Integrability and Exact results in $\mathcal{N} = 2$ gauge theories

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DESY Theory

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work in progress
Motivation: Can we go beyond perturbation theory?

Yes! If we add more symmetry to the problem: SUSY

The most symmetric gauge theory in 4D is $\mathcal{N} = 4$ super Yang Mills.

**Noether:** 

**Symmetry** $\rightarrow$ **Conservation law**

*The more symmetry the easier it is to solve the problem.*

- Yes! If we add **more symmetry** to the problem: SUSY
Motivation: The success story for $\mathcal{N} = 4$ SYM

Possible to compute observables in the strong coupling regime and in some cases to even obtain **Exact results** (for any value of the coupling).

- **AdS/CFT** (gravity/sigma model description)

- **Integrability** (The spectral problem is solved) *at large* $N_c$

- **Localization** (Exact results: e.x. Circular WL) for *any* $N_c$

Which of these properties/techniques are transferable to **more realistic gauge theories** in 4D with less SUSY?
∀ conformal $\mathcal{N} = 2$ gauge theory there is a purely gluonic subset of local operators $SU(2,1|2)$ integrable in the planar limit

$$\gamma_{\mathcal{N}=2}(g) = \gamma_{\mathcal{N}=4}(g)$$

The exact effective coupling (relative finite renormalization of $g$)

$$g^2 = f(g^2) = g^2 + g^2 (Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4})$$

we compute using localization

$$W_{\mathcal{N}=2}(g^2) = W_{\mathcal{N}=4}(g^2)$$
The $\mathcal{N} = 4$ vector multiplet in the **adjoint** of $SU(N)$:

- the **gluon** and its **SUSY partners**:
  - $\mathcal{N} = 2$ vector multiplet **adjoint** in $SU(N)$:
    
    \[
    \begin{align*}
    &A_\mu, \quad \lambda_1^\alpha, \quad \lambda_2^\alpha, \\
    &\phi_1, \quad \lambda_3^\alpha, \quad \lambda_4^\alpha, \\
    &I = 1, 2
    \end{align*}
    \]

- $\mathcal{N} = 2$ hyper multiplet in the **adjoint** of $SU(N)$:
  
  \[
  \begin{align*}
  &\phi_2, \quad \phi_3, \\
  &\Phi^I = \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix}
  \end{align*}
  \]

It is conformal $\beta = 0$ and has an **exactly marginal coupling**!

$\mathcal{N} = 4$ SYM has no quarks!
$\mathcal{N} = 2$ SuperConformal QCD (SCQCD)

- $\mathcal{N} = 2$ vector multiplet **adjoint** in $SU(N)$:

\[ A_\mu \lambda^1_\alpha \phi^2_\alpha, \quad \lambda^\mathcal{I} = \begin{pmatrix} \lambda^1_1 \\ \lambda^2_2 \end{pmatrix}, \quad \mathcal{I} = 1, 2 \]

- $\mathcal{N} = 2$ hyper multiplet **fundamental** in $SU(N)$ and $U(N_f)$:

\[ \psi^i_\alpha q_i, \quad (\tilde{q})^*_i, \quad Q^\mathcal{I} = \begin{pmatrix} q_i \\ \tilde{q}^*_i \end{pmatrix}, \quad i = 1, \ldots N_f \]

When $N_f = 2N$: $\beta = \frac{g^3_{YM}}{16\pi^2} (N_f - 2N) = 0$, **exactly marginal coupling**!
\( \mathcal{N} = 2 \) SuperConformal QCD (SCQCD)

- \( \mathcal{N} = 2 \) vector multiplet **adjoint** in \( SU(N) \):

\[
\begin{align*}
\chi_1^1 & \quad A_\mu \\
\phi & \quad \chi_2^2 \\
\tau^I & = \begin{pmatrix} \chi_1^1 \\ \chi_2^2 \end{pmatrix}, \quad I = 1, 2
\end{align*}
\]

- \( \mathcal{N} = 2 \) hyper multiplet **fundamental** in \( SU(N) \) and \( U(N_f) \):

\[
\begin{align*}
\psi_\alpha i & \\
q_i & \quad (\tilde{q})^*_i \\
Q^I & = \begin{pmatrix} q \\ \tilde{q}^* \end{pmatrix}, \quad i = 1, \ldots N_f
\end{align*}
\]

When \( N_f = 2N \):

\[
\beta = \frac{g_{YM}^3}{16\pi^2} (N_f - 2N) = 0, \text{ exactly marginal coupling!}
\]

\( \mathcal{N} = 2 \) SCFT with \( SU(N) \times SU(N) \) gauge group: two exactly marginal \( g \) and \( \tilde{g} \):

- For \( g = \tilde{g} \) we get the \( \mathcal{N} = 4 \) result (for the observables we consider)

- In the limit \( \tilde{g} \to 0 \) obtain \( \mathcal{N} = 2 \) SCQCD with \( N_f = 2N \)
Integrability of the purely gluonic $SU(2,1|2)$ Sector
$\mathcal{N} = 4$ SYM is integrable in the planar limit for any coupling

- **Perturbation theory**: mapped to an integrable spin chain
- **Strong coupling**: integrable 2D theory on the string world-sheet

**Powerful integrability toolkit**

- The spectral problem is solved **exactly**: for any coupling

Integrability now is applied to **other observables**.
The only possible way to make diagrams with external fields in the vector mult. different from the $\mathcal{N} = 4$ ones is to make a loop with hyper’s and then in this loop let a checked vector propagate!

(EP-Christoph Sieg)

The same with $\mathcal{N} = 4$ SYM

Different from $\mathcal{N} = 4$ SYM but finite!!
The only possible way to make diagrams with external fields in the vector mult. different from the $\mathcal{N} = 4$ ones is to make a loop with hyper’s and then in this loop let a checked vector propagate! (EP-Christoph Sieg)

The same with $\mathcal{N} = 4$ SYM

Different from $\mathcal{N} = 4$ SYM but finite !!

Novel Regularization prescription:

For every individual $\mathcal{N} = 2$ diagram subtract its $\mathcal{N} = 4$ counterpart.
\[ H_{N=2}^{(3)}(\lambda) - H_{N=4}^{(3)}(\lambda) \sim H_{N=4}^{(1)}(\lambda) \quad \Rightarrow \quad H_{N=2}^{(3)}(\lambda) = H_{N=4}^{(3)}(f(\lambda)) \]

with \( f(\lambda) = \lambda + c\lambda^3 \)
Operator renormalization in the Background Field Gauge

Background Field Method: \( \varphi \rightarrow A + Q \)

where \( A \) the classical background and \( Q \) the quantum fluctuation

\[
g^\text{bare} = Z_g g^\text{ren}, \quad A^\text{bare} = \sqrt{Z_A} A^\text{ren}, \quad Q^\text{bare} = \sqrt{Z_Q} Q^\text{ren}, \quad \xi^\text{bare} = Z_\xi \xi^\text{ren}
\]

In the Background Field Gauge \( Z_g \sqrt{Z_A} = 1 \) and \( Z_Q = Z_\xi \)
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In the Background Field Gauge \( Z_g \sqrt{Z_A} = 1 \) and \( Z_Q = Z_\xi \)

- Compute \( \langle O(y)A(x_1) \cdots A(x_L) \rangle \) for \( O \sim \text{tr} (\varphi^L) \).

Wick contact \( O_{i}^{\text{ren}} (Q_{\text{ren}}, A_{\text{ren}}) = \sum_j Z_{ij} O_{j}^{\text{bare}} \left( Z_Q^{1/2} Q, Z_A^{1/2} A \right) \)
Background Field Method: No $Q$'s outside, no $A$'s inside!

\[ A(x_1)A(x_m) \quad A(x_{m+1})A(x_L) \]

1. $\langle QQAA \rangle$ renormalize as $Z_Q^{2/2} Z_A^{2/2} \langle QQAA \rangle$

2. The $Q$ propagators as $Z_Q^{-1}$

3. the $O^{ren}$ has two more $Z_Q^{1/2}$

4. all $Z_Q$ will cancel (We knew it - gauge invariance!)

5. Only $Z = Z_g^2 = Z_A^{-1}$, the combinatorics the same as in $\mathcal{N} = 4$:

\[
H_{\mathcal{N}=2} (g) = H_{\mathcal{N}=4} (g) \quad \text{with} \quad g^2 = f(g^2, \bar{g}^2) = g^2 + g^2 (Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4})
\]
New vertices cannot contribute

\[ \Gamma = \Gamma_{\text{ren. tree}} + \Gamma_{\text{new}} \]

- \( \Gamma_{\text{ren. tree}} \): vertex and self-energy renormalization
  - all encoded in \( \delta Z = Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4} \)

- New vertices cannot contribute due to the non-renormalization theorem (Fiamberti, Santambrogio, Sieg, Zanon)
Localization and Exact Effective couplings
\[ Z_{S^4} = \int [D\Phi] e^{-S[\Phi]} = \int da |Z(a)|^2 \]

The **path integral** localizes to an **ordinary integral**

*(Cancelations due to supersymmetry)*

We can do an ordinary integral.
Compute the path integral exactly.
For **any value of the coupling constant**.

(Pestun)
\( \mathcal{N} = 4 \) SYM:

\[
W_{\mathcal{N}=4}(g) = \frac{l_1(4\pi g)}{2\pi g}
\]

\( \mathcal{N} = 2 \) theories:

\[
W_{\mathcal{N}=2}(g, \check{g}) = W_{\mathcal{N}=4}(f(g, \check{g}))
\]

\[
f(g, \check{g}) = \begin{cases} 
  g^2 + 2(\check{g}^2 - g^2) \left[ 6\zeta(3)g^4 - 20\zeta(5)g^4 (\check{g}^2 + 3g^2) \right] + O(g^{10}) \\
  \frac{2g\check{g}}{g + \check{g}} + O(1)
\end{cases}
\]

- Checked with Feynman diagrams calculation (up to 4-loops)

- Agrees with AdS/CFT (strong coupling)
Conclusions

- ∀ observable in the purely gluonic $SU(2, 1|2)$ sector
take the $\mathcal{N} = 4$ answer and replace $g^2 \rightarrow \mathbf{g}^2 = f(g^2)$

We need more checks!! (EP-Mitev), (Leoni-Mauri-Santambrogio) and (Fraser)

**Lesson**: Think of the $\mathcal{N} = 4$ SYM as a regulator!!

The integrable $\mathcal{N} = 4$ model knows all about the combinatorics.

For $\mathcal{N} = 2$: relative finite renormalization encoded in $\mathbf{g}^2 = f(g^2)$.

- In asymptotically conformal $\mathcal{N} = 2$ theories and $\mathcal{N} = 1$ SCFTs
all loop statement: purely gluonic $SU(2, 1|1)$ sector (EP-Roček)