \\
+ & \frac{64}{9}H_0 \zeta_2 - \frac{64}{3}H_3 \right] + L_Q \left[ \frac{2360}{9} + \frac{4508}{27}H_0 - \frac{160}{3}H_0^2 - \frac{224}{9}H_0^3 - \frac{320}{9}H_1^3 \right] \\
+ & \left( \frac{4556}{27} - \frac{3680}{9}H_0 - 128H_0^2 \right) H_1 + \left( -\frac{512}{3} - 128H_0 \right) H_1^2 + 128H_0H_{0,0,1} \\
+ & \left( \frac{48}{3} + \frac{64}{3}H_0 + \frac{64}{3}H_1 \right) H_{0,0,1} - \frac{1256}{9}H_{0,0,0,1} - 64H_{0,0,1} + \left( \frac{2608}{9} + \frac{832}{3}H_0 \right) \\
+ & \left( \frac{448}{3} + \frac{320}{3} \zeta_3 \right] - \frac{14197}{54} - \frac{3262}{27}H_0 + \frac{4}{3}H_0^2 + \frac{196}{27}H_0^3 + \frac{380}{81}H_0^3 \\
+ & \left( \frac{302}{9} + \frac{13624}{81}H_0 + \frac{1628}{27}H_0^2 + \frac{304}{27}H_0^3 \right) H_1 + \left( \frac{448}{9} + \frac{80}{9}H_0 - \frac{8}{9}H_0^2 \right) H_1^2 \\
+ & \frac{128}{27}H_0^3 + \left( \frac{112}{9} - \frac{1304}{27}H_0 - \frac{32}{3}H_0^2 + \frac{160}{9}H_0H_1 - \frac{128}{9}H_1^2 \right) H_{0,1} \right]
\end{align*}
\[
\begin{align*}
&+ \left( \frac{1184}{27} + \frac{128}{9} H_0 - \frac{256}{9} H_1 \right) H_{0,0,1} + \left( -16 - \frac{32}{3} H_0 + \frac{64}{3} H_1 \right) H_{0,1,1} \\
&- \frac{16}{9} H_{0,1}^2 - \frac{128}{9} H_{0,0,1} + \frac{64}{3} H_{0,0,1,1} + \left( -\frac{2488}{27} - \frac{1192}{27} H_0 - \frac{80}{9} H_0^2 + \left( -\frac{160}{9} \right) \right) H_1 + \frac{128}{9} H_1^2 - \frac{32}{9} H_{0,1} \right) c_2 + \left( \frac{3088}{27} - \frac{128}{9} H_0 + \frac{160}{9} H_1 \right) c_3 + \frac{64}{3} B_4 \\
&- \frac{664}{9} \zeta_4 + C_F T_F^2 \left[ L_M^3 \frac{128}{27} + L_Q^3 \frac{64}{27} + L_M^2 \left[ \frac{320}{27} + \frac{64}{9} H_0 \right] + L_Q^2 \left[ -\frac{464}{27} \right] - \frac{128}{9} H_0 - \frac{64}{9} H_1 \right] + L_M \left[ \frac{1984}{81} + L_Q \left[ \frac{3760}{81} + \frac{2144}{27} H_0 + \left( \frac{928}{27} + \frac{128}{9} H_0 \right) H_1 \right] + \frac{64}{3} H_0^2 + \frac{64}{9} H_1^2 + \frac{128}{9} H_{0,1} - \frac{256}{9} \zeta_2 \right] \right] - \frac{12064}{729} + \frac{64}{81} H_0 - \frac{160}{81} H_0^2 - \frac{32}{9} H_0^3 + \frac{896}{27} \zeta_3 \right]
\end{align*}
\]

\[+ C_F N_F T_F^2 \left[ L_M^3 \frac{64}{27} + L_Q^3 \frac{128}{27} + L_Q^2 \left[ -\frac{928}{27} - \frac{256}{9} H_0 - \frac{128}{9} H_1 \right] + L_M \left[ \frac{2176}{81} - \frac{320}{27} H_0 - \frac{32}{9} H_0^2 \right] + L_Q \left[ \frac{7520}{81} + \frac{4288}{27} H_0 + \frac{128}{3} H_0^2 + \left( \frac{1856}{27} \right) \right] + \frac{256}{9} H_0 \right] H_1 + \frac{128}{9} H_1^2 + \frac{256}{9} H_{0,1} - \frac{512}{9} \zeta_2 \right] + \frac{24064}{729} + \frac{128}{81} H_0 - \frac{320}{81} H_0^2 - \frac{64}{81} H_0^3 - \frac{512}{27} \zeta_3 \right]
\]

\[+ \delta(1 - x) \left( C_F C_F T_F \left[ L_M^3 \frac{44}{9} - L_Q^3 \frac{88}{9} + L_M^2 \left[ \frac{34}{3} \right] \right] - \frac{16}{3} \zeta_3 \right] + \frac{L_Q^2 \left[ \frac{938}{3} - \frac{16}{3} \zeta_3 \right] + L_M \left[ -\frac{1595}{27} + \frac{272}{9} \zeta_3 + \frac{68}{3} \zeta_4 \right] + L_Q \left[ -\frac{11032}{27} \right] \right] - \frac{32}{3} \zeta_2 + \frac{1024}{9} \zeta_3 - \frac{196}{9} \zeta_4 \right] + \frac{55}{243} - \frac{10045}{81} \zeta_3 - \frac{16}{9} \zeta_2 \zeta_3 + \frac{2624}{27} \zeta_4 - \frac{176}{9} \zeta_5 \right]
\]

\[-8 B_4 \right) + C_F^2 T_F \left[ L_M^3 6 + L_M^2 L_Q 6 + L_M^2 \left[ -10 + \frac{32}{3} \zeta_3 \right] + L_Q^2 \left[ -48 + \frac{32}{3} \zeta_3 \right] \right] + L_M L_Q 2 + L_M \left[ -5 - \frac{112}{9} \zeta_3 - \frac{136}{3} \zeta_4 \right] + L_Q \left[ \frac{368}{3} + \frac{64}{3} \zeta_2 + 16 B_4 + \frac{352}{9} \zeta_5 \right]
\]

\[+ C_F T_F^2 \left[ L_M^3 \frac{32}{9} + L_Q^3 \frac{16}{9} + L_M^2 8 + L_M^2 \frac{152}{9} + L_M 496 \frac{27}{27} + L_Q 1624 \frac{27}{27} - 3658 \frac{243}{243} \right] + \frac{224}{9} \zeta_3 \right] + C_F N_F T_F^2 \left[ L_M^3 \frac{16}{9} + L_Q^3 \frac{32}{9} - L_M^2 \frac{604}{9} + L_M \frac{700}{27} + L_Q \frac{3248}{27} + \frac{4732}{243} \right] - \frac{128}{9} \zeta_3 \right) + C_F C_F T_F \left[ L_M^3 \frac{88}{27} (x + 1) + L_Q^3 \frac{176}{27} (x + 1) + L_M^2 \left[ \frac{4}{9} (83x - 37) \right] \right]
\]

\[+ \frac{32}{3} (x + 1) H_0 + \frac{32}{3} x^2 + \frac{1}{3} x + \left[ H_{0,-1} - H_{-1} H_0 \right] + \frac{16}{3} x \left[ 2 \zeta_2 - H_0^2 \right]
\]
\[+L_Q \left[ -\frac{4}{27} (865x + 109) - \frac{256}{9} (x + 1) H_0 + \frac{32 x^2 + 1}{3 x + 1} \right[H_{0, -1} - H_{-1} H_0] \\
- \frac{176}{9} (x + 1) H_1 + \frac{16 x}{3 x + 1} \left[2 \zeta_2 - H_0^2 \right] \right]+ L_M \left[ -\frac{4}{81} (4577x - 4267) \right] \\
\frac{16}{27} (29x - 109) H_0 + \frac{419x^2 + 4x + 25}{x + 1} H_0^2 - \frac{32}{9} \frac{x}{x + 1} H_0^3 + \left( \frac{32}{3} (x - 1) \right) \\
+ 8 (x + 1) H_0^2 \right] H_1 + \frac{128 x^2 + 3x + 4}{x + 1} H_{0, -1} + \left[ \frac{16}{3} \right. \frac{1}{3} (x + 1) H_{0, -1} \\
\frac{32}{9} \frac{x^2 + 1}{x + 1} + \left[ \frac{4H_{0, 1} - H_0^2}{3} \right] \right] H_{-1} + \frac{64}{3} \frac{x}{x + 1} H_{0, 0, 1} \\
\frac{64 x^2 + 11}{x + 1} H_{-1} + \frac{16 x + 3x + 4}{9} \left[ 4H_{0, 1} - H_0^2 \right] \right) \right] H_{-1} + \frac{64}{3} \frac{x}{x + 1} H_{0, 0, 1} \\
\frac{64 x^2 + 1}{x + 1} H_{-1} + \frac{16 x^2 + 4x + 3}{x + 1} H_0 \right] \zeta_2 - \frac{32 x^2 + 3x + 1}{x + 1} \right] \\
+ L_Q \left[ \frac{4}{81} (12329x - 577) + \frac{64 \cdot 181x^2 + 239x + 49}{x + 1} \right] \left[ -\frac{32}{3} (x - 1) \right] \right] H_0 + \left( -\frac{8}{9} \right) \frac{12x^3 - 21x^2 - 77x - 24}{x + 1} \right] \\
+ \frac{16}{3} \frac{5x^2 - 2x + 5}{x + 1} H_{-1} \right] H_0^2 + \frac{80 x}{9 x + 1} H_0^3 + \left( \frac{8}{27} \frac{(703x + 253)}{x - 3} \right) H_0^2 \\
+ \frac{352}{9} \frac{(x + 1) H_0}{x + 1} H_1 + \left[ \frac{176}{9} \frac{(x + 1)}{x + 1} - \frac{16}{3} (x + 1) H_0 \right] H_1^2 + \left( \frac{64 x + 1}{3 x + 1} H_0 \right) \\
\frac{32}{9} \frac{x^4 + 25x^3 + 18x^2 + 25x + 6}{x + 1} + \frac{64}{3} \frac{(x - 1)^2}{x + 1} H_{-1} \right] H_{0, -1} + \left( \frac{208}{9} \right) (x + 1) \right] \\
+ \frac{16}{3} \frac{3}{x - 3} H_0 + \frac{64}{3} \frac{(x + 1)}{x + 1} H_1 \right] H_{0, 1} + \frac{32}{3} (x + 3) H_{0, 0, 1} - 32 (x + 1) H_{0, 1, 1} \right] \\
+ \frac{64}{3} \frac{(x - 1)^2}{x + 1} H_{0, 0, 1} \right] \frac{32}{3} \frac{5x^2 + 10x + 9}{x + 1} H_{0, 0, 1} + \left( -\frac{64}{3} \right) (x + 2) H_1 \right] \\
+ \frac{16}{9} \frac{12x^3 - 23x^2 - 72x - 17}{x + 1} \right] \frac{-32 (x - 1)^2}{3 x + 1} H_{-1} + \frac{163x^2 + 8x + 3}{3 x + 1} H_0 \right] \zeta_2 \\
+ \frac{64}{3} \frac{x^2 + 3x + 2}{x + 1} \right] - \frac{2}{729} \left( 108295x - 86681 \right) + \left( -\frac{4}{81} \right) \left( 995x - 2807 \right) \right] \\
\frac{32}{81} \left[ 199x^2 + 174x + 199 \right] H_{-1} + \frac{32}{9} \left[ (x + 1) H_1^2 - \frac{64 x^2 + 1}{27 x + 1} H_{-1} \right] \right] H_0 \right] \\
+ \left( \frac{4}{81} \frac{253x^2 + 391x + 586}{x + 1} H_{-1} + \frac{1619x^2 + 18x + 19}{x + 1} H_{-1} + \frac{16 x^2 + 1}{9 x + 1} H_0^2 \right] \right] H_0 \right] \\
+ \left( \frac{8}{81} \frac{22x^2 + 7x + 25}{x + 1} H_{-1} + \frac{32 x^2 + 1}{27 x + 1} H_{-1} \right] H_0^3 - \frac{16}{27} \frac{x}{x + 1} H_0^4 + \left( \frac{8}{9} \right) \left( 9x + 4 \right) H_0 \right] \right] \\
- \frac{8}{27} \left( 65x - 29 \right) + \frac{8}{9} \left( 14x + 3 \right) H_0^2 + \frac{56}{27} \left( x + 1 \right) H_0^3 \right] H_1 + \left( -\frac{4}{9} \right) \left( 43x - 46 \right) \right] \\
- \frac{8}{9} \left( 2x + 5 \right) H_0 - \frac{4}{9} \left( x + 1 \right) H_0^3 \right] H_1^2 + \frac{32}{27} \left( x + 1 \right) H_0 H_1^3 + \left( -\frac{64}{9} \right) \left( x + 1 \right) H_{-1} \right]
\[
\begin{align*}
+ \frac{32199x^2 + 174x + 199}{81} &+ \frac{64x^2 + 1}{9} H^2_{-1} H_{0,-1} &+ \left(\frac{2564x^2 + 3x + 4}{27} x + 1 \right) H_{-1} \\
- \frac{8}{27} (143x + 2) &- \frac{16}{9} (13x + 6) H_{0} &- \frac{8}{3} (x + 1) H^2_{0} &+ \left(\frac{8}{9} (11x + 20) \right) H_{-1} \\
+ \frac{64}{9} (x + 1) H_{0} &- \frac{64}{9} (x + 1) H^2_{1} H_{0,1} &+ \frac{16}{9} (7x + 1) H^2_{0,1} &+ \left(\frac{64}{9} (x + 1) \right) H_{-1} \\
- \frac{128 x^2 + 1}{9} x + 1 &H_{-1} H_{0,-1} &- \frac{2564x^2 + 3x + 4}{27} x + 1 &H_{0,-1} &+ \left(\frac{64x^2 + 1}{9} x + 1 \right) H_{-1} \\
+ \frac{32199x^2 + 18x + 19}{27} x + 1 &H_{0,0,-1} &+ \frac{8}{27} \left(9x^2 + 101x + 12 \right) x + 1 &H_{0,1} &+ \frac{64x^2 + 1}{9} x + 1 H_{-1} \\
- \frac{16}{9} (7x + 1) &H_{0} &- \frac{160}{9} (x + 1) H_{1} &H_{0,0,1} &- \frac{2564x^2 + 3x + 4}{27} x + 1 &H_{0,1,-1} \\
+ \left(\frac{16}{9} (x + 7) - \frac{128 x^2 + 1}{9} x + 1 &H_{1} &- \frac{16}{9} (x + 1) H_{0} &+ 16 (x + 1) H_{1} \right) H_{0,1,1} \\
+ \frac{64 x^2 + 1}{9} x + 1 &\left[2H_{0,-1,-1,1} + 2H_{0,-1,1,1} + H_{0,0,-1,1} - H_{0,0,-1,1} + H_{0,0,0,0,-1} \right] &+ \frac{645x^2 + 6x - 1}{9} x + 1 \right] H_{0,0,0,1} \\
- \frac{16 x^2 - 2x - 11}{9} x + 1 &H_{0,0,1,1} &- \frac{112}{9} (x + 1) H_{0,1,1,1} &+ \left(\frac{16174x^2 + 209x - 189}{81} x + 1 \right) H_{1} \\
+ \frac{32199x^2 + 18x + 29}{27} x + 1 &H_{-1} &+ \left(\frac{8}{9} (3x + 14) + \frac{80}{9} (x + 1) H_{0} \right) H_{1} \\
+ \frac{8}{27} 63x^2 + 29x + 6 &x + 1 &H_{0} &- \frac{32 x^2 + 1}{9} x + 1 &H_{-1} &H_{1} &+ \frac{64}{9} \left(x + 1\right) H^2_{1} \\
+ \frac{8}{9} x + 1 &H_{0} &- \frac{16}{9} (7x + 1) H_{0,1} &H_{2} &+ \frac{2}{27} 497x^2 + 1102x + 1085 \right] x + 1 \\
+ \frac{128 x^2 + 1}{9} x + 1 &H_{1} &+ \frac{32 6x^2 + 4x - 3}{9} x + 1 &H_{0} &+ \frac{16}{9} (x + 1) H_{1} \right] \zeta_3 &+ \frac{16}{3} \left(x + 1\right) B_4 \\
- \frac{8}{3} 36x^2 + 51x + 22 &x + 1 \right] \zeta_4 &+ C^2 F_T \left[ L^2_Q + L^2_M L_Q \right] &\left[-\frac{8}{3} (x + 5) &+ \frac{32}{3} (x + 1) H_{1} \right] \\
+ 8 (x + 1) H_{0} &+ L^2_M \left[28 (2x - 1) &+ \left(\frac{8}{3} \left(11x + 5\right) + \frac{64}{3} x^2 + 1 \right) x + 1 H_{-1} \right] H_{0} \\
- \frac{4}{3} 9x^2 + 10x + 9 &x + 1 &H_0 &+ \left(-\frac{16}{3} (2x + 1) &- \frac{64}{3} (x + 1) H_{0} \right) H_{1} &- \frac{8}{3} \left(x + 1\right) H_{0,1} \\
- 8 (x + 1) H^2_1 &- \frac{64 x^2 + 1}{3} x + 1 &H_{0,-1} &+ \frac{8}{3} 9x^2 + 10x + 9 &x + 1 &\zeta_2 &+ L^2_Q \left[\frac{4}{9} \left(161x + 130\right) \right] \\
+ \left(\frac{16}{3} \left(15x + 4\right) &+ \frac{64 x^2 + 1}{3} x + 1 H_{-1} \right) H_{0} &- 24 (x + 1) H^2_1 &+ \left(-\frac{16}{9} \left(50x + 17\right) \right) \zeta_2 \\
- \frac{160}{3} (x + 1) H_{0} &H_{1} &- \frac{4}{3} 21x^2 + 34x + 21 &x + 1 &H_0 &+ \frac{8}{3} 23x^2 + 38x + 23 \right] x + 1 \right] \zeta_2 \\
- \frac{64 x^2 + 1}{3} x + 1 &H_{0,-1} &- 8 (x + 1) H_{0,1} &+ L_M L_Q \left[\frac{4}{9} \left(19x - 85\right) &+ \frac{8}{3} \left(13x + 1\right) H_{0} \right]
\end{align*}
\]
\[+8(x + 1)H_0^2 + \left(\frac{128}{9}(4x + 1) + \frac{32}{3}(x + 1)H_0\right)H_1 - \frac{32}{3}(x + 1)\zeta_2\]
\[+L_M\left[-\frac{4}{9}(337x + 235)H_0 - \frac{4195x^2 + 238x + 123}{x + 1}H_0^2 - \frac{32x^2 + 4x + 3}{x + 1}H_0^3\right.
\[+\left(-\frac{16}{3}(7x + 3) - \frac{16}{3}(x + 1)H_0\right)H_1 + \left(\frac{184}{9}(x + 1) + \frac{32}{3}(x + 1)H_0\right)H_{0,1}\]
\[-\frac{256}{9}x^2 + 3x + 4H_{0,-1} + \frac{16}{3}x^2 - 2x + 3H_{0,0,1} + \left(\frac{256}{9}x^2 + 3x + 4\right)H_0
\[+\frac{32x^2 + 1}{x + 1}[H_0^2 - 4H_{0,1}]\right)H_1 + \frac{64x^2 + 1}{x + 1}\left[2H_{0,-1,1} - H_{0,0,-1} + 2H_{0,1,-1}\right]
\[+\left(\frac{8}{9}117x^2 + 118x + 81\right) + \frac{128x^2 + 1}{x + 1}H_{-1} + \frac{16(x + 3)(3x + 1)}{x + 1}H_0
\[+\frac{32}{3}(x + 1)H_1\zeta_2 + \frac{16x^2 + 14x + 1}{x + 1}\zeta_3\right] + L_Q\left[-\frac{8}{27}(557x + 652)\right.
\[+\left(\frac{8}{9}115x^2 + 99x + 32\right) - \frac{64x^4 + 25x^3 + 18x^2 + 25x + 6}{x + 1}H_{-1}
\[+\frac{64}{9}(x - 1)^2H_{-1}^2\right)H_0 + \frac{32}{9}x^2 + 13x + 9\frac{H_0^3}{x + 1} - \left(\frac{32}{3}5x^2 - 2x + 5\right)H_{-1}
\[+\frac{448x^3 + 519x^2 + 706x + 315}{x + 1}H_0^2 + \frac{8}{27}(908x - 19) + \frac{16}{9}(169x + 97)H_0
\[+\frac{32}{3}(7x + 5)\zeta_2\right)H_1 + \left(\frac{32}{3}(13x + 6) + 64(x + 1)H_0\right)H_1^2 + \frac{160}{9}(x + 1)H_1^3
\[+\left(\frac{64}{9}(13x + 1) + \frac{16}{3}(x + 9)H_0 - \frac{32}{3}(x + 1)H_1\right)H_0,1 + \left(-\frac{128}{3}3x + 1\right)H_0
\[+\frac{64}{9}6x^4 + 25x^3 + 18x^2 + 25x + 6\frac{H_{-1}}{(x + 1)x}
\[+\frac{128}{3}(x - 1)^2H_{-1,1} - \frac{64}{3}5x^2 + 10x + 9\right)H_{0,0,-1} + \frac{16}{3}(5x - 3)H_{0,0,1}
\[+\frac{48}{3}(x + 1)H_{0,1,1} + \left(-\frac{16}{9}24x^3 + 245x^2 + 318x + 137\right) + \frac{64}{3}(x - 1)^2H_{-1}
\[+\frac{1}{3}(12332x - 4905)\zeta_2 + \frac{32}{3}(9x + 5)H_1\zeta_2 - \frac{32}{3}121x^2 + 30x + 17\zeta_3\]
\[+\frac{1}{27}(12332x - 4905)\zeta_2 + \frac{64}{81}(x + 1)H_2^2 + \frac{64}{81}199x^2 + 174x + 199\frac{H_{-1}}{x + 1}
\[+\frac{1}{81}(-10999x - 8399) + \frac{128}{27}x^2 + 1\frac{H_3^1}{x + 1}H_0 + \left(\frac{32}{27}19x^2 + 18x + 19\right)H_{-1}
\[+\frac{32}{9}x^2 + 1\frac{H_{-1}^2}{27} - \frac{2}{81}4179x^2 + 5255x + 2868\right)H_0^2 + \left(\frac{64}{27}x^2 + 1\right)H_{-1}
\[+\frac{10}{81}177x^2 + 218x + 105\frac{H_0^3}{x + 1} - \frac{1}{27}51x^2 + 70x + 51\frac{H_0^3}{x + 1}
\[+\frac{152}{27}(x + 1)H_0^3
\]

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\[
\begin{align*}
&\quad + \frac{1}{27} \left( -3457x + 1951 \right) - \frac{16}{81} \left( 593x + 335 \right) H_0 - \frac{8}{27} \left( 146x + 71 \right) H_0^3 \\
&\quad + \left( -\frac{8}{9} \left( 3x + 55 \right) - \frac{8}{9} \left( 9x + 1 \right) H_0 + \frac{4}{9} \left( x + 1 \right) H_0^2 \right) H_1 - \frac{64}{27} \left( x + 1 \right) H_0 H_1^3 \\
&\quad + \left( -\frac{64}{9} \frac{199x^2 + 174x + 199}{x + 1} + \frac{128}{9} \left( x + 1 \right) H_{-1} - \frac{128}{9} \frac{x^2 + 1}{x + 1} H_{-1}^2 \right) H_{-1,1} \\
&\quad + \left( \frac{4}{27} \left( 251x + 407 \right) + \frac{16}{27} \left( 10x + 43 \right) H_0 + \frac{16}{3} \left( x + 1 \right) H_0^2 + \left( \frac{64}{9} \left( x - 1 \right) - \frac{80}{9} \left( x + 1 \right) H_0 \right) H_1 - \frac{512}{27} \frac{4x^2 + 3x + 4}{x + 1} H_{-1} + \frac{64}{9} \left( x + 1 \right) H_1^2 \right) H_{0,1} \\
&\quad + \frac{8}{9} \left( x + 1 \right) H_{0,1}^2 + \frac{512}{27} \frac{4x^2 + 3x + 4}{x + 1} H_{0,1} + \left( \frac{256}{9} \frac{x^2 + 1}{x + 1} H_{-1} \right) H_{0,0,1} \\
&\quad + \left( \frac{4}{27} \frac{357x^2 + 130x + 93}{x + 1} - \frac{128}{9} \frac{x^2 + 1}{x + 1} H_{-1} + \frac{64}{9} \left( x + 1 \right) \left[ 2H_1 - H_0 \right] \right) H_{0,0,1} \\
&\quad + \frac{512}{27} \frac{4x^2 + 3x + 4}{x + 1} H_{0,1} - \left( \frac{32}{9} \left( 13x + 1 \right) + \frac{16}{3} \left( x + 1 \right) \left[ H_0 - 2H_1 \right] \right) H_{0,1,1} \\
&\quad + \frac{256}{9} \frac{x^2 + 1}{x + 1} H_{-1} \left) H_{0,1,1} + \frac{128}{9} \frac{x^2 + 1}{x + 1} \left[ H_{0,0,1} - 2H_{0,-1,1} - 1 - 2H_{0,-1,1,1} \right] \\
&\quad - H_{0,0,-1,1} - H_{0,0,1,1} + H_{0,0,1,1} - 2H_{0,1,1,1} - 1 - 2H_{0,1,1,1} \right) \\
&\quad + \frac{8}{9} \frac{21x^2 + 10x + 21}{x + 1} H_{0,0,0,1} - \frac{32}{9} \frac{7x^2 + 6x + 7}{x + 1} H_{0,0,1,1} + \left( \frac{64}{9} \frac{x^2 + 1}{x + 1} H_{-1} \right) H_0 \\
&\quad + \frac{4}{81} \frac{1619x^2 + 1338x + 1511}{x + 1} + \left( \frac{4}{27} \frac{147x^2 + 298x - 9}{x + 1} + \frac{64}{9} \frac{x^2 + 1}{x + 1} H_{-1} \right) H_0 \\
&\quad + \frac{64}{27} \frac{29x^2 + 18x + 29}{x + 1} + \frac{4}{9} \frac{(x + 5)(5x + 1)}{x + 1} H_0^2 - \frac{64}{9} \left( x + 1 \right) H_1^2 \\
&\quad + \left( \frac{16}{9} \left( x + 9 \right) - \frac{16}{3} \left( x + 1 \right) H_0 \right) H_1 + \frac{16}{9} \left( x + 1 \right) H_0^2 \right) \xi_2 + \left( \frac{80}{9} \left( x + 1 \right) H_1 \right) \\
&\quad - \frac{8}{27} \frac{235x^2 + 404x + 409}{x + 1} - \frac{256}{9} \frac{x^2 + 1}{x + 1} \xi_1 + \frac{815x^2 + 22x + 15}{9} H_0 \\
&\quad + \frac{4}{81} \frac{131x^2 + 178x + 131}{x + 1} \xi_4 - \frac{32}{3} \left( x + 1 \right) B_4 \right] + C_F T_F^2 \left[ -L_3^3 \frac{64}{27} \frac{1}{x + 1} \right] \\
&\quad - L_3^3 \frac{32}{27} \left( x + 1 \right) + L_3^2 \left[ -\frac{32}{27} \left( 11x - 1 \right) - \frac{32}{9} \left( x + 1 \right) H_0 \right] + L_3^2 \left[ \frac{32}{27} \left( 14x + 5 \right) \right] \\
&\quad + \frac{64}{9} \left( x + 1 \right) H_0 + \frac{32}{9} \left( x + 1 \right) H_1 \right] - \frac{992}{81} \left( x + 1 \right) + L_3 \left[ \frac{32}{81} \left( 187x + 16 \right) \right] \\
&\quad - \frac{64}{27} \left( 28x + 13 \right) H_0 - \frac{32}{3} \left( x + 1 \right) H_0^2 + \left( \frac{64}{27} \left( 14x + 5 \right) - \frac{64}{9} \left( x + 1 \right) H_0 \right) H_1 \\
&\quad - \frac{32}{9} \left( x + 1 \right) H_1^2 - \frac{64}{9} \left( x + 1 \right) H_0 + \frac{128}{9} \left( x + 1 \right) \xi_2 \right] + \frac{16}{729} \left( 431x + 323 \right) 
\end{align*}
\]
\[ + \frac{64}{81} (6x - 7) H_0 + \frac{16}{81} (11x - 1) H_0^2 + \frac{16}{81} (x + 1) H_0^3 - \frac{448}{27} (x + 1) \zeta_3 \]
\[ + C_F N_F T_F^2 \left[ - \left[ L_M^3 + 2 L_Q^3 \right] \frac{32}{27} (x + 1) + L_Q^2 \left[ \frac{64}{27} (14x + 5) + \frac{128}{9} (x + 1) H_0 \right] + \frac{64}{9} (x + 1) H_1 \right] + L_Q \left[ - \frac{64}{81} (187x + 16) - \frac{128}{27} (28x + 13) H_0 + \frac{64}{9} (x + 1) \left[ 4\zeta_2 - 2 H_{0,1} - H_1^2 \right] - \frac{64}{3} (x + 1) H_0^2 + \left( - \frac{128}{9} (x + 1) H_0 - \frac{128}{27} (14x + 5) \right) H_1 \right] + \frac{32}{81} (x + 1) H_0^3 \]
\[ + \frac{64}{729} (161x + 215) + \frac{128}{81} (6x - 7) H_0 + \frac{32}{81} (11x - 1) H_0^2 + \frac{256}{27} (x + 1) \zeta_3 \]
\[ + \hat{C}_{q,21}^{NS,(3)} (N_F) \right\}. \quad (2.61) \]

Again, we used the short hand notation \( H_d(x) \equiv H_d \) also here. The transformation of the Wilson coefficient to the \( \overline{\text{MS}} \) scheme for the heavy quark mass affects the massive OME at 3-loops and was given in Ref. [10]; the terms are the same in the unpolarized and polarized case.

The non-singlet contributions to the structure function \( g_2(x, Q^2) \) can be obtained via the Wandzura-Wilczek relation [15]

\[ g_2(x, Q^2) = - g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2), \quad (2.62) \]

where both structure functions refer to the twist-2 contributions. This relation is implied by a relation of the OMEs in the light-cone expansion, cf. [39]. The relation has also been proven in the covariant parton model in Refs. [40–42]. For gluonic initial states, it was derived in [43]. Eq. (2.62) also holds including target mass corrections [44,45] and finite light quark contributions [45]. Furthermore, it holds in non-forward [46] and diffractive scattering, including target mass corrections [47,48].

### 3 Numerical Results

In what follows, we will choose the factorization and renormalization scale \( \mu^2 = Q^2 \). We first study the behaviour of the massive and massless Wilson coefficients in the small and large \( x \) region and then give numerical illustrations in the whole \( x \)-region.

At small \( x \), the pure massive Wilson coefficient behaves like

\[ L_{q,21}^{h,NS} (N_F + 1) - \hat{C}_{q,21}^{NS,(3)} (N_F) \propto a_s^2 4 C_F T_F \ln^2(x) + a_s^2 \left[ \frac{16}{27} C_A C_F T_F - \frac{5}{9} C_F^2 T_F \right] \ln^4(x), \quad (3.1) \]

while in the region \( x \to 1 \) one obtains

\[ L_{q,21}^{h,NS} (N_F + 1) - \hat{C}_{q,21}^{NS,(3)} (N_F) \propto a_s^2 C_F T_F \frac{8}{3} \left( \frac{\ln^2(1-x)}{1-x} \right) + a_s^3 C_F^2 T_F \left[ 16 \ln^2 \left( \frac{Q^2}{m^2} \right) \right] \]
There is a term $\ln^3(1-x)/(1-x)$ at $O(\ln(Q^2/\mu^2))$, being of relevance for different choices of the factorization scale.

The above results can be compared with the case of the massless Wilson coefficient

$$\hat{C}_{q,g_1}^{NS,(2)}(N_F) \propto a_s^2 \frac{10}{3} C_F T_F \ln^2(x)$$

$$\hat{C}_{q,g_1}^{NS,(3)}(N_F) \propto a_s^3 \left[ \frac{92}{27} C_F C_A T_F - \frac{31}{9} C_F^2 T_F \right] \ln^4(x)$$

$$\hat{C}_{q,g_1}^{NS,(2)}(N_F) \propto a_s^2 \frac{8}{3} C_F T_F \left( \frac{\ln^2(1-x)}{1-x} \right)_+$$

$$\hat{C}_{q,g_1}^{NS,(3)}(N_F) \propto a_s^3 \frac{80}{9} C_F^2 T_F \left( \frac{\ln^4(1-x)}{1-x} \right)_+ .$$

The small $x$ behaviour can be compared with leading order predictions for the non-singlet evolution kernel in Refs. [49,50]. Indeed both the massive and massless contributions follow the principle pattern $\sim c_k \alpha_{s,k+1}^3 \ln^{2k}(x)$. However, as is well known [49], less singular terms widely cancel the numerical effect of these leading terms. For the large $x$ terms the massless terms exhibit a stronger soft singularity than the massive ones.

In the following numerical illustrations we use the polarized parton distributions of Ref. [8], which are of next-to-leading order (NLO), since no next-to-next-to-leading order (NNLO) data analysis based on the anomalous dimensions calculated in Ref. [51] has been performed yet. The values of $\alpha_s$ correspond to those of the unpolarized NNLO analysis [52]. The heavy and light flavor Wilson coefficients being discussed in the following are given in Eqs. (2.2) and (2.3).

In Figure 1, the 2- and 3-loop heavy flavor corrections to the non-singlet term of the structure function $xg_1(x,Q^2)$ are calculated in the case of charm, assuming $m_c = 1.59$ GeV [53], using the formula for the Wilson coefficient Eq. (2.61), and setting $\mu^2 = Q^2$. With growing $Q^2$, the distribution diminishes at larger values of $x$ and grows towards medium values. The $O(\alpha_s^3)$ corrections lead to stronger effects if compared to those at $O(\alpha_s^2)$. We have applied the asymptotic Wilson coefficients for all the $Q^2$ values given here, which only holds for values $Q^2/m^2 \gtrsim 10$. For the heavy quark distributions we formally show also the result at $Q^2 = 4$ GeV$^2$, outside this region, indicated by dotted ($O(\alpha_s^2)$) and dash-dotted lines ($O(\alpha_s^3)$).

Figure 2 shows the effect of the Wilson coefficients comparing the contributions from $O(\alpha_s^0)$ to $O(\alpha_s^3)$ at $Q^2 = 4$ GeV$^2$ as an example, where a depletion is obtained with growing order. The 3-loop light flavor contributions to $xg_1(x,Q^2)$ ($N_F = 3$) are illustrated in Figure 3. Here the evolution is strengthened by growing $Q^2$ in the large $x$ region and depleted for lower values of $Q^2$, considering only the effects due to the Wilson coefficient.

In Figures 4 and 5 we illustrate the ratio of the flavor non-singlet charm corrections to those by the light quarks given in Eq. (2.3) up to $O(\alpha_s^2)$ and $O(\alpha_s^3)$, respectively. At $O(\alpha_s^2)$ the effect is of $O(1\%)$ and below, for the lower scales $Q^2$, but higher values are obtained for very large scales as $Q^2 \simeq 1000$ GeV$^2$ in the region $x \sim 0.003$. A qualitatively similar picture is obtained including the $O(\alpha_s^3)$ corrections. The effect on the ratio $g_1^{\text{heavy}}/g_1^{\text{light}}|_{NS}$ is about doubled. To resolve relative effects of $O(2\%)$ requires higher luminosities than available in present day experiments. They may become available in the planned experiments at a future EIC [54].

Figure 6 shows the 2- and 3-loop charm flavor non-singlet contributions to the structure function $xg_2(x,Q^2)$ according to the Wandzura-Wilczek relation (2.62) implying the oscillatory
behaviour. In size these effects are comparable to those of the structure function $xg_1(x,Q^2)$ shown in Figure 1. With growing $Q^2$ the effects become somewhat smaller. In Figure 7 we show the corresponding massless contributions to the structure function $g_2(x,Q^2)$ at $Q^2 = 4$ GeV$^2$ for the different orders in $a_s$, which slightly diminish adding higher order contributions. Taking into account the $O(a_s^3)$ corrections, the light flavor corrections to $g_2(x,Q^2)$ (1.5,2,62) grow somewhat in size with larger values of $Q^2$, see Figure 8. Similar to the case of the structure function $xg_1$ the $O(a_s^3)$ charm flavor non-singlet corrections to the structure function $xg_2(x,Q^2)$ amount to $O(1\%)$.

4 The Bjorken Sum Rule

The polarized Bjorken sum rule [55] refers to the first moment of the flavor non-singlet combination

$$\int_0^1 dx \left[ g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2) \right] = \frac{1}{6} \left. \frac{g_A}{g_V} \right| C_{BJ}(\hat{a}_s),$$

(4.1)

with $g_{A,V}$ the neutron decay constants, $g_A/g_V \approx -1.2767 \pm 0.0016$ [56] and $\hat{a}_s = a_s/\pi$. The 1- [57], 2- [58], 3- [30] and 4-loop QCD corrections [31] in the massless case are given by

$$C_{BJ}(\hat{a}_s) = 1 - \hat{a}_s + \hat{a}_s^2(-4.58333 + 0.33333 N_F) + \hat{a}_s^3(-41.4399 + 7.60729 N_F - 0.17747 N_F^2) + \hat{a}_s^4(-479.448 + 123.472 N_F - 7.69747 N_F^2 + 0.10374 N_F^3),$$

(4.2)

choosing the renormalization scale $\mu^2 = Q^2$, cf. [28] for $SU(3)_c$. Here $N_F$ denotes the number of active light flavors. The expression for general color factors was given in Ref. [31].

For the asymptotic massive corrections (2.2) only the first moments of the massless Wilson coefficients $C_{\tilde{q},\tilde{a}_i}^{(2,3),NS}(N_F)$ contribute, since the first moments of the massive non-singlet OMEs vanish due to fermion number conservation, a property holding even at higher order. Therefore, any new heavy quark changes Eq. (4.2) by a shift in $N_F \rightarrow N_F + 1$ only, for the asymptotic corrections. Different results are obtained in the tagged flavor case [5, 7] at $O(a_s^3)$, where no inclusive structure functions are considered. Corresponding power corrections were derived in [59, 60].

5 Conclusions

We calculated the heavy flavor non-singlet Wilson coefficients of the polarized inclusive structure function $g_1(x,Q^2)$ to $O(a_s^3)$ in the asymptotic region $Q^2 \gg m^2$. The first contributions of this kind are of $O(a_s^3)$. In the case of twist-2 operators the corresponding contributions to the structure function $g_2(x,Q^2)$ can be obtained using the Wandzura-Wilczek relation (2.62) [15], cf. [39–42, 45]. The asymptotic Wilson coefficient is obtained by using the factorization formula [5], Eq. (2.2), based on the massive OME [10] and the massless Wilson coefficient [29] to 3-loop order. The heavy flavor Wilson coefficient can be thoroughly represented by nested harmonic sums in Mellin-$N$ space and by harmonic polylogarithms in $x$-space. We presented numerical results corresponding to the charge weighted polarized parton contributions $\alpha \Delta f(x,Q^2) + \Delta \tilde{f}(x,Q^2)$, cf. (1.5), referring to the polarized parton distribution functions at NLO [8] for an illustration. Comparing with the corresponding massless cases the heavy flavor corrections in case of charm are of $O(1 − 2\%)$, requiring high luminosity experiments to be resolved, which are planned for the future electron-ion collider EIC [54]. We also considered the contribution
of the asymptotic Wilson coefficient to the polarized Bjorken sum-rule. Due to fermion number conservation for the massive flavor non-singlet OME in all orders in $\alpha_s$, only the first moment of the massless Wilson coefficient contributes and the effect of each heavy flavor results in a shift of $N_F$ by one unit in the expression for the massless polarized Bjorken sum-rule. The results of the present calculation could be easily applied to derive the asymptotic heavy flavor corrections to the neutral current structure function $xG_3$, [61]. However, the corresponding massless Wilson coefficient to 3-loop order has not been calculated yet.

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Figure 1: The 2- and 3-loop non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme for $m_c = 1.59$ GeV. Here we used the value of $\alpha_s(M_Z^2) = 0.1132$ and the NLO parton distribution [8] as reference. Figures 2–8 below are calculated using the same setting.

Figure 2: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ at $Q^2 = 4$ GeV$^2$ illustrating the contributions for the different orders in $\alpha_s$. 
Figure 3: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ at $O(a_s^3)$ for different values of $Q^2$.

Figure 4: The ratio $g_1^{\text{charm}} / g_1^{\text{light}}|_{\text{NS}}$ in the non-singlet case at $O(a_s^3)$ for different values of $Q^2$. 

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Figure 5: The ratio $g_1^{\text{charm}} / g_1^{\text{light}}|_\text{NS}$ in the non-singlet case at $O(a_s^2)$ for different values of $Q^2$.

Figure 6: The 2- and 3-loop non-singlet charm contributions to the twist 2 contributions of the structure function $xg_2(x, Q^2)$ by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme.
Figure 7: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_2(x, Q^2)$ at $Q^2 = 4$ GeV$^2$ illustrating the contributions for the different orders in $a_s$.

Figure 8: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_2(x, Q^2)$ at $O(a_s^3)$ for different values of $Q^2$. 
References


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J. Blümlein, M. Klein, T. Naumann and T. Riemann, Structure Functions, Quark Distributions and Λ_{QCD} at Hera, PHE-88-01.