LPENSL-TH-01/11
DESY 10-253

Long-time and large-distance asymptotic behavior of the current-current correlators in the non-linear Schrödinger model

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January 6, 2011

arXiv:1101.0844v1 [math-ph] 4 Jan 2011

Abstract

We present a new method allowing us to derive the long-time and large-distance asymptotic behavior of the correlations functions of quantum integrable models from their exact representations. Starting from the form factor expansion of the correlation functions in finite volume, we explain how to reduce the complexity of the computation in the so-called interacting integrable models to the one appearing in free fermion equivalent models. We apply our method to the time-dependent zero-temperature current-current correlation function in the non-linear Schrödinger model and compute the first few terms in its asymptotic expansion. Our result goes beyond the conformal field theory based predictions: in the time-dependent case, other types of excitations than the ones on the Fermi surface contribute to the leading orders of the asymptotics.

1 Introduction

The successful resolution of many-body quantum integrable models through Bethe Ansatz [4, 78, 83, 68] naturally raised the question of the computation of their correlation functions. In its full generality, this problem appeared first as extremely complex due to the complicated combinatorial representation of the eigenfunctions. However, for the particular values of the coupling constants for which the models are equivalent to free fermionic ones, the use of specific methods (such as the Jordan-Wigner transformation) led to numerous explicit results [66, 64, 65, 75, 70, 3, 76, 88, 74, 85, 40, 73, 72, 57, 33, 32, 34, 17, 35, 18, 19]. In this context, different types of effective representations have been obtained for the correlation functions, notably in terms of determinants. It has been possible in particular to express two-point functions in terms of Fredholm determinants of compact operators $I + V$, where the integral kernel V belongs to the class of integrable integral operators [64, 65, 57, 17, 16]. These determinants representations have been used to compute the asymptotic behavior of the correlations functions at large distances [40, 71, 33, 32, 34, 35, 18].

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It was the understanding of the underlying algebraic structures in truly interacting integrable models which finally opened the way to the computation of their correlation functions. In particular, the algebraic version of the Bethe Ansatz (ABA) [24] enabled Izergin, Korepin and their collaborators in Leningrad to obtain the first exact representations for the correlation functions of the XXZ and the non-linear Schrödinger models away from their free fermion points (see [56] and references therein). These representations, based on finite-size determinant representations for the norms of Bethe eigenstates [25, 26, 55] and for more general scalar products [82, 56], were however not completely explicit due to the introduction of dual fields to handle the combinatorial difficulties of the ABA approach. The first explicit representations (multiple integral representations for elementary building blocks of the XXZ chain at zero temperature) were obtained in the 90' by Jimbo, Miki, Miwa and Nakayashiki through q-vertex operators and solutions of the q-KZ equation [37, 39, 38]. A few years later, these representations, together with their generalization to the case of an external magnetic field, were reproduced by the Lyon group in the ABA framework thanks to the resolution of the so-called quantum inverse scattering problem [53, 54]. The Lyon group was also able to provide finite-size determinant representations for the form factors of the finite XXZ chain [53]. Developments of the latter approach led to integral representations for the two-point function [50, 52], to their generalizations in the temperature case [27, 28], in the time-dependent case [51], and in all these cases taken together [80]. On the other hand, new interesting algebraic representations for the correlation functions were also obtained recently using reduced q-KZ equations [9, 10, 11], which led to the discovery of a hidden Grassmann structure in the XXZ model [12, 8, 13, 41, 6, 7].

In this framework, one of the most challenging problem is the derivation, from these exact representations, of the long-distance asymptotic behavior of the correlation functions. In general, zero-temperature correlation functions of models with a gapless spectrum are expected to decay algebraically at large distances. The exponents governing this power-law behavior, called critical exponents, are believed to be universal, *i.e.* not depending on the microscopic details of the model, but only on its overall symmetries. Predictions concerning these exponents were obtained via the Luttinger liquid approach [69, 29, 30, 31] or the connection to Conformal Field Theory [79, 14, 1, 15, 5], together with exact computations of finite size corrections to the spectrum [20, 86, 21, 87, 22]. However, until very recently, it was not possible to confirm these (indirect) predictions via some direct computations based on exact results for an integrable model out of its free fermion point.

To tackle this problem, one may notice that, in the ABA approach, the correlation functions can be written in terms of series of multiple integrals that can be though of as some multidimensional generalizations of the Fredholm determinants which appear in the study of the free fermionic models. One may therefore attempt to extend and adapt the large arsenal of techniques available for free fermionic correlators to models away from their free fermion points. The main obstacle here comes from the highly coupled nature of the various series of multiple integrals representations for the correlation functions. Actually, until two years ago, there were only few cases out of the free fermion point where the analysis could have been performed to some extend [48, 49, 61]. A major breakthrough was achieved in [45], which proposed a systematic method to perform the asymptotic analysis of the long-distance asymptotic behavior of the correlation functions of densities in integrable models described by a six-vertex type R-matrix and solvable by the ABA. There, the idea was to extract the asymptotic behavior of the series of multiple integrals representing the correlator by making some connections with the asymptotes of Fredholm determinants. Although fully functional, the method involved several subtle steps, and in particular a quite complex summation process. The final result itself, nevertheless, happened to be quite simply given in terms of a properly normalized form factor [45, 46]. This fact naturally suggests that the method proposed in [45] might be simplified if one works directly at the level of the form factor expansion of the correlation functions.

It is the aim of the present article to expound such a simpler and more direct approach, based on the form factor expansion of the correlation functions in the ABA framework, on the finite-size determinant representation for the form factors in finite volume and on some of their main features in the thermodynamic limit. The main advantage compared to [45] is that it relies on the study of more natural physical objects: whereas in [45] the correlation function was written as a series over some elementary objects expressed in terms of bare quantities (such as energy, momentum), the so-called cycle integrals, here we chose to pursue our study of form factors a step further, and to use some of their thermodynamic properties (and in particular their representation in terms of dressed physical quantities) from the very beginning. This results into a major simplification of the summation process. Indeed, in [45], the asymptotic summation of the series of cycle integrals was quite subtle. In particular, it involved a summation of a whole class of corrections, sub-leading to each term of the series, but nevertheless contributing to the leading term in the final result: they happened to be responsible, once upon summation, for the dressing of bare physical quantities. Since in our present approach the series over form factors is already written in terms of the dressed quantities, the summation is much simpler: the series can be connected to a Fredholm determinant and *the leading asymptotic behavior of the correlation function is directly given by the leading asymptotic behavior of this Fredholm determinant*. No subtle identification of sub-leading terms is needed for the result.

Let us be slightly more precise about the different steps of our method. Its starting point is the form factor expansion of the correlation function in finite volume. Using the determinant representation of the form factors in finite volume [81, 53], one can analyze their thermodynamic limit in the particle/hole picture [82, 46, 43], and decompose their finite-size expression into some (universal) "singular" or "discrete" part, which is essentially free-fermion-like and contains the whole non-trivial scaling behavior of the form factor in the thermodynamic limit, and some (model-dependent) "regular" part, which admits a smooth thermodynamic limit [43]. The idea is to reduce the summation over form factors to a summation over their finite-size discrete parts, the (thermodynamic limit of the) regular parts being treated as some dressing functionals. The main problem is that, for interacting models, the series we consider is highly coupled. The origin of this coupling is twofold: on the one hand, the regular part is a complicated function of the particle/hole Bethe roots; on the other hand, the discrete part can be seen as a certain functional of the shift function between the considered excited state and the ground state, and hence introduces non-trivial couplings between the particle and hole-type Bethe roots. It is possible to slightly simplify the structure of the different terms by neglecting, on the level of the finite-size series, some contributions that should vanish in the thermodynamic limit. We thus obtain an effective (finite-size) form factor series with a simpler particle/hole dependence than in the original series. This series can be linked to a decoupled one, reduced to the sum of the discrete parts of the form factors with a shift function that does not depend on the excited state, exactly as in a (generalized) free fermionic model. The sum can then be computed and recast in terms of a determinant which tends to a Fredholm determinant in the thermodynamic limit. The leading asymptotic behavior of the correlation function then follows directly from the leading asymptotic behavior of the Fredholm determinant [60].

We stress that, within this method, the computation of the asymptotic expansion of the correlation functions for "truly interacting" models and for their free fermionic limits carry almost the same complexity, in as much as we offer a way of understanding the interacting case as a smooth deformation of the free fermionic one. This joins in some sense the spirit of the works [12, 8, 13, 41, 6, 7] on the hidden Grassmann structure in the XXZ model with the main difference that, in our method, the free fermionic structure is only used as an intermediate step so as to carry out certain algebraic manipulations in a simpler way and deform the range of summations to a kind of analog of the steepest descent contour.

We choose to expose our method in the framework of the quantum non-linear Schrödinger model, which is probably the best understood and simplest integrable interacting model. Note however that, with minor modifications specific to the structure of the model under investigation (more complex structure of the configuration of the pertinent Bethe roots above the ground state, the appearance of trigonometric functions ...), this method is in principle applicable to a very wide class of integrable models whose form factors are known and admit a finite-size determinant representation such as those obtained in [81, 53, 77]. We apply our method to the derivation of the first few terms in the long-distance and large-time asymptotic expansion of the correlation function of currents at zero temperature (see Section 3). The case of the reduced density matrix, together with some more precise mathematical details about the method, will be considered in a further publication [59]. We will see that *the asymptotes in the time-dependent case are not only issuing from excitations in the vicinities of the Fermi boundaries* (which correspond to the region of the spectrum taken into account by the CFT/Luttinger liquid approach), *but also from the excitations around the saddle-point λ_0 of the "plane-wave" combination $xp(\lambda) - t\varepsilon(\lambda)$, where p and ε are respectively the dressed momentum and energy of the excitations*. More precisely, the exponents governing the power-law decay of the correlation function are given in terms of the different shift functions between the ground state and excited states having one particle and one hole either on the Fermi boundaries $\pm q$ or at the saddle-point λ_0. The associated amplitudes are given by the corresponding, properly normalized, form factors of currents.

The paper is organized as follows. In Section 2, we introduce the non-linear Schrödinger (NLS) model and the different notations, relative to the description of the space of states and the thermodynamic limit of the model, that will be used throughout the article. This enables us, in Section 3, to present our main result: the leading large-distance and long-time asymptotic behavior of the (zero-temperature) correlation function of currents. The remaining part of the article is devoted to the derivation of this result. We establish in Section 4 a (finite-size) form factor series representation for the correlation function (or more precisely for its generating function). In Section 5, we explain how to relate our highly coupled series to a decoupled, free-fermion type series that can be summed up into a finite-size determinant. Finally, in Section 6, we take the thermodynamic limit of the previous result, hence representing the form factor series in terms of a Fredholm determinant, and explain how to derive the first leading terms of the long-time/large-distance asymptotic behavior of our series from those of the Fredholm determinant. Technical details are gathered in a set of Appendices: explicit representations for form factors are given in Appendix A; the summation of the free-fermion type series is performed in Appendix B; the notion of functional translation operator, which is used as a technical tool to link our series to a decoupled one, is introduced in Appendix C; in Appendix D, we show how to relate the summed-up form factor series to the type of series that was obtained in [45] by expanding the master equation; finally, in Appendix E, we explain how to control sub-leading corrections in the asymptotic series using the so-called Natte series introduced in [60].

2 The NLS model: notations and definitions

The quantum non-linear Schrödinger model is described in terms of quantum Bose fields $\Psi(x,t)$ and $\Psi^{\dagger}(x,t)$ obeying to canonical equal-time commutation relations, with Hamiltonian

$$H = \int_0^L \Big\{ \partial_x \Psi^{\dagger}\, \partial_x \Psi + c \Psi^{\dagger}\Psi^{\dagger}\Psi\Psi - h\Psi^{\dagger}\Psi \Big\}\, \mathrm{d}x. \tag{2.1}$$

The model is defined on a finite interval of length L (we take the thermodynamic limit $L \to \infty$ later on) and is subject to periodic boundary conditions. In (2.1), c denotes the coupling constant and h the chemical potential. In this article, we focus on the repulsive regime $c > 0$ in presence of a positive chemical potential $h > 0$.

Due to the conservation of the number of particles, this quantum field theory model is equivalent to a (quantum mechanical) one-dimensional many-body gas of bosons subject to point-like interactions. Such a model was first introduced and solved by Lieb and Liniger in 1963 [68, 67], where it was used as a test for Bogoliubov's theory. It was then thoroughly investigated. In particular, in 1982, Izergin and Korepin [36] (see also [56]) proposed a lattice regularization of the model allowing to implement the ABA scheme [24] in finite volume.

In this article, we consider the zero-temperature time-dependent correlation function of currents, *i.e.* the ground state expectation value $\langle\, j(x,t)\, j(0,0)\,\rangle$, where $j(x,t) = \Psi^{\dagger}(x,t)\,\Psi(x,t)$. Our aim is to evaluate, in the thermodynamic limit $L \to \infty$ of the model, its large-distance and long-time asymptotic behavior. Prior to writing down our result, we introduce some necessary notations. Namely, we describe the space of states of the model and provide definitions of several thermodynamic quantities which appear in our result.

In the algebraic Bethe ansatz framework, the eigenstates $|\,\psi(\{\lambda\})\,\rangle$ of the Hamiltonian (2.1) are parametrized by a set of spectral parameters $\lambda_1, \dots, \lambda_N$ solution to the system of Bethe equations. In their logarithmic form, these Bethe equations read

$$L p_0(\lambda_j) + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j, \qquad j = 1, \dots, N, \tag{2.2}$$

in terms of a set of integer (for N odd) or half-integer (for N even) numbers $n_1, \dots, n_N$. The bare momentum $p_0(\lambda)$ and the bare phase $\theta(\lambda)$ appearing above are given by

$$p_0(\lambda) = \lambda, \qquad \theta(\lambda) = i \log\Big(\frac{ic + \lambda}{ic - \lambda}\Big). \tag{2.3}$$

The energy of the state $|\,\psi(\{\lambda\})\,\rangle$ is

$$E(\{\lambda\}) = \sum_{j=1}^{N} \varepsilon_0(\lambda_j), \qquad \text{with} \qquad \varepsilon_0(\lambda) = \lambda^2 - h. \tag{2.4}$$

The solutions to the system of Bethe equations have been studied by Yang and Yang [89]. In particular, it has been proven that all solutions are sets of real numbers and that a set of solutions $\{\lambda_j\}$ can be uniquely parametrized by a set of (half-)integers $\{n_j\}$.

In the following, λ_j, $j = 1, \dots, N$, will denote the Bethe roots describing the ground state of the Hamiltonian (2.1). They correspond to the solution to (2.2) associated with the following choice of (half)-integers: $n_j = j - (N+1)/2$, where the number of particles N is fixed by the value of the chemical potential h. In the thermodynamic limit ($L \to \infty$, $N \to \infty$ with N/L tending to some finite average density D), the parameters λ_j densely fill a symmetric interval $[-q, q]$ of the real axis (the Fermi zone) with some density distribution $\rho(\lambda)$ solving the following linear integral equation

$$\rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \rho(\mu)\, \mathrm{d}\mu = \frac{p_0'(\lambda)}{2\pi}, \tag{2.5}$$

where K denotes the Lieb kernel

$$K(\lambda) = \theta'(\lambda) = \frac{2c}{\lambda^2 + c^2}. \tag{2.6}$$

Excited states of (2.1) correspond to other solutions of the Bethe equations. For technical purpose, it is actually convenient to consider the *twisted Bethe states* $|\, \psi_\kappa(\{\mu\})\, \rangle$ instead. These are parameterized by solutions $\mu_{\ell_1}, \ldots, \mu_{\ell_{N_\kappa}}$ of the twisted Bethe equations,

$$L p_0(\mu_{\ell_j}) + \sum_{k=1}^{N_\kappa} \theta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi\Big(\ell_j - \frac{N_\kappa + 1}{2}\Big) + i\beta, \qquad j = 1, \ldots, N_\kappa. \tag{2.7}$$

Here β is some purely imaginary parameter, $\kappa = e^\beta$, and $\ell_1 < \ell_2 < \cdots < \ell_{N_\kappa}$ are some integers. Note that, the set of roots of (2.7) being completely defined by the set of integers ℓ_j, we have chosen to label them accordingly. Eigenstates of the Hamiltonian (2.1) are obtained as limits when $\kappa \to 1$ of the twisted Bethe states $|\, \psi_\kappa(\{\mu\})\, \rangle$.

For purely imaginary β, the roots of (2.7) are real. The state parameterized by the solutions of (2.7) with $\ell_j = j$, $j = 1, \ldots, N_\kappa$, is called the κ-twisted ground state in the N_κ-sector. In the thermodynamic limit, excitations above this κ-twisted ground state correspond to solutions such that most of the ℓ_j's coincide with their value for the ground state: $\ell_j = j$ except for n integers (with n remaining finite in the $N_\kappa \to \infty$ limit) $h_1, \ldots, h_n \in \{1, \ldots, N_\kappa\}$ for which $\ell_{h_k} = p_k \notin \{1, \ldots, N_\kappa\}$. The integers h_k represent "holes" with respect to the distributions of integers for the ground state in the N_κ-sector, whereas the integers p_k represent "particles". For such an excited state $|\, \psi_\kappa(\{\mu_{\ell_j}\})\, \rangle$, the counting function

$$\widehat{\xi}_\kappa(\omega) \equiv \widehat{\xi}_\kappa(\omega \,|\, \{\mu_{\ell_j}\}) = \frac{1}{2\pi} p_0(\omega) + \frac{1}{2\pi L} \sum_{k=1}^{N_\kappa} \theta(\omega - \mu_{\ell_k}) + \frac{N_\kappa + 1}{2L} - \frac{i\beta}{2\pi L}, \tag{2.8}$$

which is monotonously increasing, defines unambiguously a set of real "background" parameters μ_j, $j \in \mathbb{Z}$, as the unique solutions to $\widehat{\xi}_\kappa(\mu_j) = j/L$. Among this set of parameters, the N_κ solutions of the twisted Bethe equations with integers ℓ_j correspond to the solutions $\widehat{\xi}_\kappa(\mu_{\ell_j}) = \ell_j/L$, $j = 1, \ldots, N_\kappa$, and the particle rapidities μ_{p_a} (respectively the hole rapidities μ_{h_a}), $a = 1, \ldots, n$, correspond to the solutions $\widehat{\xi}_\kappa(\mu_{p_a}) = p_a/L$ (respectively $\widehat{\xi}_\kappa(\mu_{h_a}) = h_a/L$).

Due to the particular type of correlation function we consider in this article (current-current correlation function), we will restrict our study to states having the same number of parameters as the ground state, *i.e.* $N_\kappa = N$. To describe the position of the roots of an excited state $|\, \psi_\kappa(\{\mu_{\ell_j}\})\, \rangle$ with respect to the ground state roots $\{\lambda_j\}$, it is convenient to define the (finite-size) shift function

$$\widehat{F}(\omega) \equiv \widehat{F}(\omega | \{\mu_{\ell_j}\}) = L\big[\widehat{\xi}(\omega) - \widehat{\xi}_\kappa(\omega)\big] = \frac{1}{2\pi} \sum_{k=1}^{N} \big[\theta(\omega - \lambda_k) - \theta(\omega - \mu_{\ell_k})\big] + \frac{i\beta}{2\pi}, \tag{2.9}$$

in which $\widehat{\xi}_\kappa$ is the excited state's counting function (2.8) whereas $\widehat{\xi}$ denotes the ground state counting function (at $\kappa = 1$). The spacing between the root λ_j for the ground state and the background parameters μ_j defined[3] by $\widehat{\xi}_\kappa(\mu_j) = j/L$ is then given by

$$\mu_j - \lambda_j = \frac{F(\lambda_j)}{L\rho(\lambda_j)} + O\big(L^{-2}\big), \qquad j = 1, \dots, N, \tag{2.10}$$

in which $F(\lambda)$ is the thermodynamic limit of the shift function, solution of the integral equation

$$F(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda-\mu)\, F(\mu)\, \mathrm{d}\mu = \frac{i\beta}{2\pi} - \frac{1}{2\pi}\sum_{k=1}^{n}\big[\theta(\lambda-\mu_{p_k}) - \theta(\lambda-\mu_{h_k})\big]. \tag{2.11}$$

Introducing the dressed phase $\phi(\lambda, \mu)$ and the dressed charge $Z(\lambda)$ as the respective solutions of the linear integral equations

$$\phi(\lambda,\mu) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda-\omega)\, \phi(\omega,\mu)\, \mathrm{d}\omega = \frac{\theta(\lambda-\mu)}{2\pi}, \tag{2.12}$$

$$Z(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda-\mu)\, Z(\mu)\, \mathrm{d}\mu = 1, \tag{2.13}$$

we get

$$F(\lambda) = \frac{i\beta}{2\pi} Z(\lambda) - \sum_{a=1}^{n}\big[\phi(\lambda,\mu_{p_a}) - \phi(\lambda,\mu_{h_a})\big]. \tag{2.14}$$

Other important thermodynamic quantities that we need to introduce in order to formulate our result are the dressed energy $\varepsilon(\lambda)$ and the dressed momentum $p(\lambda)$, defined as

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda-\mu)\, \varepsilon(\mu)\, \mathrm{d}\mu = \varepsilon_0(\lambda), \qquad \text{with} \quad \varepsilon(\pm q) = 0, \tag{2.15}$$

$$p(\lambda) = p_0(\lambda) + \int_{-q}^{q} \theta(\lambda-\mu)\, \rho(\mu)\, \mathrm{d}\mu = 2\pi\int_{0}^{\lambda} \rho(\mu)\, \mathrm{d}\mu. \tag{2.16}$$

Note that expression (2.16) enables us to express, in the large L limit, the counting function (2.8) of any excited state with a finite number of particles and holes above the N-particle ground state in the following form:

$$\widehat{\xi}_\kappa(\omega \,|\, \{\mu_{\ell_j}\}) = \xi(\omega) + \mathrm{O}(L^{-1}), \qquad \text{with} \quad \xi(\omega) = \frac{1}{2\pi}p(\omega) + \frac{D}{2}, \tag{2.17}$$

D being the average density in the thermodynamic limit ($N/L \to D$). In particular this means that, in the thermodynamic limit and up to $\mathrm{O}(1/L)$ corrections, the counting function does not depend on the particular localization of the corresponding Bethe roots $\{\mu_{\ell_j}\}$: it is the same for all particle/hole-type excited states.

[3] Since the counting function depends on the particular excited state we consider, so does the set of background parameters: these are therefore defined for *each* excited state $\{\mu_{\ell_j}\}$ separately.

3 The main result: large-distance/long-time asymptotic behavior of the correlation function of currents

We are now in position to formulate our main result, namely the leading asymptotic behavior of the zero-temperature correlation function of currents $\langle\, j(x,t)\, j(0,0)\,\rangle$ in the large-distance and long-time regime, with $x/t = \text{const}$.

Let $u(\lambda) = p(\lambda) - \frac{t}{x}\varepsilon(\lambda)$, where p is the dressed momentum (2.16) and ε the dressed energy (2.15), the ratio t/x being fixed (and non-zero). We assume that this function has a unique saddle-point[4] λ_0 on $\mathbb{R}$ (*i.e.* u' admits a unique zero λ_0 on $\mathbb{R}$, which moreover satisfies $u''(\lambda_0) < 0$). One distinguishes two different regimes[5] according to whether $\lambda_0 \in]-q, q[$ (time-like regime) or $\lambda_0 \notin [-q, q]$ (space-like regime). Then, at large-distance and long-time ($x \to +\infty$, $t \to +\infty$, $x/t = \text{const}$) and in the space-like regime,

$$\begin{aligned}
\langle\, j(x,t)\, j(0,0)\,\rangle &= \Big(\frac{p_F}{\pi}\Big)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\big[x^2-t^2v_F^2\big]^2}\,(1+\mathrm{o}(1))\\
&+ \frac{2\cos(2xp_F)\cdot\big|\mathcal{F}_{-q}^{q}\big|^2}{[-i(x-v_Ft)]^{\mathcal{Z}^2}\,[i(x+v_Ft)]^{\mathcal{Z}^2}}\,(1+\mathrm{o}(1))\\
&+ \frac{\sqrt{2\pi}\,\mathrm{e}^{-i\frac{\pi}{4}}\,p'(\lambda_0)}{\sqrt{t\varepsilon''(\lambda_0)-xp''(\lambda_0)}}\,\frac{\mathrm{e}^{ix[p(\lambda_0)-p_F]-it\varepsilon(\lambda_0)}\cdot\big|\mathcal{F}_{q}^{\lambda_0}\big|^2}{[-i(x-v_Ft)]^{[F_q^{\lambda_0}(q)-1]^2}\,[i(x+v_Ft)]^{[F_q^{\lambda_0}(-q)]^2}}\,(1+\mathrm{o}(1))\\
&\quad+ \frac{\sqrt{2\pi}\,\mathrm{e}^{-i\frac{\pi}{4}}\,p'(\lambda_0)}{\sqrt{t\varepsilon''(\lambda_0)-xp''(\lambda_0)}}\,\frac{\mathrm{e}^{ix[p(\lambda_0)+p_F]-it\varepsilon(\lambda_0)}\cdot\big|\mathcal{F}_{-q}^{\lambda_0}\big|^2}{[-i(x-v_Ft)]^{[F_{-q}^{\lambda_0}(q)]^2}\,[i(x+v_Ft)]^{[F_{-q}^{\lambda_0}(-q)+1]^2}}\,(1+\mathrm{o}(1)), \qquad (3.1)
\end{aligned}$$

whereas in the time-like regime,

$$\begin{aligned}
\langle\, j(x,t)\, j(0,0)\,\rangle &= \Big(\frac{p_F}{\pi}\Big)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\big[x^2-t^2v_F^2\big]^2}\,(1+\mathrm{o}(1))\\
&+ \frac{2\cos(2xp_F)\cdot\big|\mathcal{F}_{-q}^{q}\big|^2}{[-i(x-v_Ft)]^{\mathcal{Z}^2}\,[i(x+v_Ft)]^{\mathcal{Z}^2}}\,(1+\mathrm{o}(1))\\
&+ \frac{\sqrt{2\pi}\,\mathrm{e}^{i\frac{\pi}{4}}\,p'(\lambda_0)}{\sqrt{t\varepsilon''(\lambda_0)-xp''(\lambda_0)}}\,\frac{\mathrm{e}^{-ix[p(\lambda_0)-p_F]+it\varepsilon(\lambda_0)}\cdot\big|\mathcal{F}_{\lambda_0}^{q}\big|^2}{[-i(x-v_Ft)]^{[F_{\lambda_0}^{q}(q)+1]^2}\,[i(x+v_Ft)]^{[F_{\lambda_0}^{q}(-q)]^2}}\,(1+\mathrm{o}(1))\\
&\quad+ \frac{\sqrt{2\pi}\,\mathrm{e}^{i\frac{\pi}{4}}\,p'(\lambda_0)}{\sqrt{t\varepsilon''(\lambda_0)-xp''(\lambda_0)}}\,\frac{\mathrm{e}^{-ix[p(\lambda_0)+p_F]+it\varepsilon(\lambda_0)}\cdot\big|\mathcal{F}_{\lambda_0}^{-q}\big|^2}{[-i(x-v_Ft)]^{[F_{\lambda_0}^{-q}(q)]^2}\,[i(x+v_Ft)]^{[F_{\lambda_0}^{-q}(-q)-1]^2}}\,(1+\mathrm{o}(1)). \qquad (3.2)
\end{aligned}$$

In these expressions, $p_F = \pm p(\pm q) = \pi D$ is the Fermi momentum, $v_F = \varepsilon'(q)/p'(q)$ is the Fermi velocity, whereas $\mathcal{Z} = Z(\pm q)$ is the value of the dressed charge on the Fermi surface. The exponents of the last two terms in (3.1)-(3.2) are given in terms of the values at the Fermi boundaries $\pm q$ of some specific shift functions (2.14)$F_{\mu_h}^{\mu_p}$ (at $\beta = 0$) between the ground state and an excited state with one particle at μ_p and one hole at μ_h: the shift functions

$$F_{\pm q}^{\lambda_0}(\lambda) = -[\phi(\lambda,\lambda_0) - \phi(\lambda,\pm q)] \qquad (3.3)$$

[4]It is clear that the bare counterpart $u_0 = p_0(\lambda) - \frac{t}{x}\varepsilon_0(\lambda)$ of this function fulfills this property: u_0' admits a unique zero λ_0 on $\mathbb{R}$, and moreover $u_0''(\lambda_0) < 0$. Considering the form of the integral equations (2.15) and (2.16), one easily sees that u' admits at least one zero. The case of several saddle-points can in principle be treated similarly, and will give rise to several contributions.

[5]We do not consider here the case $\lambda_0 = \pm q$. Indeed, this specific case would demand some further analysis.

between the ground state and an excited state with one particle at λ_0 and one hole at $\pm q$ in (3.1) (space-like regime), and the shift functions

$$F_{\lambda_0}^{\pm q}(\lambda) = -[\phi(\lambda, \pm q) - \phi(\lambda, \lambda_0)] \tag{3.4}$$

between the ground state and an excited state with one particle at $\pm q$ and one hole at λ_0 in (3.2) (time-like regime).

The corresponding amplitudes are given in terms of some properly normalized form factors $\mathcal{F}_{\mu_h}^{\mu_p}$ of the current operator between the ground state and an excited state containing one particle at μ_p and one hole at μ_h. The connexion

$$\left|\mathcal{F}_{-q}^{q}\right|^2 = \lim_{L\to+\infty} \left(\frac{L}{2\pi}\right)^{2\mathcal{Z}^2} \left| \frac{\langle\, \psi_{-q}^{q} \,|\, j(0,0) \,|\, \psi_g \,\rangle}{||\psi_{-q}^{q}|| \cdot ||\psi_g||} \right|^2, \tag{3.5}$$

between the amplitude $\left|\mathcal{F}_{-q}^{q}\right|^2$ and the square of the norm of an Umklapp form factor of the density operator between the ground state $|\, \psi_g \,\rangle$ and an excited state $|\, \psi_{-q}^{q} \,\rangle$ containing one particle and one hole at the opposite ends of the Fermi boundaries was already noticed in [45, 46]. A similar phenomenon happens for the other terms. More precisely, the amplitudes $\left|\mathcal{F}_{\pm q}^{\lambda_0}\right|^2$ appearing in (3.1) (space-like regime) correspond to the thermodynamic limit of the properly normalized norm squared of the form factor of the current operator taken between the ground state $|\, \psi_g \,\rangle$ and an eigenstate $|\, \psi_{\pm q}^{\lambda_0} \,\rangle$ with a particle at λ_0 and a hole at $\pm q$:

$$\left|\mathcal{F}_{q}^{\lambda_0}\right|^2 = \lim_{L\to+\infty} \left(\frac{L}{2\pi}\right)^{1+[F_q^{\lambda_0}(q)-1]^2+[F_q^{\lambda_0}(-q)]^2} \left| \frac{\langle\, \psi_{q}^{\lambda_0} \,|\, j(0,0) \,|\, \psi_g \,\rangle}{||\psi_{q}^{\lambda_0}|| \cdot ||\psi_g||} \right|^2, \tag{3.6}$$

$$\left|\mathcal{F}_{-q}^{\lambda_0}\right|^2 = \lim_{L\to+\infty} \left(\frac{L}{2\pi}\right)^{1+[F_{-q}^{\lambda_0}(q)]^2+[F_{-q}^{\lambda_0}(-q)+1]^2} \left| \frac{\langle\, \psi_{-q}^{\lambda_0} \,|\, j(0,0) \,|\, \psi_g \,\rangle}{||\psi_{-q}^{\lambda_0}|| \cdot ||\psi_g||} \right|^2. \tag{3.7}$$

Similarly, the amplitudes $\left|\mathcal{F}_{\lambda_0}^{\pm q}\right|^2$ appearing in (3.2) (time-like regime) correspond to the thermodynamic limit of the properly normalized norm squared of the form factor of the current operator between the ground state $|\, \psi_g \,\rangle$ and an eigenstate $|\, \psi_{\lambda_0}^{\pm q} \,\rangle$ with a particle at $\pm q$ and a hole at λ_0:

$$\left|\mathcal{F}_{\lambda_0}^{q}\right|^2 = \lim_{L\to+\infty} \left(\frac{L}{2\pi}\right)^{1+[F_{\lambda_0}^{q}(q)+1]^2+[F_{\lambda_0}^{q}(-q)]^2} \left| \frac{\langle\, \psi_{\lambda_0}^{q} \,|\, j(0,0) \,|\, \psi_g \,\rangle}{||\psi_{\lambda_0}^{q}|| \cdot ||\psi_g||} \right|^2, \tag{3.8}$$

$$\left|\mathcal{F}_{\lambda_0}^{-q}\right|^2 = \lim_{L\to+\infty} \left(\frac{L}{2\pi}\right)^{1+[F_{\lambda_0}^{-q}(q)]^2+[F_{\lambda_0}^{-q}(-q)-1]^2} \left| \frac{\langle\, \psi_{\lambda_0}^{-q} \,|\, j(0,0) \,|\, \psi_g \,\rangle}{||\psi_{\lambda_0}^{-q}|| \cdot ||\psi_g||} \right|^2. \tag{3.9}$$

The explicit expressions of (3.5)-(3.9) are given in Appendix A.3.

We now comment this result. First of all, the first two lines of each expression appear as a direct generalization, involving the relativistic combinations $x \pm v_F t$, of the large-distance asymptotic behavior of the corresponding static correlation function. They are in agreement (up to the amplitude of the third term, computed in [45, 46]) with the predictions coming from the CFT/Luttinger liquid approximations. This is indeed not surprising, since such terms involve particle/hole excitations on the Fermi boundary and are therefore correctly taken into account by a linearization of the spectrum around this point. However, when time is of the same order of magnitude as distance (*i.e.* $x/t \sim \mathrm{O}(1)$), other types of excitations also contribute to the asymptotics, namely those involving particles (for the space-like regime) or holes (for the time-like regime) with rapidities located at the saddle-point λ_0. These contributions appear in the last two lines of (3.1), (3.2).

Remark 3.1. This result is also valid when $x/t \to 0$ ($t >> x$). In the $t/x \to 0$ limit (*i.e.* $t << x$), one has $\lambda_0 \to +\infty$, and the last two terms of (3.1)-(3.2) exhibit a very quick oscillation except around $x = 0$, which reconstructs the $\delta(x)$-part of the equal-time correlation function. Hence, as expected, the contribution of these terms vanishes at large distances.

Remark 3.2. It is easy to see that our result coincides, at the free fermion point $c = +\infty$, with the leading asymptotic behavior obtained through a direct computation of the correlation function. In that case $\lambda_0 = x/(2t)$ and the combination of the last two terms of (3.1)-(3.2) reduces to:

$$i\sqrt{\frac{\pi}{t}}\,\mathrm{e}^{i\alpha\frac{\pi}{4}}\,\mathrm{e}^{i\alpha(\frac{x^2}{4t}+th)}\left\{\frac{\mathrm{e}^{-i\alpha[xq-t(q^2-h)]}}{x-2qt}-\frac{\mathrm{e}^{i\alpha[xq+t(q^2-h)]}}{x+2qt}\right\}, \tag{3.10}$$

where $\alpha = 1$ in the space-like regime and $\alpha = -1$ in the time-like regime. Note that this contribution is dominant with respect to the other power-law decaying terms.

4 Form factor expansion for the correlation function in finite volume

The derivation of the result we have just announced is based on the form factor expansion for the correlation function. In this section, we write down the form factor series representation for the current-current correlation function in finite volume, or more precisely for its generating function. Each term of the series can be expressed by using the finite-size determinant representation for the form factors of the model and their relation with the overlap scalar products. Such a series will be the starting point for our study.

Let us consider the zero-temperature correlation function of currents $\langle\, j(x,t)\, j(0,0)\,\rangle$, with $j(x,t) = \mathrm{e}^{itH} j(x,0)\,\mathrm{e}^{-itH}$. Inserting the sum over the complete (see [23]) set of Bethe states $|\,\psi'\,\rangle$ between the two operators we obtain

$$\langle\, j(x,t)\, j(0,0)\,\rangle = \sum_{\psi'} \mathrm{e}^{-it\mathcal{E}_{\mathrm{ex}}}\frac{\langle\,\psi_g\,|\, j(x,0)\,|\,\psi'\,\rangle\,\langle\,\psi'\,|\, j(0,0)\,|\,\psi_g\,\rangle}{||\psi_g||^2\cdot||\psi'||^2}, \tag{4.1}$$

in which $\mathcal{E}_{\mathrm{ex}}$ denotes the difference of energies between the excited state $|\,\psi'\,\rangle$ and the ground state $|\,\psi_g\,\rangle$. The matrix elements of $j(x,0)$ can be easily obtained in terms of the matrix elements of the operator $\mathrm{e}^{\beta Q_x}$, with $Q_x = \int_0^x j(y,0)\mathrm{d}y$:

$$\frac{\langle\,\psi_g\,|\, j(x,0)\,|\,\psi'\,\rangle}{||\psi_g||\cdot||\psi'||} = \partial_x\partial_\beta\frac{\langle\,\psi(\{\lambda_j\})\,|\,\mathrm{e}^{\beta Q_x}\,|\,\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi(\{\lambda_j\})\|\cdot||\psi_\kappa(\{\mu_{\ell_j}\})||}\Bigg|_{\beta=0}. \tag{4.2}$$

In their turn, the latter can be computed from the results of [44, 77, 58] in terms of the scalar products:

$$\langle\,\psi(\{\lambda_j\})\,|\,\mathrm{e}^{\beta Q_x}\,|\,\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle = \mathrm{e}^{ix\mathcal{P}_{\mathrm{ex}}^{\kappa}}\langle\,\psi(\{\lambda_j\})\,\,|\,\,\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle, \tag{4.3}$$

with $\mathcal{P}_{\mathrm{ex}}^{\kappa} = \sum_{j=1}^{N}[p_0(\mu_{\ell_j}) - p_0(\lambda_j)]$. In these expressions, $\{\lambda\}$ parametrizes the ground state whereas $\{\mu_{\ell_j}\}$ is a set of solutions to the twisted Bethe equations (2.7) associated with the set of integers $\{\ell_j\}$, such that $|\,\psi_\kappa(\mu_{\ell_j})\,\rangle \to |\,\psi'\,\rangle$ when $\kappa \to 1$ (we recall that $\kappa = \mathrm{e}^{\beta}$). Note that it is enough to consider states with $N_\kappa = N$, as otherwise, the corresponding matrix element is zero.

Therefore, (4.1) can be rewritten as

$$\langle\, j(x,t)\, j(0,0)\,\rangle = \sum_{\ell_1<\ldots<\ell_N} \mathrm{e}^{ix\mathcal{P}_{\mathrm{ex}}-it\mathcal{E}_{\mathrm{ex}}} \left| \partial_y \partial_\beta \left\{ \mathrm{e}^{iy\mathcal{P}^{\kappa}_{\mathrm{ex}}} \frac{\langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi(\{\lambda_j\})\| \cdot \|\psi_\kappa(\{\mu_{\ell_j}\})\|} \right\}_{y,\beta=0} \right|^2, \tag{4.4}$$

in which we have set $\mathcal{P}_{\mathrm{ex}} = \lim_{\kappa\to 1} \mathcal{P}^{\kappa}_{\mathrm{ex}}$. Using the fact that $\mathcal{P}_{\mathrm{ex}} = 0$ if $|\,\psi(\{\lambda_j\})\,\rangle = |\,\psi(\{\mu_{\ell_j}\})\,\rangle$ and that $\langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle = 0$ if $|\,\psi(\{\lambda_j\})\,\rangle \neq |\,\psi(\{\mu_{\ell_j}\})\,\rangle$ in virtue of the orthogonality of Bethe states [23], we obtain

$$\begin{aligned} \langle\, j(x,t)\, j(0,0)\,\rangle =& -\left[\partial_\beta \mathcal{P}^{\kappa}_{\mathrm{ex}}|_{\beta=0}\right]^2 \\ -& \sum_{\psi'\neq\psi_g} \mathrm{e}^{ix\mathcal{P}_{\mathrm{ex}}-it\mathcal{E}_{\mathrm{ex}}}\, \mathcal{P}^2_{\mathrm{ex}} \frac{\partial_\beta \langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle_{\beta=0} \cdot \partial_\beta \langle\, \psi_\kappa(\{\mu_{\ell_j}\}) \mid \psi(\{\lambda_j\})\,\rangle_{\beta=0}}{||\psi_g||^2 \cdot ||\psi'||^2}. \end{aligned} \tag{4.5}$$

On the other hand, setting $\mathcal{E}^{\kappa}_{\mathrm{ex}} = \sum_{a=1}^{N} [\varepsilon_0(\mu_{\ell_a}) - \varepsilon_0(\lambda_a)]$ and using similar arguments,

$$\begin{aligned} \partial_x^2 \partial_\beta^2 \left\{ \mathrm{e}^{ix\mathcal{P}^{\kappa}_{\mathrm{ex}}-it\mathcal{E}^{\kappa}_{\mathrm{ex}}} \cdot \left| \frac{\langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{||\psi_\kappa(\{\mu_{\ell_j}\})|| \cdot ||\psi(\{\lambda_j\})||} \right|^2 \right\}_{\beta=0} &= -2\delta_{\psi',\psi_g} \left[\partial_\beta \mathcal{P}^{\kappa}_{\mathrm{ex}}|_{\beta=0}\right]^2 \\ -2(1-\delta_{\psi',\psi_g})\, \mathrm{e}^{ix\mathcal{P}_{\mathrm{ex}}-it\mathcal{E}_{\mathrm{ex}}}\, \mathcal{P}^2_{\mathrm{ex}} \frac{\partial_\beta \langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle_{\beta=0} \cdot \partial_\beta \langle\, \psi_\kappa(\{\mu_{\ell_j}\}) \mid \psi(\{\lambda_j\})\,\rangle_{\beta=0}}{||\psi_g||^2 \cdot ||\psi'||^2}&. \end{aligned} \tag{4.6}$$

Above we have used the fact that, for $\beta \in i\mathbb{R}$, the solutions of the κ-twisted Bethe equations (2.7) are real, which means that

$$\overline{\langle\, \psi(\{\lambda\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle} = \langle\, \psi_\kappa(\{\mu_{\ell_j}\}) \mid \psi(\{\lambda\})\,\rangle. \tag{4.7}$$

Let us define

$$\mathcal{Q}^{\kappa}_N(x,t) = \sum_{\ell_1<\cdots<\ell_N} \mathrm{e}^{ix\sum_{a=1}^{N}[u_0(\mu_{\ell_a})-u_0(\lambda_a)]} \left| \frac{\langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi_\kappa(\{\mu_{\ell_j}\})\| \cdot \|\psi(\{\lambda_j\})\|} \right|^2, \tag{4.8}$$

with $u_0(\lambda) = p_0(\lambda) - \frac{t}{x}\varepsilon_0(\lambda)$. Comparing (4.5) and (4.6), we get that $\mathcal{Q}^{\kappa}_N(x,t)$ is the generating function of the time and space-dependent zero-temperature current-current correlation function of the NLS model in finite volume:

$$\langle\, j(x,t)\, j(0,0)\,\rangle = \frac{1}{2} \partial_x^2 \partial_\beta^2 \mathcal{Q}^{\kappa}_N(x,t)\big|_{\beta=0}. \tag{4.9}$$

We recall that, in (4.8), the set of parameters $\{\lambda_j\}$ denotes the solution of the Bethe equations (2.2) parametrizing the ground state of the Hamiltonian (2.1), whereas $\{\mu_{\ell_j}\}$ stands for the unique solution of the κ-twisted logarithmic Bethe equations (2.7) defined by the choice of integers $\ell_1 < \cdots < \ell_N$. The sum runs here through all the possible choices of ordered N-tuples of integers $\ell_1 < \cdots < \ell_N$, and therefore through all the excited states $|\,\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle$ in the N-particle sector [23].

Each term of the form factor series (4.8) involves a normalized scalar product between a Bethe state and a twisted Bethe state. For the model in finite volume, such scalar products (and form factors) admit finite-size determinant representations [81, 58]. When the size L becomes large, these scalar products (and form factors) exhibit a non-trivial behavior with respect to the size of the system [82, 2]. For scalar products between the ground state and an excited state with a finite number of particles and holes such as those described in Section 2, it was shown in [46, 43] how to extract the leading large L asymptotic behavior from their finite size determinant representations.

The detailed representation for the products of form factor that appear in each term of the series (4.8), as well as its leading behavior in the thermodynamic limit, is recalled in Appendix A. In finite volume, the corresponding normalized scalar product takes the following form:

$$\left| \frac{\langle\, \psi(\{\lambda_j\}) \mid \psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi_\kappa(\{\mu_{\ell_j}\})\| \cdot \|\psi(\{\lambda_j\})\|} \right|^2 = \widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) \cdot \widehat{\mathcal{G}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}), \tag{4.10}$$

in which

$$\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \prod_{j=1}^{N} \frac{\sin^2[\pi \widehat{F}(\lambda_j)]}{\pi^2 L^2\, \widehat{\xi}'(\lambda_j)\, \widehat{\xi}'_\kappa(\mu_{\ell_j})} \cdot \left[\det_N \frac{1}{\lambda_j - \mu_{\ell_k}} \right]^2 \tag{4.11}$$

is the so-called discrete part of the form factor, which contains the whole non-trivial singular part of the form factor with respect to the system-size (and is quite universal), whereas $\widehat{\mathcal{G}}_N(\{\lambda_j\},\{\mu_{\ell_j}\})$ is a dressing function which, for a n-particle/n-hole excited state as defined in Section 2, admits a smooth thermodynamic limit:

$$\lim_{L,N\to\infty} \widehat{\mathcal{G}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \mathcal{G}_n\begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix}. \tag{4.12}$$

Here $\mathcal{G}_n$ is a holomorphic function of the particle/hole rapidities $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$. The microscopic details of this dressing function (see Appendix A) depend on the model and do not really matter for our study. We will only use the fact that $\mathcal{G}_n$ can be represented in terms of a functional

$$\mathcal{G}_n\begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix} \equiv \mathscr{G}\left[\varpi_n\left(\cdot \,\middle|\, \begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) \right] \tag{4.13}$$

of the function

$$\varpi_n\left(\lambda \,\middle|\, \begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) = \sum_{a=1}^{n} \left\{ \frac{1}{\lambda - \mu_{p_a}} - \frac{1}{\lambda - \mu_{h_a}} \right\}. \tag{4.14}$$

The behavior of the thermodynamic limit of the discrete part is more difficult to obtain (see [46, 43] and Appendix A). Its knowledge is however unnecessary for our purpose, since our study will directly rely on the finite-size formula (4.11). We will simply use the fact that (4.11) can be understood as a functional of the finite-size shift function $\widehat{F}$, depending moreover on the extra sets of particle/hole integers $\{p_a\}$ and $\{h_a\}$:

$$\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) \equiv \widehat{D}_{N,n}\begin{pmatrix} \{p_a\} \\ \{h_a\} \end{pmatrix} [\widehat{F}]. \tag{4.15}$$

More precisely, the function $\widehat{\xi}$ being fixed and the set of parameters λ_j being defined as the pre-image of the set $\{j/L : j = 1, \ldots, N\}$ by this function, it means that:

- the function $\widehat{\xi}_\kappa$ in (4.11) should be understood as a functional of the shift function $\widehat{F}$ through the relation $\widehat{\xi}_\kappa = \widehat{\xi} - L^{-1}\widehat{F}$;
- the set of integers ℓ_j being fixed by the choice of $\{p_a\}$ and $\{h_a\}$ (and *vice-versa*), the parameters μ_{ℓ_j} are obtained as their pre-image by the function $L\widehat{\xi}_\kappa$.

5 Effective decoupling of the finite-size form factor series

The structure of the summations in (4.8) is extremely intricate. Indeed, the sets $\{\mu_{\ell_j}\}$ of Bethe parameters over which we sum up are implicit functions of the N integers ℓ_j labelling the corresponding excited states. Although one can build on such a description so as to use the multidimensional residue theory to recast this complex sum into a contour integral (the so-called *master equation* [52, 51, 44]), the later still remains difficult to handle. In particular, one cannot perform the thermodynamic limit directly on the level of the master equation. Therefore, one has to expand the contour integral into yet another series (the so-called multidimensional Fredholm series) that has a presumably well defined thermodynamic limit. The asymptotic analysis of the obtained series, decomposed in terms of cycle integrals, then relies on a non-trivial summation process, resulting in particular into the dressing of bare quantities (energy, momentum) into dressed ones. Although such an approach was successfully applied in [45] to produce the leading asymptotic behavior of the time-independent correlation functions of the XXZ chain and of the NLS model (see also [63] for the temperature-dependent case), we wish here, so as to bypass some of the subtleties of [45], to propose an alternative line of though by pursuing our study directly on the form factor series.

The idea of our method is the following: the series (4.8) can be related to an essentially free fermion type series (see Appendix B), this *via* some reasonable physical assumptions concerning the contribution of each of its terms in the thermodynamic limit, and *via* some formal manipulations. Indeed, it is shown in Appendix B that if, for each excited state,

- the rapidities of the particles and holes are decoupled, *i.e.* they are determined by a counting function that does not depend on the position of the roots of the corresponding excited state,
- the dressing part $\widehat{\mathcal{G}}_N$ of the form factors is decoupled,

then the series in finite volume can be summed up into a finite-size determinant. We explain in this section how to reduce the study of our highly coupled series to this simple case. It means that, in our setting, the key-role in the summation is played by the universal discrete part $\widehat{D}_N$ of the form factors exactly as in the free fermion case. This fact should be put in parallel with the key-role played, in the master equation-based asymptotic analysis [45], by the Cauchy determinant part of the form factors.

5.1 An effective form factor series

Considering that we will finally be interested in the thermodynamic limit $L \to +\infty$ of the generating function (4.8), we now slightly simplify its form factor series. Our simplifications rely on two kinds of assumptions:

(i) the contribution to the sum (4.8) of a state having a macroscopically (with respect to L) different energy and momentum from the ground state should vanish in the thermodynamic limit;

(ii) for each term of the series, "non-discrete" quantities (*i.e.* parts of the form factor expression behaving smoothly at the thermodynamic limit) should contribute to the thermodynamic limit of the sum (4.8) only through their leading order in L.

Assumption (i) can be attributed in particular to the extremely quick oscillation of the phase factors for states having large excitation momenta and energies. It means notably that:

- we can use the particle/hole picture to describe the large-L behavior of the form factors, keeping in mind that the relevant (*i.e.* the one not vanishing in the $L \to +\infty$ limit) part of the form factor expansion corresponds to a summation over states with a *finite* (or at least growing much less than L) number of particle/hole excitations above the ground state;
- we can introduce a "cut-off" (with respect to L) of the range of integers on which we sum up in (4.8) since, for very large integers, momentum and energy of the corresponding state become also very large.

Doing this, we roughly speaking neglect correcting terms in the lattice size L. It is thus reasonable to assume that, on the same ground, one can also neglect some finite-size corrections to the leading thermodynamic behavior of the form factors, as stated in (ii). More precisely, we make the following simplifications in the series (4.8):

- we replace the finite-size shift function $\widehat{F}$, the smooth part $\widehat{G}_N$ and the oscillating exponent $\sum_{a=1}^{N}[u_0(\mu_{\ell_a}) - u_0(\lambda_a)]$ by their leading contribution in the thermodynamic limit;
- in the obtained expression, we now understand the rapidities of the particles μ_{p_a} (resp. of the holes μ_{h_a}), corresponding to a particular choice of integers $\ell_1 < \cdots < \ell_N$, as being defined in terms of the thermodynamic limit F of the shift function (and no longer in terms of its finite-size counterpart (2.9)) as the pre-images of p_a/L (resp. h_a/L) by the counting function $\widehat{\xi}_F = \widehat{\xi} - F/L$; in other words, the $2n$ parameters $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$ are obtained as the unique solutions to the system of $2n$ equations

$$\widehat{\xi}(\mu_{p_a}) - \frac{1}{L} F\left(\mu_{p_a} \,\middle|\, \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right) = \frac{p_a}{L} \quad \text{and} \quad \widehat{\xi}(\mu_{h_a}) - \frac{1}{L} F\left(\mu_{h_a} \,\middle|\, \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right) = \frac{h_a}{L}\ , \tag{5.1}$$

  with $a = 1, \ldots, n$.

These simplifications result into an effective series that we assume to have the *same value at the thermodynamic limit* as the original series. In other words, we conjecture that,

$$\lim_{L,N\to\infty} \mathcal{Q}_N^\kappa(x,t)_{\text{eff}} = \lim_{L,N\to\infty} \mathcal{Q}_N^\kappa(x,t) \equiv \mathcal{Q}^\kappa(x,t), \tag{5.2}$$

with

$$\mathcal{Q}_N^\kappa(x,t)_{\text{eff}} = \mathrm{e}^{-\frac{x\beta p_F}{\pi}} \sum_{\substack{\ell_1<\cdots<\ell_N \\ \ell_a\in\mathcal{B}_L}} \mathrm{e}^{ix\sum\limits_{a=1}^{n}[u(\mu_{p_a})-u(\mu_{h_a})]} \cdot \widehat{D}_{N,n}\begin{pmatrix}\{p_a\}\\ \{h_a\}\end{pmatrix} \left[F\left(\cdot \,\middle|\, \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right)\right]$$
$$\times\, \mathscr{G}\left[\varpi_n\left(\cdot \,\middle|\, \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right)\right]. \tag{5.3}$$

In (5.3), the sums are restricted to the set $\mathcal{B}_L = \big\{\ell \in \mathbb{Z} \mid -w_L \leq \ell \leq w_L\big\}$, where w_L is a cut-off growing as $L^{1+\epsilon}$, for some $\epsilon > 0$. For a given set of integers $\{\ell_j\}$, the parameters

p_a and h_a define the position of particles and holes as explained in Section 2. The (finite-size) representation for the form factor has been partly replaced by its thermodynamic limit. More precisely, the dressing function and the phase factor, which both behave smoothly at the thermodynamic limit, have been replaced by their leading equivalents. Note that, as a consequence, it is now the dressed energy and momentum that appear in the phase factor (*cf* Appendix A): $u(\lambda) = p(\lambda) - \frac{t}{x}\varepsilon(\lambda)$. In (5.3), we have kept the finite-size expression of the discrete part, but we have nevertheless replaced the expression of the finite-size shift function $\widehat{F}$ by its limiting value F (2.14). Finally, the particles and holes' rapidities associated to a given set of integers $\ell_1 < \cdots < \ell_N$ are obtained (in terms of F) *via* the system of $2n$ equations (5.1).

Remark 5.1. We stress that, in (5.3), we are still in "finite volume" (*i.e.* for the moment, L and N are kept finite). We have only modified the original series (4.8) arguing that it should not affect the value of its thermodynamic limit.

Remark 5.2. The introduction of the cut-off w_L is convenient to avoid problems of convergence in our further manipulations of the series. Indeed, as far as we remain in finite volume, we now deal with finite sums only.

5.2 Towards a free-fermion type series

Although the effective series (5.3) is already simpler that the original one it is still highly coupled:

- the thermodynamic limit of the shift function (2.14) still depends on the position of particles and holes,
- the rapidities of the particles and holes have to be computed for each excited state separately (*i.e.* they are excited state dependent),
- the expression of the functional $\mathscr{G}$ is extremely intricate.

Our aim is now to relate $\mathcal{Q}_N^\kappa(x,t)_{\text{eff}}$ to a decoupled series, so as to reduce its analysis to the one of the generalized free-fermion case that is carried out in Appendix B. This can be done by understanding the coupling between variables as the result of the action of some functional translation operators[6] (see Appendix C).

The function ϖ_n (4.14) depends *linearly* on a function of the rapidities of the particles and of the holes. This means that one can formally express this dependence by means of the action of a functional translation operator, as explained in Appendix C: for any sufficiently smooth functional $\mathcal{F}$ supported on a neighborhood $\mathcal{U}$ of the real axis,

$$\mathcal{F}\Bigg[\varpi_n\bigg(\cdot\,\bigg|\begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\bigg)\Bigg] = \prod_{a=1}^{n} \exp\Bigg\{\int\limits_{\mathbb{R}} \mathrm{d}\lambda \left[\frac{1}{\lambda-\mu_{p_a}} - \frac{1}{\lambda-\mu_{h_a}}\right] \frac{\delta}{\delta\omega(\lambda)}\Bigg\} \cdot \mathcal{F}[\omega]\Bigg|_{\omega=0}. \tag{5.4}$$

The thermodynamic limit of the shift function (2.14) also depends *linearly* on a function of the rapidities of the particles and holes. However, the situation is slightly more complicated than in (5.4) since, in virtue of (5.1), the parameters $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$ are themselves functionals of the shift function. As explained in Appendix C, one can still represent any smooth functional of

[6]The use of functional translation operators is simply a convenient and compact way to manipulate generalized Lagrange series (see Appendix C of [45]). In fact, the whole reasoning that follows can instead be performed by writing explicitly the corresponding series.

this shift function in terms of translation operators, provided that one imposes some operator ordering $:\ \cdot\ :$ and that one takes into account the contribution of the Jacobian coming from the summation of the corresponding multi-dimensional Lagrange series (see formulae (C.5) and (C.11)):

$$\mathcal{F}\Bigg[F\bigg(\cdot\,\bigg|\begin{matrix}\{\bar{\mu}_{p_a}\}\\ \{\bar{\mu}_{h_a}\}\end{matrix}\bigg)\Bigg] =: \prod_{a=1}^{n}\exp\Bigg\{\int_{\mathbb{R}}\mathrm{d}\lambda\,[\phi(\lambda,\bar{\mu}_{h_a})-\phi(\lambda,\bar{\mu}_{p_a})]\,\frac{\delta}{\delta\tau(\lambda)}\Bigg\}\cdot\mathcal{F}[\nu_\tau]\ :\ \Big|_{\tau=0} J. \quad (5.5)$$

Above the operator ordering $:\ \cdot\ :$ is such that, in the formal series expansion in powers of $\delta/\delta\tau(\lambda)$, all functional derivative operators are located on the left. ν_τ is the function

$$\nu_\tau(\lambda) = i\frac{\beta Z(\lambda)}{2\pi} + \tau(\lambda), \quad (5.6)$$

and J is the Jacobian $J = \det_{\mathbb{R}}\big[I - \delta\Gamma[\nu_\tau](\lambda)/\delta\tau(\mu)\big]_{|\tau=F}$ of the functional

$$\Gamma[\nu_\tau](\lambda) = \sum_{a=1}^{n}[\phi(\lambda,\bar{\mu}_{h_a})-\phi(\lambda,\bar{\mu}_{p_a})], \qquad \text{where} \quad \bar{\mu}_\ell \equiv \widehat{\xi}_{\nu_\tau}^{-1}\Big(\frac{\ell}{L}\Big) \quad \text{with} \quad \ell\in\mathbb{Z}\,. \quad (5.7)$$

This Jacobian is evaluated at $\tau = F$, where F coincides with the shift function occuring in the *l.h.s.* of (5.5). We stress that the bar occuring in the parameters $\bar{\mu}_{p_a}$ (resp. $\bar{\mu}_{h_a}$) appearing in the *r.h.s.* of (5.5) indicates that these are to be understood as the pre-images of p_a/L (resp. h_a/L) by the counting function $\widehat{\xi}_{\nu_\tau}$, as in (5.7).

It is easy to convince oneself that, provided that the number of particle/hole excitations is finite, $J = 1 + \mathrm{O}(L^{-1})$. In the light of our previous arguments, only this sector of excitations is expected to contribute to the $L\to+\infty$ limit of the form factor expansion. As a consequence, in order to be consistent with our previous approximations, we also drop the finite-size corrections due to this Jacobian by considering from now on that our effective series (still abusively denoted by the same symbol $\mathcal{Q}_N^{\kappa}(x,t)_{\mathrm{eff}}$) is in fact given by the expression (5.3) in which each term is multiplied by the inverse of this Jacobian.

Hence, the use of the functional derivative leads to the following representation:

$$\widehat{D}_{N,n}\begin{pmatrix}\{p_a\}\\ \{h_a\}\end{pmatrix}\Bigg[F\bigg(\cdot\,\bigg|\begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\bigg)\Bigg]\cdot\mathscr{G}\Bigg[\varpi_n\bigg(\cdot\,\bigg|\begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\bigg)\Bigg]\cdot J^{-1}$$
$$=: \prod_{a=1}^{n}\mathrm{e}^{\int_{\mathbb{R}}\mathrm{d}\lambda\,[\phi(\lambda,\bar{\mu}_{h_a})-\phi(\lambda,\bar{\mu}_{p_a})]\,\frac{\delta}{\delta\tau(\lambda)}}\prod_{a=1}^{n}\mathrm{e}^{\int_{\mathbb{R}}\mathrm{d}\lambda\,\big[\frac{1}{\lambda-\bar{\mu}_{p_a}}-\frac{1}{\lambda-\bar{\mu}_{h_a}}\big]\,\frac{\delta}{\delta\omega(\lambda)}}$$
$$\cdot\,\widehat{D}_{N,n}\begin{pmatrix}\{p_a\}\\ \{h_a\}\end{pmatrix}[\nu_\tau(\cdot)]\cdot\mathscr{G}[\omega(\cdot)]\ :\ \Big|_{\substack{\tau=0\\ \omega=0}}. \quad (5.8)$$

Here we have insisted on the fact that $\widehat{D}_{N,n}$ and $\mathscr{G}$ are functionals acting on the argument $\cdot$ of $\nu_\tau(\cdot)$ and $\omega(\cdot)$. Also, on the *l.h.s.* of (5.8), we have explicitly pointed out the parametric dependence of F and ϖ_n on the parameters $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$.

In the *r.h.s.* of (5.8), the functional $\widehat{D}_{N,n}$ acts on a shift function ν_τ that does not depend anymore on the particle/hole rapidities. In other words, the effective shift function ν_τ becomes independent of the summation variables and hence mimics the one appearing in the generalized free fermion model studied in Appendix B. Moreover, the equations defining the position of the particle's/hole's rapidities also become decoupled: the rapidity $\bar{\mu}_\ell$ is now the unique solution to $\widehat{\xi}_{\nu_\tau}(\bar{\mu}_\ell) = \ell/L$, and its value does not depend anymore on the choice of the other integers describing the excited state, but only on the function τ.

It is convenient to express, on a formal level, each term of the pre-factor in (5.8) as a ratio of two exponents $\mathrm{e}^{\widehat{g}(\bar{\mu}_{p_a})}/\mathrm{e}^{\widehat{g}(\bar{\mu}_{h_a})}$, where $\widehat{g}$ is an operator valued function (we have used the hat so as to insist on this property):

$$\widehat{g} = \widehat{g}_\tau + \widehat{g}_\omega, \quad \text{with} \quad \widehat{g}_\tau(\lambda) = -\int_{\mathbb{R}} \mathrm{d}\mu\, \phi(\mu,\lambda)\, \frac{\delta}{\delta\tau(\mu)}, \quad \widehat{g}_\omega(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu}{\mu-\lambda}\, \frac{\delta}{\delta\omega(\lambda)}. \tag{5.9}$$

In the following, we set

$$\widehat{E}_-^2 = \mathrm{e}^{-ixu-\widehat{g}}. \tag{5.10}$$

The operator order $:\cdot:$ being linear, we can exchance it with a *finite* summation symbol such as the one appearing in the expression (5.3) for $\mathcal{Q}_N^\kappa(x,t)_{\mathrm{eff}}$. Hence, we obtain

$$\begin{aligned}\mathcal{Q}_N^\kappa(x,t)_{\mathrm{eff}} &= \mathrm{e}^{-\frac{\beta x p_F}{\pi}} \; : \sum_{\substack{\ell_1<\cdots<\ell_N \\ \ell_j\in\mathcal{B}_L}} \prod_{a=1}^{n} \frac{\widehat{E}_-^2(\bar{\mu}_{h_a})}{\widehat{E}_-^2(\bar{\mu}_{p_a})} \cdot \widehat{D}_{N,n}\begin{pmatrix}\{p_a\}\\ \{h_a\}\end{pmatrix}[\nu_\tau]\cdot \mathscr{G}[\omega] \; : \Big|_{\substack{\tau=0\\ \omega=0}} \\ &= \mathrm{e}^{-\frac{\beta x p_F}{\pi}} \; : \prod_{a=1}^{N} \frac{\widehat{E}_-^2(\bar{\mu}_a)}{\widehat{E}_-^2(\lambda_a)} \cdot X_N\big[\nu_\tau, \widehat{E}_-^2\big]\cdot \mathscr{G}[\omega] \; : \Big|_{\substack{\tau=0\\ \omega=0}},\end{aligned} \tag{5.11}$$

in which X_N is the generalized free-fermionic functional (B.1) studied in Appendix B.

5.3 Representation in terms of a finite-size determinant

In Appendix B we have recast the generalized free-fermionic functional $X_N[\nu, E_-^2]$ into a finite-size determinant (B.2), this without any further approximations. Such a representation is obtained through *purely algebraic* manipulations on the initial definition (B.1) for $X_N[\nu, E_-^2]$. These two representations are still equal when applied to the (non-commutative) operator valued functions ν_τ and $\widehat{E}_-$: it is indeed not a problem to implement the appropriate operator order at any step of the computation performed in Appendix B. One may therefore use the determinant representation (B.2) to obtain new representations for our series (5.11), provided that one imposes an operator order on the entries of the columns of the determinant[7] [84]. This leads to

$$\mathcal{Q}_N^\kappa(x,t)_{\mathrm{eff}} = \mathrm{e}^{-\frac{\beta x p_F}{\pi}} \; : \prod_{a=1}^{N} \frac{\widehat{E}_-^2(\bar{\mu}_a)}{\widehat{E}_-^2(\lambda_a)} \cdot \det_N\left[\delta_{jk} + \frac{\widehat{V}^{(L)}(\lambda_j,\lambda_k)}{L\,\widehat{\xi}'(\lambda_k)}\right]\cdot \mathscr{G}[\omega] \; : \Big|_{\substack{\tau=0\\ \omega=0}}, \tag{5.12}$$

where the (operator valued) kernel is given as

$$\widehat{V}^{(L)}(\lambda_k,\lambda_j) = 4\frac{\sin[\pi\nu_\tau(\lambda_k)]\,\sin[\pi\nu_\tau(\lambda_j)]}{2i\pi(\lambda_k-\lambda_j)}\Big\{\widehat{E}_+^{(L)}(\lambda_k)\,\widehat{E}_-(\lambda_j) - \widehat{E}_+^{(L)}(\lambda_j)\,\widehat{E}_-(\lambda_k)\Big\}, \tag{5.13}$$

with

$$\widehat{E}_+^{(L)}(\lambda) = i\widehat{E}_-(\lambda)\left\{\int_{\bar{A}_L}^{\bar{B}_L} \frac{\mathrm{d}\mu}{2\pi}\frac{\widehat{E}_-^{-2}(\mu)}{\mu-\lambda} + \frac{\widehat{E}_-^{-2}(\lambda)}{2}\cot[\pi\nu_\tau(\lambda)] + I_L\big[\nu_\tau, \widehat{E}_-^{-2}\big](\lambda)\right\}. \tag{5.14}$$

[7] Such an order can for instance be implemented explicitly by expanding the determinant into a sum over permutations.

The expression of the functional I_L can be found in (B.5).

Such an expression provides a formal re-summation of the effective form factor series (5.3). In the next section, we show how to take the thermodynamic limit of (5.12).

6 Thermodynamic limit and asymptotic analysis

We now build on the results of the previous section so as to derive the leading large-distance/long-time asymptotic behavior of the thermodynamic limit $\mathcal{Q}^{\kappa}(x,t)$ of the generating function $\mathcal{Q}^{\kappa}_N(x,t)$.

6.1 Representation in terms of a Fredholm determinant in the thermodynamic limit

The next step of our study is to obtain a convenient representation for the thermodynamic limit $N, L \to +\infty$ of the generating function (4.8). Recall at this point that the effective series (5.3) was built so as to approach, in this limit, the same value $\mathcal{Q}^{\kappa}(x,t)$ as the original form factor series (4.8). We will therefore use the representation (5.12) to derive a suitable expression for $\mathcal{Q}^{\kappa}(x,t)$.

In principle, prior to taking the thermodynamic limit of a formal representation such as (5.12), one should first compute the effect of the translation operators: the operation of taking the thermodynamic limit is indeed *a priori* only allowed on expressions defined in terms of explicit holomorphic functions (*i.e.* which do not contain any operator valued functions). In Appendix D, such a procedure is explicitly performed so as to obtain, starting from (5.12), a particular series representation of the thermodynamic limit $\mathcal{Q}^{\kappa}(x,t)$ of $\mathcal{Q}^{\kappa}_N(x,t)$. More precisely, in Appendix D, we expand the finite-size determinant in (5.12) into a sum over determinants of all sub-matrices of $V(\lambda_j,\lambda_k)$, factorize the Cauchy part of each of these determinants, compute the effects of the translation operators, and *then* take the thermodynamic limit. This enables us to express $\mathcal{Q}^{\kappa}(x,t)$ as a series of multiple integrals coinciding, in the equal-time case, with the series expansion derived from the master equation representation in [45].

In fact, it is easy to convince oneself, by carrying out the computation in the reverse order, that the operation of taking the thermodynamic limit $N, L \to +\infty$ and the one of computing the effect of the translation operators do actually commute. Indeed, let us consider the expression (5.12) in which we have sent directly $N, L \to +\infty$, hence replacing sums by integrals in the prefactor and the finite-size determinant representation for X_N by the Fredholm determinant (B.11) :

$$\mathrm{e}^{\frac{-\beta x p_F}{\pi}} \;\boldsymbol{:}\; \mathrm{e}^{-\int\limits_{-q}^{q} \nu_\tau(\lambda)\,[ixu'(\lambda)+\widehat{g}'(\lambda)]\,\mathrm{d}\lambda} \det\left[I + V[\nu_\tau, u, \widehat{g}]\right] \cdot \mathscr{G}[\omega] \;\boldsymbol{:}\;\Big|_{\substack{\tau=0\\ \omega=0}}, \tag{6.1}$$

where

$$V[\nu,u,g](\lambda,\mu) = 4\frac{\sin[\pi\nu(\lambda)]\,\sin[\pi\nu(\mu)]}{2i\pi(\lambda-\mu)}\Big\{E_+(\lambda)\,E_-(\mu) - E_+(\mu)\,E_-(\lambda)\Big\} \tag{6.2}$$

with

$$E_-(\lambda) = \mathrm{e}^{-ix\frac{u(\lambda)}{2} - \frac{g(\lambda)}{2}}, \tag{6.3}$$

$$E_+(\lambda) = iE_-(\lambda)\left\{ \fint_{\mathbb{R}} \frac{\mathrm{d}\mu}{2\pi}\,\frac{E_-^{-2}(\mu)}{\mu-\lambda} + \frac{E_-^{-2}(\lambda)}{2}\cot[\pi\nu(\lambda)] \right\}. \tag{6.4}$$

Then, if we decompose the Fredholm determinant in (6.1) into its Fredholm series and subsequently compute the effect of the functional translation operators, we obtain *the same representation* (D.24) as by performing all these operations in the opposite order. This means (provided that the series (D.24) is convergent, which is not completely obvious but seems nevertheless a reasonable assumption) that the thermodynamic limit of the effective series (5.12), which is supposed to give the thermodynamic limit of the original form factor series (4.8), is effectively given by the expression (6.1), *i.e.*

$$\mathcal{Q}^{\kappa}(x,t) = \mathrm{e}^{\frac{-\beta x p_F}{\pi}} \,:\, \mathrm{e}^{-\int_{-q}^{q} \nu_{\mathcal{T}}(\lambda)\,[ixu'(\lambda)+\widehat{g}'(\lambda)]\,\mathrm{d}\lambda} \operatorname{det}\big[I+V[\nu_{\mathcal{T}},u,\widehat{g}]\big]\cdot\mathscr{G}[\omega]\,:\Big|_{\substack{\mathcal{T}=0\\ \omega=0}}, \tag{6.5}$$

with a kernel V given by (6.2)-(6.4). It also means that we can now use any other existing representation for the Fredholm determinant so as to compute the effect of the translation operators and recover standard scalar-valued functions. In fact, it is not very convenient for our purpose to expand the determinant into its Fredholm series like in Appendix D since the latter does not provide any information on its large-x asymptotic behavior. This large-x asymptotic behavior was studied in [60] precisely with the goal of computing the asymptotic behavior of (6.5), and a much more convenient (with respect to the $x \to +\infty$ limit) representation for the Fredholm determinant was obtained there.

6.2 Large-x asymptotic behavior of the Fredholm determinant

The large x asymptotic analysis of the Fredholm determinant with kernel (6.2)-(6.4) was performed in [60] using Riemann-Hilbert problem-based techniques. There, it was proven that, under some hypothesis about the regularity and behavior of the functions ν, u and g defined on some open neighborhood of the real axis (see [60] for more precisions), and provided that the function u has a unique saddle-point λ_0 on $\mathbb{R}$ (with $\lambda_0 \neq \pm q$),

$$\begin{aligned}\operatorname{det}_{[-q,q]}\big[I+V\big][\nu,u,g] &= \exp\left\{\int_{-q}^{q}\big[ixu'(\lambda)+g'(\lambda)\big]\,\nu(\lambda)\,\mathrm{d}\lambda\right\}\\ \times\Bigg\{\mathcal{B}_x[\nu,u] &+ \sum_{\epsilon=\pm1}\mathrm{e}^{i\epsilon x[u(q)-u(-q)]+\epsilon[g(q)-g(-q)]}\,\mathcal{B}_x[\nu+\epsilon,u]\\ &+\frac{1}{x^{\frac{3}{2}}}\sum_{\epsilon=\pm1}\mathrm{e}^{i\alpha x[u(\lambda_0)-u(\epsilon q)]+\alpha[g(\lambda_0)-g(\epsilon q)]}\,b_1^{(\epsilon,\alpha)}[\nu,u]\,\mathcal{B}_x[\nu,u]+\mathcal{R}_x[\nu,u,g]\Bigg\},\end{aligned} \tag{6.6}$$

with $\alpha = +1$ in the space-like regime[8] ($\lambda_0 > q$) and $\alpha = -1$ in the time-like regime ($\lambda_0 \in]-q,q[$). In (6.6), the functional $\mathcal{B}_x$ reads

$$\mathcal{B}_x[\nu,u] = \frac{\mathrm{e}^{i\frac{\pi}{2}[\nu^2(q)-\nu^2(-q)]}\,\widetilde{\mathcal{B}}[\nu]}{[2qx\,(u'(q)+i0^+)]^{\nu^2(q)}\,[2qx\,u'(-q)]^{\nu^2(-q)}}, \tag{6.7}$$

[8] The case $\lambda_0 < -q$ was not considered in [60].

with $\widetilde{\mathcal{B}}[\nu]$ given by (A.22), and

$$b_1^{(\epsilon,\alpha)}[\nu,u] = \mathrm{e}^{-i\alpha\frac{\pi}{4}} \frac{[2qx\,u'(\epsilon q)+i0^+]^{2\epsilon\alpha\nu(\epsilon q)}}{\sqrt{-2\pi u''(\lambda_0)}\,u'(\epsilon q)} \frac{(-\epsilon)\,\nu(\epsilon q)}{(\lambda_0-\epsilon q)^2} \left(\frac{\lambda_0+q}{\lambda_0-q-i0^+}\right)^{-2\alpha\nu(\lambda_0)} \\ \times (\mathrm{e}^{-2i\pi\nu(\lambda_0)}-1)^{1-\alpha}\,(\mathrm{e}^{-2i\pi\nu(\epsilon q)}-1)^{\alpha} \frac{\Gamma(1-\epsilon\alpha\nu(\epsilon q))}{\Gamma(1+\epsilon\alpha\nu(\epsilon q))}\,\mathrm{e}^{\alpha\widetilde{J}[\nu](\lambda_0)-\alpha\widetilde{J}[\nu](\epsilon q)}, \quad (6.8)$$

where $\widetilde{J}$ is given by (A.25). $\mathcal{R}_x[\nu,u,g]$ is a remainder which is uniformly of order $\mathrm{O}\big(\frac{\log x}{x}\big)$ in what concerns the non-oscillating corrections, of order $\mathrm{O}\big(\mathcal{B}_x[\nu+\epsilon,u]\,\frac{\log x}{x}\big)$ in what concerns the oscillating corrections at $\mathrm{e}^{i\epsilon x[u(q)-u(-q)]}$, and of order $\mathrm{O}\big(\mathcal{B}_x[\nu,u]\,b_1^{(\epsilon,\alpha)}[\nu,u]\,\frac{\log x}{x^{5/2}}\big)$ in what concerns the oscillating corrections at $\mathrm{e}^{i\alpha x[u(\lambda_0)-u(\epsilon q)]}$. It was shown in [60] how to obtain a series representation for this remainder, the so-called Natte series, which possesses the property of being well ordered with respect to its large-x behavior (in can be shown that its n-th term is (uniformly) a $\mathrm{O}(x^{-na})$, for some $0<a<1$ that depends on ν, u and g). Hence, this series is well adapted for the study of the asymptotic behavior of the determinant (on the contrary to its Fredholm series). The form of such a Natte series is recalled in Appendix E.

Remark 6.1. Note that for $|\Re(\nu)|<1/2$ the first (non-oscillating) term $\mathcal{B}_x[\nu,u]$ in (6.6) is always leading, at large x, with respect to the other ones. This leading term will produce the leading asymptotic behavior of the generating function $\mathcal{Q}^{\kappa}(x,t)$, as we will see in the next subsection. However, recall that we have to differentiate twice with respect to x and with respect to β at $\beta=0$ in order to obtain the correlation function $\langle\, j(x,t)\, j(0,0)\,\rangle$. By such a process, the first oscillating corrections may become leading with respect to non-oscillating terms. This is the reason why we also consider these corrections in (6.6).

6.3 Asymptotic behavior of the correlation function

So as to obtain the asymptotic behavior of the correlation function, it remains to compute the effect of the functional translation operators on the representation (6.6) of (6.5). First, we observe that the exponential pre-factor of (6.5) is canceled by the one in (6.6):

$$\exp\left\{-\int_{-q}^{q}\nu_\tau(\mu)\,\widehat{g}'(\mu)\,\mathrm{d}\mu\right\}\exp\left\{\int_{-q}^{q}\nu_\tau(\mu)\,\widehat{g}'(\mu)\,\mathrm{d}\mu\right\}=1. \quad (6.9)$$

Such a simplification is justified by the fact that, when applying (6.9) on any functionals of ν_τ or ω, we observe an *algebraic* cancellation for each term of the $\boldsymbol{:}\cdot\boldsymbol{:}$ ordered series expansion in $\delta/\delta\tau$ and $\delta/\delta\omega$ for these exponents [9]. Therefore,

$$\mathcal{Q}^{\kappa}(x,t) = \mathrm{e}^{-\frac{\beta x p_F}{\pi}}\,\boldsymbol{:}\,\Bigg\{\mathcal{B}_x[\nu_\tau,u] + \sum_{\epsilon=\pm1}\mathrm{e}^{i\epsilon x[u(q)-u(-q)]+\epsilon[\widehat{g}(q)-\widehat{g}(-q)]}\,\mathcal{B}_x[\nu_\tau+\epsilon,u] \\ +\frac{1}{x^{\frac{3}{2}}}\sum_{\epsilon=\pm1}\mathrm{e}^{i\alpha x[u(\lambda_0)-u(\epsilon q)]+\alpha[\widehat{g}(\lambda_0)-\widehat{g}(\epsilon q)]}\,b_1^{(\epsilon,\alpha)}[\nu_\tau,u]\,\mathcal{B}_x[\nu_\tau,u] + \mathcal{R}_x[\nu_\tau,u,\widehat{g}]\Bigg\}\cdot\mathscr{G}[\omega]\,\boldsymbol{:}\,\Big|_{\substack{\tau=0\\ \omega=0}}. \quad (6.10)$$

[9] Such an algebraic cancellation is very similar to the one occuring when computing $(\sqrt{I-A})^2$ through a Taylor series expansion around zero.

It can be proved that, by computing the effect of the functional translation operators occuring in the remainder $\mathcal{R}_x[\nu_\tau, u, \widehat{g}]$, one obtains corrections that are of the same order (*i.e.* $\mathrm{O}(\frac{\log x}{x})$ corrections to each of the terms already present in the asymptotics) as originally in (6.6). This is explicitly done in Appendix E by using the so-called Natte series representation of the Fredholm determinant derived in [60]. More precisely, it is shown in Appendix E that the effect of the translation operators do not mix the orders in x among the different terms of this (well-ordered) series. Therefore, *the leading asymptotic behavior of the generating function follows directly from the above leading asymptotic behavior of the Fredholm determinant.*

Since no translation operator is applied on the first (non-oscillating) term of (6.10), we simply need to set $\tau = 0$ and $\omega = 0$ into the corresponding expressions. The action of the translation operators $\mathrm{e}^{\epsilon[\widehat{g}(q)-\widehat{g}(-q)]}$ on the second term of (6.10) results into replacing the function $\mathscr{G}[\omega]$ by

$$\mathscr{G}\left[\varpi_1\left(\cdot\,\middle|\begin{matrix}\epsilon q\\-\epsilon q\end{matrix}\right)\right] \equiv \mathcal{G}_1\begin{pmatrix}\epsilon q\\-\epsilon q\end{pmatrix}, \tag{6.11}$$

and the function ν_τ by the shift function

$$F\left(\cdot\,\middle|\begin{matrix}\epsilon q\\-\epsilon q\end{matrix}\right) = \frac{i\beta}{2\pi}Z - \big[\phi(\cdot,\epsilon q) - \phi(\cdot,-\epsilon q)\big] = \Big(\frac{i\beta}{2\pi} + \epsilon\Big)Z - \epsilon \tag{6.12}$$

associated to a state with one particle and one hole located on the opposite ends of the Fermi zone. Similarly, the action of the translation operators $\mathrm{e}^{\alpha[\widehat{g}(\lambda_0)-\widehat{g}(\epsilon q)]}$ results into replacing $\mathscr{G}[\omega]$ and ν_τ respectively by

$$\mathscr{G}\left[\varpi_1\left(\cdot\,\middle|\begin{matrix}\lambda_0\\\epsilon q\end{matrix}\right)\right] \equiv \mathcal{G}_1\begin{pmatrix}\lambda_0\\\epsilon q\end{pmatrix} \quad \text{and} \quad F\left(\cdot\,\middle|\begin{matrix}\lambda_0\\\epsilon q\end{matrix}\right) = \frac{i\beta}{2\pi}Z - \big[\phi(\cdot,\lambda_0) - \phi(\cdot,\epsilon q)\big] \tag{6.13}$$

in the space-like regime $\alpha = +1$, and by

$$\mathscr{G}\left[\varpi_1\left(\cdot\,\middle|\begin{matrix}\epsilon q\\\lambda_0\end{matrix}\right)\right] \equiv \mathcal{G}_1\begin{pmatrix}\epsilon q\\\lambda_0\end{pmatrix} \quad \text{and} \quad F\left(\cdot\,\middle|\begin{matrix}\epsilon q\\\lambda_0\end{matrix}\right) = \frac{i\beta}{2\pi}Z - \big[\phi(\cdot,\epsilon q) - \phi(\cdot,\lambda_0)\big] \tag{6.14}$$

in the time-like regime $\alpha = -1$. Therefore, we get

$$\begin{aligned}\mathcal{Q}^{\kappa}(x,t) = \mathrm{e}^{-\frac{\beta x p_F}{\pi}} \Bigg\{ &\mathcal{B}_x\Big[\frac{i\beta}{2\pi}Z, u\Big]\, \mathcal{G}_0\, \Big(1 + \mathrm{O}\Big(\frac{\log x}{x}\Big)\Big) \\ &+ \sum_{\epsilon=\pm1} \mathrm{e}^{i\epsilon x[u(q)-u(-q)]}\, \mathcal{B}_x\Big[\Big(\frac{i\beta}{2\pi}+\epsilon\Big)Z, u\Big]\, \mathcal{G}_1\begin{pmatrix}\epsilon q\\-\epsilon q\end{pmatrix}\, \Big(1 + \mathrm{O}\Big(\frac{\log x}{x}\Big)\Big) \\ &+ \frac{1}{x^{\frac{3}{2}}} \sum_{\epsilon=\pm1} \mathrm{e}^{ix[u(\lambda_0)-u(\epsilon q)]}\, b_1^{(\epsilon,+1)}\Big[F\left(\cdot\,\middle|\begin{matrix}\lambda_0\\\epsilon q\end{matrix}\right), u\Big]\, \mathcal{B}_x\Big[F\left(\cdot\,\middle|\begin{matrix}\lambda_0\\\epsilon q\end{matrix}\right), u\Big]\, \mathcal{G}_1\begin{pmatrix}\lambda_0\\\epsilon q\end{pmatrix}\, \Big(1 + \mathrm{O}\Big(\frac{\log x}{x}\Big)\Big) \Bigg\}\end{aligned} \tag{6.15}$$

in the space-like regime, and similar expressions in the time-like regime. Namely, in the time-like regime, the exponent in the last term changes sign and $b_1^{(\epsilon,+1)}$, $F\big(\cdot\,|\,{}^{\lambda_0}_{\epsilon q}\big)$, $\mathcal{G}_1\binom{\lambda_0}{\epsilon q}$ are replaced, respectively, by $b_1^{(\epsilon,-1)}$, $F\big(\cdot\,|\,{}^{\epsilon q}_{\lambda_0}\big)$ and $\mathcal{G}_1\binom{\epsilon q}{\lambda_0}$.

In order to obtain the leading asymptotic behavior of the correlation function of currents, it remains finally to compute the second x-derivative and second β-derivative at $\beta = 0$ of the previous result, *cf* (4.9). The derivatives of the first term (the non-oscillating one) in (6.15) produce the constant and the non-oscillating term appearing in (3.1) and (3.2) (we have used (A.20)). In their turns, the derivatives of the two types of oscillating corrections produce the corresponding oscillating terms in (3.1), (3.2), with amplitudes given by (A.28)-(A.30).

7 Conclusion

In this article, we have proposed a new method to derive, starting from first principles, the leading asymptotic behavior of the two-point correlation functions of quantum integrable systems. To explain the main steps of this method, we chose to focus on the example of the current-current correlation function of the quantum non-linear Schrödinger model. The case of the field/conjugated field correlator $\langle \Psi^\dagger(x,t)\,\Psi(0,0)\,\rangle$, together with a rigorous setting for carrying out all the manipulations with operator valued determinants, will be given in [59].

Our result goes beyond the CFT/Luttinger liquid based predictions: the saddle-point contributions that appear in the asymptotic behavior (3.1)-(3.2) involve excitations away from the Fermi surface and cannot be neglected for x/t finite.

Compared to the approach of [45], based on the master equation representation for the correlation functions, the present study relies directly on their form factor expansion. We would like to conclude this article by making here a few comments about the similarities and differences between the two methods.

Of course, since the master equation can be understood as the result of a summation over the form factors, the spirit of the two approaches is essentially the same. In particular, in both approaches, the asymptotic analysis relies essentially on the *singular part* of the form factor which is explicitly extracted (in the form of a Cauchy determinant squared), whereas the (model-dependent) regular part is treated as a dressing part which is formally decoupled (or linked to decoupled functions) to make the analysis possible. This enables one to draw a link between the quantity to estimate and the Fredholm determinant of a generalized sine kernel. The asymptotic analysis of this Fredholm determinant then leads to the asymptotic behavior of the correlation function.

However, there exist some essential differences between the two approaches. In [45], the correlation function was expanded into a series whose building blocks were the so-called *cycle integrals*. These cycle integrals could then be related to a Fredholm determinant, which allowed one to access to their asymptotic behavior. The physical interpretation of these objects was however not clear, and they happened to be quite indirectly related to the correlation functions. In fact, once the asymptotic behavior of these cycle integrals established, one had to sum up the series so as to obtain the asymptotic behavior of the correlation function. In this process, the main problem was that *not only the leading asymptotic behavior of individual cycle integrals was contributing to the leading order for the correlation functions*: one had to perform a fine and non-trivial study [47, 45, 62] of the asymptotic series so as to gather terms *at all order* that finally, when summed up, were contributing to the leading order. In fact, all these terms were rearranging themselves into some generalization of a multiple Lagrange series producing, when summed up, a *dressing of bare quantities* (energy and momentum) *into dressed ones*. On the contrary, here, one deals from the very beginning with objects having a clear physical interpretation, the form factors. In this context, *the dressed quantities appear naturally when considering the thermodynamic limit of these form factors*. Hence, performing the summation over the form factors, we can connect the series to a Fredholm determinant that is already expressed in terms of these dressed quantities. Therefore, the asymptotic study is much simpler: it happens that *the leading asymptotic behavior of the Fredholm determinant gives directly the leading asymptotic behavior of the correlation function*. In other words, there is no need to resort to highly non-trivial summation as in [45].

In fact, we would like to stress that the whole process described in the core of this article is quite simple and direct. Once the form factor series written down and the effective contributing part of the form factor established, the result of the summation can be expressed, in the thermodynamic limit, in terms of a Fredholm determinant. The leading asymptotic behavior of the correlation function follows directly from the leading asymptotic behavior of this Fredhom determinant. Even if, in the course of the computations, we use some functional translation operators to relate our series to a decoupled one, whenever the action of these functional translation operators has to be computed, it is quite straightforward, *i.e.* it produces a simple translation of the functional on which it acts. On the contrary, if one wants to recover, starting from the form factor expansion, a series similar to the one studied in [45], the computations are much more involved (see Appendix D). In particular, the action of the translation operators on the series of Appendix D produces non-trivial effects: one has to deal with summations of generalized Lagrange series, which results into an undressing of the dressed quantities into bare ones. It is therefore clear that most of the mathematical complexity corresponding to this non-trivial summation has already been taken into account by the fact that we had some precise description (the particle-hole picture) of our form factors, which allowed us from the very beginning to deal with dressed quantities instead of bare ones. Note that the simpler setting of our method enables us here to consider the time-dependent case, which is not so obvious within the approach of [45]. Note also that, in principle, there is no intrinsic obstruction preventing us from obtaining higher order terms in the asymptotic expansion for the correlation functions. For this it is enough to refine the asymptotic expansion of the Fredholm determinant.

Of course, there is a price to pay for this simplicity: the need of some clear picture to describe the form factors. Whereas in [45] all kinds of Bethe roots were automatically taken into account within the master equation framework (and this without any precise study of the spectrum), here we strongly rely on the fact that the spectrum of the model we consider is particularly simple: all excited states can be described in terms of particles and holes. In order to apply our method to the case of the XXZ spin chain, for example, one has also to take into account the contribution of complex solutions. Although it seems that, for the time-independent case, these solutions do not contribute to the leading asymptotic behavior of correlation functions (see [42]), the question remains open in the time-dependent case. This will be the subject of a further study.

Acknowledgements

V. T. is supported by CNRS and by the ANR grant "DIADEMS". K. K. K. is supported by the EU Marie-Curie Excellence Grant MEXT-CT-2006-042695. K. K. K. would like to thank the Theoretical Physics group of the Laboratory of Physics at ENS Lyon for hospitality, which makes this collaboration possible. V.T. would like to thank LPTHE (Paris VI University) for hospitality. We would like to thank N. Kitanine, J. M. Maillet and N. A. Slavnov for stimulating discussions and comments. V. T. would also like to thank M. Civelli and S. Teber for their interest in this work.

A The form factors and their thermodynamic limit

In this appendix, we present the explicit expression and the leading thermodynamic behavior of the combinations of finite-volume form factors which appear in each term of the series (4.1), *i.e.*, with the notations of Section 4, of

$$\mathrm{e}^{-it\mathcal{E}_{\mathrm{ex}}}\frac{\langle\,\psi_g\,|\,j(x,0)\,|\,\psi'\,\rangle\,\langle\,\psi'\,|\,j(0,0)\,|\,\psi_g\,\rangle}{||\psi_g||^2\cdot||\psi'||^2}$$
$$=\frac{1}{2}\partial_x^2\partial_\beta^2\bigg\{\mathrm{e}^{ix\mathcal{P}^\kappa_{\mathrm{ex}}-it\mathcal{E}^\kappa_{\mathrm{ex}}}\bigg|\frac{\langle\,\psi(\{\lambda_j\})\mid\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi_\kappa(\{\mu_{\ell_j}\})\|\cdot\|\psi(\{\lambda_j\})\|}\bigg|^2\bigg\}_{\beta=0}. \quad \text{(A.1)}$$

These results being the complete analogues, in the case of the NLS model, of those derived in [46, 43] for the XXZ chain, we skip the details of the computations.

A.1 Thermodynamic limit of the space and time-dependent phase factor

The thermodynamic limit of $\mathrm{e}^{ix\mathcal{P}^\kappa_{\mathrm{ex}}-it\mathcal{E}^\kappa_{\mathrm{ex}}}$ generates dressed quantities. This limit being completely smooth, it is easy to see that

$$\begin{aligned}\mathrm{e}^{ix\mathcal{P}^\kappa_{\mathrm{ex}}-it\mathcal{E}^\kappa_{\mathrm{ex}}}&=\exp\bigg\{ix\int_{-q}^{q}u_0'(\lambda)\,F(\lambda)\,\mathrm{d}\lambda+ix\sum_{a=1}^{n}\big[u_0(\mu_{p_a})-u_0(\mu_{h_a})\big]\bigg\}(1+\mathrm{O}(1/L))\\&=\exp\bigg\{-\frac{x\beta}{2\pi}\int_{-q}^{q}u_0'(\lambda)\,Z(\lambda)\,\mathrm{d}\lambda+ix\sum_{a=1}^{n}\big[u(\mu_{p_a})-u(\mu_{h_a})\big]\bigg\}(1+\mathrm{O}(1/L)),\end{aligned}$$

in which we have used the explicit form (2.14) of the shift function F, and where u is the following combination of dressed quantities $u(\lambda)=p(\lambda)-t\varepsilon(\lambda)/x$. This identification was possible due to

$$u(\lambda)=u_0(\lambda)-\int_{-q}^{q}u_0'(\mu)\,\phi(\mu,\lambda)\,\mathrm{d}\mu. \quad \text{(A.2)}$$

Note that, due to the fact that ε_0' is an odd function whereas the dressed charge Z is even, one has

$$\int_{-q}^{q}\mathrm{d}\lambda\,Z(\lambda)\,u_0'(\lambda)=2p_F. \quad \text{(A.3)}$$

A.2 Representation of the normalized scalar product

Let $\{\lambda_j\}$ be the solution of the system of Bethe equations (2.2) parametrizing the ground state of (2.1) and let $\{\mu_{\ell_j}\}$ be a set of Bethe roots of (2.7) parametrizing an excited state above the κ-twisted ground state in the N-particle sector. Then the normalized modulus squared of the corresponding overlap scalar product in finite volume can be represented as

$$\bigg|\frac{\langle\,\psi(\{\lambda_j\})\mid\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle}{\|\psi_\kappa(\{\mu_{\ell_j}\})\|\cdot\|\psi(\{\lambda_j\})\|}\bigg|^2=\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\})\cdot\widehat{\mathcal{W}}_N(\{\lambda_j\},\{\mu_{\ell_j}\})\cdot\widehat{\mathcal{A}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}), \quad \text{(A.4)}$$

where $\widehat{D}_N$, $\widehat{\mathcal{W}}_N$ and $\widehat{\mathcal{A}}_N$ are given by

$$\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\})=\prod_{j=1}^{N}\frac{\sin^2[\pi\widehat{F}(\lambda_j)]}{\pi^2L^2\,\widehat{\xi}'(\lambda_j)\,\widehat{\xi}_\kappa'(\mu_{\ell_j})}\cdot\bigg[\det_N\frac{1}{\lambda_j-\mu_{\ell_k}}\bigg]^2, \quad \text{(A.5)}$$

$$\widehat{\mathcal{W}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \prod_{j,k=1}^{N} \frac{(\lambda_j-\mu_{\ell_k}-ic)\,(\mu_{\ell_j}-\lambda_k-ic)}{(\lambda_j-\lambda_k-ic)\,(\mu_{\ell_j}-\mu_{\ell_k}-ic)}, \tag{A.6}$$

$$\widehat{\mathcal{A}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \frac{(1-\kappa)^2}{(1-\mathrm{e}^{-2i\pi\widehat{F}(\theta_1)})(1-\mathrm{e}^{-2i\pi\widehat{F}(\theta_2)})} \prod_{a=1}^{N} \frac{(\theta_1-\lambda_a+ic)(\theta_2-\lambda_a+ic)}{(\theta_1-\mu_{\ell_a}+ic)(\theta_2-\mu_{\ell_a}+ic)}$$
$$\times \frac{\det_\Gamma\big[I+\frac{1}{2i\pi}\widehat{U}^{(\lambda)}_{\theta_1}(\omega,\omega')\big]\cdot\det_\Gamma\big[I+\frac{1}{2i\pi}\widehat{U}^{(\lambda)}_{\theta_2}(\omega,\omega')\big]}{\det_N\Big[\delta_{jk}-\frac{K(\lambda_j-\lambda_k)}{2\pi L\,\widehat{\xi}'(\lambda_k)}\Big]\cdot\det_N\Big[\delta_{jk}-\frac{K(\mu_{\ell_j}-\mu_{\ell_k})}{2\pi L\,\widehat{\xi}'_\kappa(\mu_{\ell_k})}\Big]}. \tag{A.7}$$

We recall that $\widehat{\xi}_\kappa$ and $\widehat{\xi}$ denote respectively the excited state counting function (2.8) and the ground state counting function at $\kappa=1$, whereas $\widehat{F}$ is the finite-size shift function (2.9). Here θ_1 and θ_2 are some arbitrary real parameters. The integral operator $I+\frac{1}{2i\pi}\widehat{U}^{(\lambda)}_\theta$ acts on the closed contour Γ surrounding the ground state roots $\{\lambda_j\}$ and no other singularity of the kernel. This kernel reads

$$\widehat{U}^{(\lambda)}_\theta(\omega,\omega') = -\prod_{a=1}^{N}\frac{(\omega-\mu_{\ell_a})(\omega-\lambda_a+ic)}{(\omega-\lambda_a)(\omega-\mu_{\ell_a}+ic)}\cdot\frac{K_\kappa(\omega-\omega')-K_\kappa(\theta-\omega')}{1-\mathrm{e}^{-2i\pi\widehat{F}(\omega)}}, \tag{A.8}$$

with

$$K_\kappa(\omega) = \frac{1}{\omega+ic}-\frac{\kappa}{\omega-ic}. \tag{A.9}$$

Remark A.1. Although each individual term, in (A.7), does depend on the set of auxiliary parameters θ_k, the overall combination $\widehat{\mathcal{A}}_N$ does not. This was proven in [45].

The factor $\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\})$ (A.5) is the so-called discrete part of the form factor. It contains all the non-trivial "singular part" of the form factor (see [43]). On the contrary, the factor

$$\widehat{\mathcal{G}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) \equiv \widehat{\mathcal{W}}_N(\{\lambda_j\},\{\mu_{\ell_j}\})\cdot\widehat{\mathcal{A}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) \tag{A.10}$$

admits a smooth thermodynamic limit. It can be thought of as a dressing function. It is equal to 1 at the free-fermion point.

In the case of an excited state $|\,\psi_\kappa(\{\mu_{\ell_j}\})\,\rangle$ with a finite number of particles and holes as described in Section 2, it is easy to compute the thermodynamic limit of the dressing function $\widehat{\mathcal{G}}_N$ (see [43]):

$$\lim_{L,N\to\infty}\widehat{\mathcal{G}}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \mathcal{G}_n\begin{pmatrix}\{\mu_{p_a}\}\\\{\mu_{h_a}\}\end{pmatrix} \equiv \mathcal{W}_n\begin{pmatrix}\{\mu_{p_a}\}\\\{\mu_{h_a}\}\end{pmatrix}\cdot\mathcal{A}_n\begin{pmatrix}\{\mu_{p_a}\}\\\{\mu_{h_a}\}\end{pmatrix}, \tag{A.11}$$

with

$$\mathcal{W}_n\begin{pmatrix}\{\mu_{p_a}\}\\\{\mu_{h_a}\}\end{pmatrix} = \prod_{a,b=1}^{n}\frac{(\mu_{p_a}-\mu_{h_b}-ic)(\mu_{h_a}-\mu_{p_b}-ic)}{(\mu_{p_a}-\mu_{p_b}-ic)(\mu_{h_a}-\mu_{h_b}-ic)}$$
$$\times\,\mathrm{e}^{-2i\pi\sum_{a=1}^{n}\sum_{\epsilon=\pm}\{\mathcal{C}[F](\mu_{p_a}+i\epsilon c)-\mathcal{C}[F](\mu_{h_a}+i\epsilon c)\}+C_0[F]}, \tag{A.12}$$

$$\mathcal{A}_n\begin{pmatrix}\{\mu_{p_a}\}\\\{\mu_{h_a}\}\end{pmatrix} = (1-\kappa)^2\frac{\mathrm{e}^{-2i\pi\{\mathcal{C}[F](\theta_1+ic)+\mathcal{C}[F](\theta_2+ic)\}}}{(1-\mathrm{e}^{-2i\pi F(\theta_1)})(1-\mathrm{e}^{-2i\pi F(\theta_2)})}\prod_{j=1}^{n}\frac{(\theta_1-\mu_{h_j}+ic)(\theta_2-\mu_{h_j}+ic)}{(\theta_1-\mu_{p_j}+ic)(\theta_2-\mu_{p_j}+ic)}$$
$$\times\frac{\det\big[I+\frac{1}{2i\pi}U_{\theta_1}\big]\,\det\big[I+\frac{1}{2i\pi}U_{\theta_2}\big]}{\det^2\big[I-\frac{1}{2\pi}K\big]}. \tag{A.13}$$

In these expressions, F denotes the shift function F (2.11), $\mathcal{C}[\nu]$ is the rational Cauchy transform of the function ν on $[-q,q]$,

$$\mathcal{C}[\nu](\lambda) = \int_{-q}^{q} \frac{\mathrm{d}\mu}{2i\pi} \frac{\nu(\mu)}{\mu-\lambda}, \tag{A.14}$$

whereas C_0 is the following functional:

$$C_0[\nu] = -\int_{-q}^{q} \mathrm{d}\lambda\,\mathrm{d}\mu\, \frac{\nu(\lambda)\,\nu(\mu)}{(\lambda-\mu-ic)^2}. \tag{A.15}$$

The integral kernel U_θ takes the form

$$U_\theta(\omega,\omega') = -\prod_{a=1}^{n} \frac{(\omega-\mu_{p_a})(\omega-\mu_{h_a}+ic)}{(\omega-\mu_{h_a})(\omega-\mu_{p_a}+ic)} \frac{\mathrm{e}^{2i\pi\mathcal{C}[F](\omega)}\{K_\kappa(\omega-\omega')-K_\kappa(\theta-\omega')\}}{\mathrm{e}^{2i\pi\mathcal{C}[F](\omega+ic)}(1-\mathrm{e}^{-2i\pi F(\omega)})}. \tag{A.16}$$

Remark A.2. The dependence on the particle/hole rapidities of $\mathcal{G}_n$ is twofold: explicitly in the above expressions, and also in the shift function F. The latter is a holomorphic function of $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$, *cf* (2.14). Therefore, $\mathcal{G}_n$ is itself a holomorphic function of $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$. This function can, in fact, be understood as a functional $\mathscr{G}[\varpi_n]$ of the function ϖ_n (4.14) (note that the dependence in n is exclusively contained in ϖ_n). For this it is enough to observe that

$$F(\lambda) = \frac{i\beta}{2\pi} Z(\lambda) - \int_{\Gamma(\mathbb{R})} \frac{\mathrm{d}\mu}{2i\pi} \phi(\lambda,\mu)\,\varpi_n\left(\mu \middle| \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right), \tag{A.17}$$

where the contour $\Gamma(\mathbb{R})$ surrounds the real axis counterclockwise, and that

$$\prod_{a=1}^{n} \mathrm{e}^{f(\mu_{p_a})-f(\mu_{h_a})} = \exp\left\{ \int_{\Gamma(\mathbb{R})} \frac{\mathrm{d}\lambda}{2i\pi} f(\lambda)\,\varpi_n\left(\lambda \middle| \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right) \right\} \tag{A.18}$$

for any holomorphic functions f with a sufficiently mild growth at infinity on $\Gamma(\mathbb{R})$. In particular, we have

$$\begin{aligned}\prod_{a,b=1}^{n} &\frac{(\mu_{p_a}-\mu_{h_b}-ic)(\mu_{h_a}-\mu_{p_b}-ic)}{(\mu_{p_a}-\mu_{p_b}-ic)(\mu_{h_a}-\mu_{h_b}-ic)} \\ &= \exp\left\{ -\int_{\Gamma(\mathbb{R})} \frac{\mathrm{d}\lambda\,\mathrm{d}\mu}{(2i\pi)^2} \log(\lambda-\mu-ic)\,\varpi_n\left(\lambda \middle| \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right) \varpi_n\left(\mu \middle| \begin{matrix}\{\mu_{p_a}\}\\ \{\mu_{h_a}\}\end{matrix}\right) \right\}.\end{aligned} \tag{A.19}$$

Remark A.3. The explicit factor $(1-\kappa)^2$ in (A.13) stresses that, for generic configurations of the parameters $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$, $\mathcal{G}_n$, considered as a function of β, has in general a zero of order 2 at $\beta = 0$. Note however that there exist some exceptions, in particular at the free fermion point $c = +\infty$, when $\mathcal{G}$ is identically 1 or, for general c, when the excited state coincides with the ground state in the limit $\beta \to 0$ (*i.e.* for $n = 0$). In the latter case, the limit can be taken as in [45] and we obtain

$$\lim_{\beta\to 0} \mathcal{G}_0 = 1. \tag{A.20}$$

The study of the thermodynamic behavior of the factor $\widehat{D}_N$ (A.5) is slightly more technical. Proceeding as in [43], one obtains that, for a n particle/hole excited state,

$$\widehat{D}_N(\{\lambda_j\},\{\mu_{\ell_j}\}) = \left(\frac{L}{2\pi}\right)^{-F^2(q)-F^2(-q)-2n} [2q\, p'(q)]^{-F^2(q)-F^2(-q)}\, \widetilde{\mathcal{B}}[F]$$
$$\times\, \Gamma^2\left(\begin{matrix}\{p_j\},\{p_j-N+F(\mu_{p_j})\},\{h_j+F(\mu_{h_j})\},\{N+1-h_j-F(\mu_{h_j})\}\\ \{p_j-N\},\{p_j+F(\mu_{p_j})\},\{h_j\},\{N+1-h_j\}\end{matrix}\right)$$
$$\times\, \mathrm{det}_n^2\left[\frac{1}{\mu_{p_j}-\mu_{h_k}}\right] \prod_{j=1}^{n} \frac{\sin^2[\pi F(\mu_{h_j})]\, \mathrm{e}^{J[F](\mu_{p_j})-J[F](\mu_{h_j})}}{\pi^2\, p'(\mu_{p_j})\, p'(\mu_{h_j})} \left(1+\mathrm{O}\left(\frac{\log L}{L}\right)\right). \quad \text{(A.21)}$$

The expression (A.21) is given in terms of the below functional of the shift function (2.14):

$$\widetilde{\mathcal{B}}[\nu] = \mathrm{e}^{C_1[\nu]}\, \frac{G^2(1+\nu(q))\, G^2(1-\nu(-q))}{(2\pi)^{\nu(q)-\nu(-q)}} \quad \text{(A.22)}$$

where G is the Barnes G function and

$$C_1[\nu] = \frac{1}{2}\int_{-q}^{q} \mathrm{d}\lambda\, \mathrm{d}\mu\, \frac{\nu'(\lambda)\nu(\mu)-\nu'(\mu)\nu(\lambda)}{\lambda-\mu}$$
$$+\, \nu(q)\int_{-q}^{q} \frac{\nu(q)-\nu(\lambda)}{q-\lambda}\mathrm{d}\lambda + \nu(-q)\int_{-q}^{q} \frac{\nu(-q)-\nu(\lambda)}{q+\lambda}\mathrm{d}\lambda. \quad \text{(A.23)}$$

We have also defined

$$J[F](\lambda) = 2F(\lambda)\, \log\left[\frac{\lambda-q}{\lambda+q}\frac{p(\lambda)-p(-q)}{p(\lambda)-p(q)}\right] + \widetilde{J}[F](\lambda), \quad \text{(A.24)}$$

with

$$\widetilde{J}[\nu](\lambda) = 2\int_{-q}^{q} \frac{\nu(\mu)-\nu(\lambda)}{\mu-\lambda}\, \mathrm{d}\mu. \quad \text{(A.25)}$$

Finally,

$$\Gamma\left(\begin{matrix}\{a_k\}\\ \{b_k\}\end{matrix}\right) = \prod_{k=1}^{n} \frac{\Gamma(a_k)}{\Gamma(b_k)}, \quad \text{(A.26)}$$

with the prescription that, should some of the particles in (A.21) have their rapidities to the left of the Fermi zone, the arguments of the Γ-functions have to be understood as limits

$$\frac{\Gamma(p_k)}{\Gamma(p_k-N)} = \lim_{\epsilon\to 0} \frac{\Gamma(p_k+\epsilon)}{\Gamma(p_k-N+\epsilon)}. \quad \text{(A.27)}$$

Note that in (A.21) the thermodynamic limit has only been taken partly. Indeed, since the complete thermodynamic behavior of (A.21) depends on whether the corresponding particles and holes remain or not at a finite distance from the Fermi boundaries. In the next subsection, we particularize these expressions to the special form factors with one particle and one hole which appear in the results (3.1), (3.2).

A.3 Explicit value of the amplitudes

We collect here the explicit values of the non-universal amplitudes appearing in (3.1)-(3.2), which correspond to the normalized one particle/hole form factors (3.5)-(3.9). They can be easily obtained from the expressions (A.12)-(A.21) and read

$$\begin{aligned}
\left|\mathcal{F}^{q}_{-q}\right|^2 &= \left|\mathcal{F}^{-q}_{q}\right|^2 \\
&= -\frac{2p_F^2}{\pi^2}\,\frac{\widetilde{\mathcal{B}}\big[F^{q}_{-q}\big]\,\mathrm{e}^{\widetilde{J}[F^{q}_{-q}](q)-\widetilde{J}[F^{q}_{-q}](-q)}}{[2q\,p'(q)]^{[F^{q}_{-q}(q)+1]^2+[F^{q}_{-q}(-q)+1]^2}}\,\Gamma^2\big(1+F^{q}_{-q}(q)\big)\,\Gamma^2\big(1+F^{q}_{-q}(-q)\big) \\
&\qquad\qquad \times \partial_\beta^2 \sin^2\Big[i\frac{\beta}{2}\mathcal{Z}+\pi F^{q}_{-q}(-q)\Big]\,\mathcal{G}_1\begin{pmatrix} q \\ -q\end{pmatrix}\bigg|_{\beta=0} \\
&= -\frac{2p_F^2}{\pi^2}\,\frac{\widetilde{\mathcal{B}}\big[F^{-q}_{q}\big]\,\mathrm{e}^{\widetilde{J}[F^{-q}_{q}](-q)-\widetilde{J}[F^{-q}_{q}](q)}}{[2q\,p'(q)]^{[F^{q}_{-q}(q)-1]^2+[F^{q}_{-q}(-q)-1]^2}}\,\Gamma^2\big(1-F^{-q}_{q}(q)\big)\,\Gamma^2\big(1-F^{-q}_{q}(-q)\big) \\
&\qquad\qquad \times \partial_\beta^2 \sin^2\Big[i\frac{\beta}{2}\mathcal{Z}+\pi F^{-q}_{q}(q)\Big]\,\mathcal{G}_1\begin{pmatrix} -q \\ q\end{pmatrix}\bigg|_{\beta=0},
\end{aligned} \tag{A.28}$$

in terms of the shift functions $F^{\pm q}_{\mp q}(\lambda) = \phi(\lambda, \mp q) - \phi(\lambda, \pm q) = \pm Z(\lambda) \mp 1$;

$$\begin{aligned}
\left|\mathcal{F}^{\lambda_0}_{\epsilon q}\right|^2 &= -\left(\frac{p(\lambda_0)-\epsilon p_F}{\lambda_0-\epsilon q}\right)^2 \left(\frac{\lambda_0-q}{\lambda_0+q}\right)^{2F^{\lambda_0}_{\epsilon q}(\lambda_0)} \frac{[2q\,p'(q)]^{-[F^{\lambda_0}_{\epsilon q}(q)]^2-[F^{\lambda_0}_{\epsilon q}(-q)]^2+2\epsilon F^{\lambda_0}_{\epsilon q}(\epsilon q)}}{2\pi^2\,p'(\lambda_0)\,p'(q)} \\
&\qquad \times \widetilde{\mathcal{B}}\big[F^{\lambda_0}_{\epsilon q}\big]\,\mathrm{e}^{\widetilde{J}[F^{\lambda_0}_{\epsilon q}](\lambda_0)-\widetilde{J}[F^{\lambda_0}_{\epsilon q}](\epsilon q)}\,\Gamma^2\big(1-\epsilon F^{\lambda_0}_{\epsilon q}(\epsilon q)\big) \\
&\qquad\qquad \times \partial_\beta^2 \sin^2\Big[i\frac{\beta}{2}\mathcal{Z}+\pi F^{\lambda_0}_{\epsilon q}(\epsilon q)\Big]\,\mathcal{G}_1\begin{pmatrix} \lambda_0 \\ \epsilon q\end{pmatrix}\bigg|_{\beta=0},
\end{aligned} \tag{A.29}$$

in terms of the shift functions $F^{\lambda_0}_{\epsilon q} = \phi(\lambda, \epsilon q) - \phi(\lambda, \lambda_0)$, with $\epsilon = \pm 1$;

$$\begin{aligned}
\left|\mathcal{F}^{\epsilon q}_{\lambda_0}\right|^2 &= -\left(\frac{p(\lambda_0)-\epsilon p_F}{\lambda_0-\epsilon q}\right)^2 \left(\frac{q-\lambda_0}{q+\lambda_0}\right)^{-2F^{\epsilon q}_{\lambda_0}(\lambda_0)} \frac{[2q\,p'(q)]^{-[F^{\epsilon q}_{\lambda_0}(q)]^2-[F^{\epsilon q}_{\lambda_0}(-q)]^2-2\epsilon F^{\epsilon q}_{\lambda_0}(\epsilon q)}}{2\pi^2\,p'(\lambda_0)\,p'(q)} \\
&\qquad \times \widetilde{\mathcal{B}}\big[F^{\epsilon q}_{\lambda_0}\big]\,\mathrm{e}^{\widetilde{J}[F^{\epsilon q}_{\lambda_0}](\epsilon q)-\widetilde{J}[F^{\epsilon q}_{\lambda_0}](\lambda_0)}\,\Gamma^2\big(1+\epsilon F^{\epsilon q}_{\lambda_0}(\epsilon q)\big) \\
&\qquad\qquad \times \partial_\beta^2 \sin^2\Big[i\frac{\beta}{2}Z(\lambda_0)+\pi F^{\epsilon q}_{\lambda_0}(\lambda_0)\Big]\,\mathcal{G}_1\begin{pmatrix} \epsilon q \\ \lambda_0\end{pmatrix}\bigg|_{\beta=0},
\end{aligned} \tag{A.30}$$

in terms of the shift functions $F^{\epsilon q}_{\lambda_0} = \phi(\lambda, \lambda_0) - \phi(\lambda, \epsilon q)$, with $\epsilon = \pm 1$. We recall that the functionals $\widetilde{\mathcal{B}}$ and $\widetilde{J}$ are respectively given by (A.22) and (A.25), and that the expressions of the dressing functions $\mathcal{G}_1$ are obtained through (A.12)-(A.13).

Remark A.4. For generic parameters, the β-derivatives will apply directly to the factor $(1-\kappa)^2$ of $\mathcal{G}_1$ (see remark A.3). However, in the free fermion point $c = +\infty$, they will apply on the sinus squared, since in that case the corresponding shift functions vanish (and $\mathcal{G}_1 \equiv 1$). There may exist other particular values of the parameters for which the shift function becomes an integer but the overall expressions are anyway smooth (we may then have to apply the β-derivatives to other factors such as the Barnes functions).

B Study of a generalized free fermionic generating function

In this appendix, we consider the case of a generalized free-fermionic model for which the effective generating function (5.3) can be represented as a finite-size determinant. Namely, we suppose here that the shift function does not depend on the position of the roots parametrizing the corresponding excited state and that the dressing function $\mathcal{G}_n$ is separated (which happens in particular at the free fermion point $c = +\infty$ of the NLS model).

More precisely, for a given counting function $\widehat{\xi}$ (for example the ground state counting function of the NLS model), which defines a set of real parameters λ_j, $j = 1, \dots, N$ by $\widehat{\xi}(\lambda_j) = j/L$, we consider the functional

$$X_N\big[\nu, E_-^2\big] = \sum_{\substack{\ell_1 < \dots < \ell_N \\ \ell_j \in \mathcal{B}_L}} \prod_{j=1}^{N} \frac{E_-^2(\lambda_j)}{E_-^2(\mu_{\ell_j})} \cdot \prod_{j=1}^{N} \frac{\sin^2[\pi\nu(\lambda_j)]}{\pi^2 L^2\, \widehat{\xi}'(\lambda_j)\, \widehat{\xi}'_\nu(\mu_{\ell_j})} \cdot \left[\det_N \frac{1}{\lambda_j - \mu_{\ell_k}}\right]^2, \tag{B.1}$$

where E_-^{-1} and ν are holomorphic functions on some open neighborhood of $\mathbb{R}$. The above expression should be understood as follows:

- the function $\widehat{\xi}_\nu$ is defined by $\widehat{\xi}_\nu = \widehat{\xi} - L^{-1}\nu$;
- the multiple sum runs through all the possible choices of N-tuples of integers $\ell_1 < \dots < \ell_N$ belonging to the set $\mathcal{B}_L = \{j \in \mathbb{Z} \mid -w_L < j < w_L\}$, where $w_L \sim L^{1+\epsilon}$, $\epsilon > 0$, is some cut-off which goes to $+\infty$ with L.
- for a given set of integers ℓ_j, the parameters μ_{ℓ_j} are obtained as the pre-image of ℓ_j/L by the function $\widehat{\xi}_\nu$. This means, in particular, that they depend on ν.

Moreover, we restrict our study those functions ν that make $\widehat{\xi}_\nu$ a "good" counting function (*i.e.* ensuring a one-to-one correspondence between the rapidities and the integers).

Proposition B.1. *The functional $X_N\big[\nu, E_-^2\big]$ can be recast as the following finite size determinant:*

$$X_N\big[\nu, E_-^2\big] = \det_N \left[\delta_{jk} + \frac{V^{(L)}(\lambda_j, \lambda_k)}{L\,\widehat{\xi}'(\lambda_k)}\right]. \tag{B.2}$$

The corresponding finite-size kernel is given as

$$V^{(L)}(\lambda, \mu) = 4\frac{\sin[\pi\nu(\lambda)]\,\sin[\pi\nu(\mu)]}{2i\pi(\lambda - \mu)} \Big\{E_+^{(L)}(\lambda)\, E_-(\mu) - E_+^{(L)}(\mu)\, E_-(\lambda)\Big\}, \tag{B.3}$$

where

$$E_+^{(L)}(\lambda) = iE_-(\lambda) \left\{ \fint_{A_L}^{B_L} \frac{d\mu}{2\pi} \frac{E_-^{-2}(\mu)}{\mu - \lambda} + \frac{E_-^{-2}(\lambda)}{2} \cot[\pi\nu(\lambda)] + I_L\big[\nu, E_-^{-2}\big](\lambda) \right\}, \tag{B.4}$$

In this expression, the last term $I_L\big[\nu, E_-^{-2}\big](\lambda)$ corresponds to the integral

$$I_L\big[\nu, E_-^{-2}\big](\lambda) = \int_{\mathscr{C}_\uparrow} \frac{dz}{2\pi} \frac{E_-^{-2}(z)}{z - \lambda} \frac{1}{1 - e^{-2i\pi L \widehat{\xi}_\nu(z)}} + \int_{\mathscr{C}_\downarrow} \frac{dz}{2\pi} \frac{E_-^{-2}(z)}{z - \lambda} \frac{1}{e^{2i\pi L \widehat{\xi}_\nu(z)} - 1}. \tag{B.5}$$

Finally, the integration endpoints A_L and B_L are such that $L\widehat{\xi}_\nu(A_L) = -w_L - 1/2$ and $L\widehat{\xi}_\nu(B_L) = w_L + 1/2$, and $\mathscr{C}_{\uparrow/\downarrow}$ are some oriented contours, included in the joint domain of holomorphy of ν and E_-^{-1}. Theses contours join the points A_L and B_L through the upper/lower half plane respectively, it has been depicted on Fig. 1.

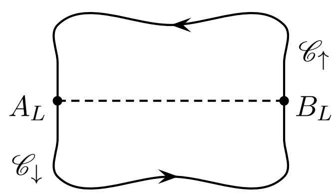

Figure 1: Contour $\mathscr{C}_{\uparrow/\downarrow}$.

Proof — The summand in (B.1) is a symmetric function of the N summation variables μ_{ℓ_j} which vanishes on the diagonals $\ell_j = \ell_k$, $j \neq k$. Therefore, we can replace the summation over the fundamental simplex $\ell_1 < \cdots < \ell_N$ in the N^{th} power cartesian product $\mathcal{B}_L^N$ by a summation over the whole space $\mathcal{B}_L^N$, provided that we divide the result by $N!$. The summation domain being now symmetric, we can invoke the antisymmetry of the determinants so as to replace one of the Cauchy determinants by $N!$ times the product of its diagonal entries. This last operation produces a separation of variables, which enables us to recast the result into a single determinant by introducing the sum over ℓ_j into the j^{th} line of the determinant:

$$X_N\big[\nu, E_-^2\big] = \prod_{j=1}^{N} \frac{4\sin^2[\pi\nu(\lambda_j)]}{\widehat{\xi}'(\lambda_j)} \cdot \det_N M, \tag{B.6}$$

with

$$M_{jk} = \delta_{j,k} \sum_{\ell \in \mathcal{B}_L} \frac{E_-^2(\lambda_j) \cdot E_-^{-2}(\mu_\ell)}{4\pi^2 L^2\, \widehat{\xi}_\nu'(\mu_\ell)\, (\mu_\ell - \lambda_j)^2} \\ + (1-\delta_{j,k}) \sum_{\ell \in \mathcal{B}_L} \frac{E_-^2(\lambda_j) \cdot E_-^{-2}(\mu_\ell)}{4\pi^2 L^2\, \widehat{\xi}_\nu'(\mu_\ell)\, (\lambda_j - \lambda_k)} \bigg[\frac{1}{\mu_\ell - \lambda_j} - \frac{1}{\mu_\ell - \lambda_k} \bigg]. \tag{B.7}$$

The above discrete sums can be expressed in terms of Hilbert transforms (plus corrections that vanish in the $L \to +\infty$ limit). Indeed,

$$\begin{aligned} \sum_{\ell \in \mathcal{B}_L} \frac{E_-^{-2}(\mu_\ell)}{2\pi L\, \widehat{\xi}_\nu'(\mu_\ell)(\mu_\ell - \lambda)} &= \frac{-iE_-^{-2}(\lambda)}{\mathrm{e}^{2i\pi L\, \widehat{\xi}_\nu(\lambda)} - 1} + \int_{\mathscr{C}_\uparrow \cup \mathscr{C}_\downarrow} \frac{E_-^{-2}(z)\, \mathrm{d}z}{2\pi(z-\lambda)(\mathrm{e}^{2i\pi L\, \widehat{\xi}_\nu(z)} - 1)} \\ &= \frac{-iE_-^{-2}(\lambda)}{\mathrm{e}^{2i\pi L\, \widehat{\xi}_\nu(\lambda)} - 1} - \int_{\mathscr{C}_\uparrow} \frac{E_-^{-2}(z)\, \mathrm{d}z}{2\pi(z-\lambda)} + I_L\big[\nu, E_-^{-2}\big](\lambda) \\ &= \fint_{A_L}^{B_L} \frac{E_-^{-2}(z)\, \mathrm{d}z}{2\pi(z-\lambda)} - \frac{E_-^{-2}(\lambda)}{2} \cot\big[\pi L \widehat{\xi}_\nu(\lambda)\big] + I_L\big[\nu, E_-^{-2}\big](\lambda). \end{aligned} \tag{B.8}$$

Differentiating the above expression with respect to λ leads to

$$\sum_{\ell\in\mathcal{B}_L}\frac{E_-^{-2}(\mu_\ell)}{2\pi L\,\widehat{\xi}_\nu'(\mu_\ell)(\mu_\ell-\lambda)^2}=\frac{\partial}{\partial\lambda}\fint_{A_L}^{B_L}\frac{E_-^{-2}(z)\,\mathrm{d}z}{2\pi(z-\lambda)}-\frac{\partial_\lambda E_-^{-2}(\lambda)}{2}\cot\big[\pi L\widehat{\xi}_\nu(\lambda)\big]$$
$$+\frac{E_-^{-2}(\lambda)\,\pi L\widehat{\xi}_\nu'(\lambda)}{2\sin^2[\pi L\widehat{\xi}_\nu(\lambda)]}+\partial_\lambda I_L\big[\nu,E_-^{-2}\big](\lambda). \quad \text{(B.9)}$$

By applying (B.8), (B.9) in (B.7), and using the fact that $L\widehat{\xi}_\nu(\lambda_j)=L\widehat{\xi}(\lambda_j)-\nu(\lambda_j)=j-\nu(\lambda_j)$, we obtain

$$\begin{aligned}M_{jk}&=\delta_{j,k}\frac{\widehat{\xi}'(\lambda_j)}{4\sin^2[\pi\nu(\lambda_j)]}+\frac{E_-(\lambda_j)}{E_-(\lambda_k)}\frac{E_+^{(L)}(\lambda_j)\,E_-(\lambda_k)-E_+^{(L)}(\lambda_k)\,E_-(\lambda_j)}{2i\pi L(\lambda_j-\lambda_k)}\\&=\frac{E_-(\lambda_j)}{E_-(\lambda_k)}\cdot\frac{\widehat{\xi}'(\lambda_k)}{4\sin[\pi\nu(\lambda_j)]\,\sin[\pi\nu(\lambda_k)]}\left\{\delta_{j,k}+\frac{V^{(L)}(\lambda_j,\lambda_k)}{L\,\widehat{\xi}'(\lambda_k)}\right\},\end{aligned} \quad \text{(B.10)}$$

in which $E_+^{(L)}$ is given by (B.4). Note that we recovered the derivative of the λ-type counting function $\widehat{\xi}$ thanks to the identity $\widehat{\xi}_\nu=\widehat{\xi}-L^{-1}\nu$. This ends the proof of the proposition. ■

Let us now suppose that, in the thermodynamic limit $L,N\to+\infty$ with $N/L\to D$ (D finite), the roots λ_j condensate on some symmetric interval $[-q,q]$ of the real axis with the density $\rho(\lambda)=\lim_{L,N\to\infty}\widehat{\xi}'(\lambda)$ of the NLS model. Let us moreover suppose that, in this limit, the counting function $\widehat{\xi}_\nu$ takes the form (2.17) in terms of the dressed momentum $p(\lambda)$ of the NLS model. We demand in addition that $\log E_-^{-2}$ has at most a polynomial growth in λ when $\Re(\lambda)\to\pm\infty$. Then the thermodynamic limit of $X_N\big[\nu,E_-^2\big]$ is well defined and given by the following Fredholm determinant:

$$\lim_{M,N\to\infty}X_N\big[\nu,E_-^2\big]=\det{}_{[-q,q]}[I+V] \quad \text{(B.11)}$$

with kernel

$$V(\lambda,\mu)=4\frac{\sin[\pi\nu(\lambda)]\,\sin[\pi\nu(\mu)]}{2i\pi(\lambda-\mu)}\big\{E_+(\lambda)\,E_-(\mu)-E_+(\mu)\,E_-(\lambda)\big\}. \quad \text{(B.12)}$$

Here, the function $E_+(\lambda)$ reads

$$E_+(\lambda)=iE_-(\lambda)\left\{\fint_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\frac{E_-^{-2}(\mu)}{\mu-\lambda}+\frac{E_-^{-2}(\lambda)}{2}\cot[\pi\nu(\lambda)]\right\}. \quad \text{(B.13)}$$

Indeed, it follows from the form (2.17) of the counting function $\widehat{\xi}_\nu$ that A_L and B_L tend respectively to $-\infty$ and $+\infty$ in the thermodynamic limit. Moreover, it can easily be shown that $I_L[\nu,E_-^2]=\mathrm{O}(L^{-1})$. The thermodynamic limit of $X_N[\nu,E_-^2]$ is therefore a direct consequence of the fact that $\det_N[\delta_{k\ell}+\mathrm{o}(L^{-1})]\to 1$, whenever the $\mathrm{o}(L^{-1})$ symbol is uniform in the entries.

C Functional Translation operator

In this appendix, we provide a heuristic approach to the notion of functional translation operator that we use in Section 5.2 to express our highly coupled form factor series in terms of a decoupled one. As we will see, such an object is in fact a convenient tool to manipulate generalized multidimensional Lagrange series (see Appendix C of [45]). We refer to [59] for a more explicit and rigorous construction.

The notion of one-dimensional translation operator, which acts on some holomorphic function g as

$$\mathrm{e}^{\alpha\partial_\omega} \cdot g(\omega)\big|_{\omega=0} = \sum_{n\geq 0} \frac{\alpha^n}{n!}\, g^{(n)}(0) = g(\alpha), \tag{C.1}$$

can easily be extended to the multidimensional case:

$$\prod_{p=1}^{s} \mathrm{e}^{\alpha_p\partial_{\omega_p}} \cdot g_s(\omega_1,\ldots,\omega_s)\big|_{\omega_p=0} = g_s(\alpha_1,\ldots,\alpha_s). \tag{C.2}$$

By analogy, we can thus define a translation operator acting on functionals as

$$T_\gamma \cdot U[\tau]\big|_{\tau=0} = \exp\left\{\int_{\mathcal{C}} \mathrm{d}\omega\, \gamma(\omega)\, \frac{\delta}{\delta\tau(\omega)}\right\} U[\tau]\Big|_{\tau=0} = U[\gamma]. \tag{C.3}$$

Here, the integral is taken on a contour $\mathcal{C}$ (typically, an interval of the real axis), and U is a functional acting on holomorphic functions defined on a neighborhood $\mathcal{U}$ of $\mathcal{C}$.

Formula (C.3) can be understood as the result of the action of finite dimensional translation operators on a finite dimensional approximation of U. More precisely, let us consider a discretization $t_1,\ldots,t_s$ of $\mathcal{C}$ and a collection of holomorphic functions $U_s(\{z_i\}_{i=1}^s)$ such that, for any holomorphic function γ, $U_s(\{\gamma(t_i)\}_{i=1}^s) \underset{s\to+\infty}{\longrightarrow} U[\gamma]$. Then, we can represent the translation operator in terms of limits of the finite variable case:

$$T_\gamma \cdot U[\tau]\big|_{\tau=0} = \lim_{s\to+\infty} \prod_{p=1}^{s} \exp\left\{\gamma(t_p)\, \frac{\partial}{\partial\tau(t_p)}\right\}\; U_s\big(\{\tau(t_k)\}_1^s\big)\Big|_{\tau=0}. \tag{C.4}$$

Such discretizations allow us to compute the action of more complicated translation operators. Let us define, as in Section 5.2, the operator ordered version $\boldsymbol{:}\,\mathcal{O}\,\boldsymbol{:}$ of some expression $\mathcal{O}$ containing functional derivatives of the type $\delta/\delta\tau(\lambda)$ (such as in (C.3)) as being the expression where all functional derivative operators are placed on the left (in each term of the Taylor series expansion of $\mathcal{O}$). Then, let us consider, for some functionals Γ and U, the following generalization of (C.3):

$$\mathcal{L} = \boldsymbol{:}\, \exp\left\{\int_{\mathcal{C}} \Gamma[\tau](\mu)\, \frac{\delta}{\delta\tau(\mu)}\, \mathrm{d}\mu\right\} U[\tau] \,\boldsymbol{:}\, \Big|_{\tau=0} \tag{C.5}$$

$$\equiv \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\mathcal{C}} \mathrm{d}^n\mu \prod_{i=1}^{n} \frac{\delta}{\delta\tau(\mu_i)} \cdot \left\{\prod_{p=1}^{n} \Gamma[\tau](\mu_p)\, U[\tau]\right\}\Bigg|_{\tau=0}. \tag{C.6}$$

After discretization one gets

$$\mathcal{L} = \lim_{s\to+\infty} \mathcal{L}_s \tag{C.7}$$

with

$$\mathcal{L}_s = \boldsymbol{:} \prod_{p=1}^{s} \exp\left\{\Gamma_s(\{\tau_k\}_1^s)(\tau_p)\, \partial_{\tau_p}\right\} U_s(\{\tau_k\}_1^s)\Big|_{\tau_k=0} \boldsymbol{:} \tag{C.8}$$

$$\equiv \sum_{\substack{n_1,\dots,n_s\\=0}}^{+\infty} \prod_{\ell=1}^{s} \frac{\partial_{\tau_\ell}^{n_\ell}}{n_\ell!} \cdot \left\{ \prod_{p=1}^{s} \Gamma_s^{n_p}(\{\tau_k\})(\tau_p)\, U_s(\{\tau_k\}) \right\}\Bigg|_{\tau_k=0}, \tag{C.9}$$

in which we have set $\tau_k \equiv \tau(t_k)$. The last series is a multi-dimensional Lagrange series which can be computed as

$$\mathcal{L}_s = \frac{U_s(\{\nu(t_k)\})}{\det_s\left[\delta_{jk} - \partial_{\tau_k}\Gamma_s(\{\tau_\ell\})(t_j)\right]_{\tau_k=\nu(t_k)}}, \tag{C.10}$$

where $\nu(t_k)$, $k=1,\dots,s$, are obtained as the solutions of the system $\nu(t_k) = \Gamma_s(\{\nu(t_\ell)\})(t_k)$, $k=1,\dots,s$. The $s\to+\infty$ limit can be taken, at least formally. It yields

$$\mathcal{L} = \frac{U[\nu]}{\det_{\mathcal{C}}\left[I - \dfrac{\delta}{\delta\tau(\mu)}\Gamma[\tau](\lambda)\right]_{\tau=\nu}}, \tag{C.11}$$

in which ν is the solution to the equation

$$\nu(\mu) = \Gamma[\nu](\mu). \tag{C.12}$$

D The Master equation issued-like series representation

In this appendix we obtain, starting from (5.12)-(5.14), an alternative series representation for $\mathcal{Q}_N^{\kappa}(x,t)_{\text{eff}}$. This representation is in the spirit of the series obtained in [45] for the equal-time correlation functions.

D.1 The finite-size series

Expanding the determinant in (5.12) into its finite Fredholm series (this expansion is an immediate consequence of the Laplace expansion for determinants), we obtain:

$$\mathcal{Q}_N^{\kappa}(x,t)_{\text{eff}} = \mathrm{e}^{-\frac{\beta p_F}{\pi}x} \boldsymbol{:} \prod_{j=1}^{N} \frac{\widehat{E}_-^2(\bar{\mu}_j)}{\widehat{E}_-^2(\lambda_j)} \sum_{n=0}^{N} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} \prod_{s=1}^{n} \frac{1}{L\widehat{\xi}'(\lambda_{i_s})} \det_n\left[\widehat{V}^{(L)}(\lambda_{i_j},\lambda_{i_k})\right] \mathscr{G}\left[\omega\right] \boldsymbol{:}\Big|_{\substack{\tau=0\\ \omega=0}}, \tag{D.1}$$

in which $\hat{V}^{(L)}(\lambda,\mu)$ is given by (5.13). It is convenient to use the analytic properties of the operator valued mappings $\widehat{E}_+^{(L)}$ and $\widehat{E}_-$ so as to re-cast this finite-size kernel $V^{(L)}$ into a more compact contour integral representation:

$$\widehat{V}^{(L)}(\lambda,\mu) = 4\sin[\pi\nu_\tau(\lambda)]\sin[\pi\nu_\tau(\mu)]\,\widehat{E}_-(\lambda)\,\widehat{E}_-(\mu) \oint_{\mathscr{C}_q} \frac{\mathrm{d}z}{(2i\pi)^2} \frac{\widehat{E}_+^{(L)}(z)\,\widehat{E}_-^{-1}(z)}{(z-\lambda)(z-\mu)}, \tag{D.2}$$

where $\mathscr{C}_q$ surrounds $[-q,q]$ (see Fig. 2), $\lambda,\mu\in[\,-q\,;q\,]$ and

$$\widehat{E}_{+}^{(L)}(z)\,\widehat{E}_{-}^{-1}(z)=i\int\limits_{\mathscr{C}^{(L)}}\frac{\mathrm{d}y}{2\pi}\frac{f^{(L)}(y,\nu_\tau(y))}{y-z}\,\widehat{E}_{-}^{-2}(y). \tag{D.3}$$

In this last expression the contour $\mathscr{C}^{(L)}=\mathscr{C}_E^{(L)}\cup\widetilde{\mathscr{C}}_q\cup\mathscr{C}_\uparrow\cup\mathscr{C}_\downarrow$ consists of a union of four contours. Two of them, $\mathscr{C}_{\uparrow/\downarrow}$, are as depicted in Fig. 1, and the remaining two, $\mathscr{C}_E^{(L)}\cup\widetilde{\mathscr{C}}_q$, appear in Fig. 2. In particular, the contour $\widetilde{\mathscr{C}}_q$ encircles the contour $\mathscr{C}_q$ of (D.2), which enables us to recast the integral kernel $\widehat{V}^{(L)}(\lambda,\mu)$ in a more compact form. The integrand in (D.3) involves the function

$$f^{(L)}(y,\nu_\tau(y))=\mathbf{1}_{\mathscr{C}_E^{(L)}}(y)-\frac{\mathbf{1}_{\widetilde{\mathscr{C}}_q}(y)}{\mathrm{e}^{-2i\pi\nu_\tau(y)}-1}+\frac{\mathbf{1}_{\mathscr{C}_\downarrow}(y)}{\mathrm{e}^{2i\pi L\widehat{\xi}_{\nu_\tau}(y)}-1}+\frac{\mathbf{1}_{\mathscr{C}_\uparrow}(y)}{1-\mathrm{e}^{-2i\pi L\widehat{\xi}_{\nu_\tau}(y)}}\,. \tag{D.4}$$

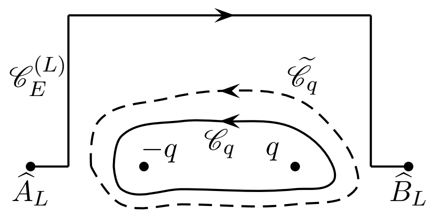

Figure 2: Contours $\mathscr{C}_E^{(L)}$, $\mathscr{C}_q$ and $\widetilde{\mathscr{C}}_q$. The endpoints $\widehat{A}_L$ and $\widehat{B}_L$ are such that $L\widehat{\xi}_{\nu_\tau}(\widehat{A}_L)=-w_L-1/2$ and $L\widehat{\xi}_{\nu_\tau}(\widehat{B}_L)=w_L+1/2$.

Factorizing the integrals over z (D.2) out of the determinant in (D.1) and then using the symmetry of the summand in order to reconstruct a second Cauchy determinant, we get

$$\mathscr{Q}_N^{\kappa}(x,t)_{\mathrm{eff}}=\mathrm{e}^{-\frac{\beta p_F}{\pi}x}\,\boldsymbol{:}\prod_{\ell=1}^{N}\frac{\widehat{E}_{-}^{2}(\bar{\mu}_\ell)}{\widehat{E}_{-}^{2}(\lambda_\ell)}\sum_{n=0}^{N}\frac{1}{(n!)^2}\sum_{i_1,\dots,i_n=1}^{N}\prod_{s=1}^{n}\left\{\frac{4\sin^2[\pi\nu_\tau(\lambda_{i_s})]}{L\widehat{\xi}'(\lambda_{i_s})}\,\widehat{E}_{-}^{2}(\lambda_{i_s})\right\}$$
$$\times\oint\limits_{\mathscr{C}_q}\frac{\mathrm{d}^n z}{(2i\pi)^{2n}}\int\limits_{\mathscr{C}^{(L)}}\frac{\mathrm{d}^n y}{(-2i\pi)^n}\prod_{s=1}^{n}\left\{\frac{f^{(L)}(y_s,\nu_\tau(y_s))}{y_s-z_s}\widehat{E}_{-}^{-2}(y_s)\right\}\cdot\det{}_n^2\left[\frac{1}{z_j-\lambda_{i_k}}\right]\,\mathscr{G}[\omega]\,\boldsymbol{:}\Big|_{\substack{\tau=0\\ \omega=0}}. \tag{D.5}$$

Since

$$\widehat{E}_{-}^{2}=\mathrm{e}^{-ixu-\widehat{g}_\tau-\widehat{g}_\omega}, \tag{D.6}$$

it remains to compute the effect of the two types of functional translation operators occuring in (D.5). We start by taking the ω-translation into account, namely

$$\prod_{a=1}^{n}\frac{\mathrm{e}^{\widehat{g}_\omega(y_a)}}{\mathrm{e}^{\widehat{g}_\omega(\lambda_{i_a})}}\prod_{a=1}^{N}\frac{\mathrm{e}^{\widehat{g}_\omega(\lambda_a)}}{\mathrm{e}^{\widehat{g}_\omega(\bar{\mu}_a)}}\cdot\mathscr{G}[\omega]\Big|_{\omega=0}=\mathscr{G}\left[\varpi_{N+n}\left(\cdot\,\middle|\,\begin{array}{c}\{\lambda_j\}_1^N\cup\{y_s\}_{s=1}^n\\ \{\bar{\mu}_j\}_1^N\cup\{\lambda_{i_s}\}_{s=1}^n\end{array}\right)\right]$$
$$=\mathcal{G}_N\begin{pmatrix}\{\lambda_j\}_1^N\setminus\{\lambda_{i_s}\}_{s=1}^n\cup\{y_s\}_{s=1}^n\\ \{\bar{\mu}_j\}_1^N\end{pmatrix}. \tag{D.7}$$

Above, we have first computed the action of the translation operators according to (5.4), then used the equality (4.13). We should now compute the action of the second set of translation operators involving $\widehat{g}_\tau$. Since the parameters $\bar{\mu}_\ell \equiv \bar{\mu}_\ell[\nu_\tau] = \widehat{\xi}_{\nu_\tau}^{-1}(\ell/L)$, $\ell = 1, \dots, N$, at which the $\widehat{g}_\tau$ are evaluated, are themselves functionals of τ, we should take care of the operator ordering. The expression we have to compute is of the type:

$$
:\sum_{n=0}^{N} \frac{1}{(n!)^2} \sum_{i_1,\dots,i_n=1}^{N} \oint_{\mathscr{C}_q} \frac{\mathrm{d}^n z}{(2i\pi)^{2n}} \int_{\mathscr{C}^{(L)}} \frac{\mathrm{d}^n y}{(-2i\pi)^n} \exp\left\{ \int_{\mathbb{R}} \Gamma_{\{i_a\}}[\tau](\omega) \frac{\delta}{\delta\tau(\omega)} \mathrm{d}\omega \right\} \\
\cdot \mathcal{R}^{(n)}_{\{i_a\}} \begin{pmatrix} \{y_s\}_1^n \\ \{z_s\}_1^n \end{pmatrix} [\nu_\tau] : \Big|_{\tau=0}, \quad \text{(D.8)}
$$

in which $\mathcal{R}^{(n)}_{\{i_a\}}$ is a smooth functional of ν_τ that, moreover, depends on the integration variables $\{y_s\}$ and $\{z_s\}$, and the functional $\Gamma_{\{i_a\}}[\tau](\lambda)$ driving the functional translation reads

$$
\Gamma_{\{i_a\}}[\tau](\lambda) = \sum_{\ell=1}^{N} \left[\phi(\lambda, \bar{\mu}_\ell[\nu_\tau]) - \phi(\lambda, \lambda_\ell)\right] + \sum_{s=1}^{n} \left[\phi(\lambda, \lambda_{i_s}) - \phi(\lambda, y_s)\right]. \quad \text{(D.9)}
$$

According to Appendix C (see (C.5)), computing the action of a functional translation operator with weigths $\Gamma_{\{i_a\}}[\tau](\lambda)$ as above amounts to summing up a multi-dimensional Lagrange series. In our case, the result is obtained by replacing the functional argument ν_τ of $\mathcal{R}^{(n)}_{\{i_a\}}$ by the function $\nu_{\widehat{\mathfrak{t}}}$, in which $\widehat{\mathfrak{t}}$ is the solution to the non-linear functional equation $\widehat{\mathfrak{t}}(t) = \Gamma_{\{i_a\}}[\widehat{\mathfrak{t}}](t)$, and then by dividing the obtained expression by the corresponding functional Jacobian $J_{\{i_a\}}[\widehat{\mathfrak{t}}] = \det_{\mathbb{R}}\left[I - \delta\Gamma_{\{i_a\}}[\tau](\lambda)/\delta\tau(\lambda)\right]\big|_{\tau=\widehat{\mathfrak{t}}}$ (see (C.11)).

Finally, $\mathcal{Q}_N^\kappa(x,t)_{\text{eff}}$ admits the following representation:

$$
\mathcal{Q}_N^\kappa(x,t)_{\text{eff}} = \mathrm{e}^{-\frac{\beta p_F}{\pi}x} \sum_{n=0}^{N} \frac{1}{(n!)^2} \sum_{i_1,\dots,i_n=1}^{N} \oint_{\mathscr{C}_q} \frac{\mathrm{d}^n z}{(2i\pi)^{2n}} \int_{\mathscr{C}^{(L)}} \frac{\mathrm{d}^n y}{(-2i\pi)^n} \frac{\mathrm{e}^{ix\mathcal{U}^{(L)}_{\{i_a\}}(\{\lambda_a\},\{\bar{\mu}_a\},\{y_s\})}}{J_{\{i_a\}}[\widehat{\mathfrak{t}}]} \\
\times \prod_{s=1}^{n} \left\{ \frac{4\sin^2[\pi\nu_{\widehat{\mathfrak{t}}}(\lambda_{i_s})]}{L\widehat{\xi}'(\lambda_{i_s})\,(y_s - z_s)} f^{(L)}(y_s, \nu_{\widehat{\mathfrak{t}}}(y_s)) \right\} \\
\times \det{}_n^2\left[\frac{1}{z_j - \lambda_{i_k}}\right] \mathcal{G}_N\begin{pmatrix} \{\lambda_j\}_1^N \setminus \{\lambda_{i_s}\}_{s=1}^n \cup \{y_s\}_{s=1}^n \\ \{\bar{\mu}_j[\nu_{\widehat{\mathfrak{t}}}]\}_1^N \end{pmatrix}. \quad \text{(D.10)}
$$

In (D.10), we have set

$$
\mathcal{U}^{(L)}_{\{i_a\}}(\{\lambda_a\},\{\bar{\mu}_a[\nu_{\widehat{\mathfrak{t}}}]\},\{y_s\}) = \sum_{\ell=1}^{N} \left[u(\lambda_\ell) - u(\bar{\mu}_\ell[\nu_{\widehat{\mathfrak{t}}}])\right] + \sum_{s=1}^{n} \left[u(y_s) - u(\lambda_{i_s})\right] \quad \text{(D.11)}
$$

and explicitly insisted on the fact that $\bar{\mu}_\ell \equiv \bar{\mu}_\ell[\nu_{\widehat{\mathfrak{t}}}]$, $\ell = 1, \dots, N$, are now functionals of the solution $\widehat{\mathfrak{t}}$ to $\widehat{\mathfrak{t}}(t) = \Gamma_{\{i_a\}}[\widehat{\mathfrak{t}}](t)$.

D.2 Taking the thermodynamic limit

For finite N, (D.10) gives a rather implicit representation. We are however interested in computing the thermodynamic limit $L, N \to +\infty$ of this expression (in fact it is only in this limit that the effective series (D.10) is supposed to coincide with the original form factor series (4.8)). In this limit of interest (D.10) can be considerably simplified.

In particular, the non-linear functional equation $\widehat{\mathfrak{r}}(t)=\Gamma_{\{i_a\}}[\widehat{\mathfrak{r}}](t)$ for $\widehat{\mathfrak{r}}$ turns into a linear integral equation for $\mathfrak{r}(t)$, with $\widehat{\mathfrak{r}}(t) \underset{N,L\to\infty}{\longrightarrow} \mathfrak{r}(t)$:

$$\mathfrak{r}(t)=\nu_{\mathfrak{r}}(t)-i\frac{\beta Z(t)}{2\pi}=\int_{-q}^{q}\partial_2\phi(t,\lambda)\,\nu_{\mathfrak{r}}(\lambda)\,\mathrm{d}\lambda+\sum_{s=1}^{n}[\phi(t,\lambda_{i_s})-\phi(t,y_s)]. \tag{D.12}$$

Indeed, in the thermodynamic limit, the Bethe roots λ_j for the ground state condensate on $[-q,q]$ with the density $\rho(\lambda)$, whereas the parameters $\bar{\mu}_j[\nu_{\widehat{\mathfrak{r}}}]$ (defined in terms of the counting function $\widehat{\xi}_{\nu_{\widehat{\mathfrak{r}}}}$), are shifted with respect to the λ_j's according to:

$$\bar{\mu}_j[\nu_{\widehat{\mathfrak{r}}}]-\lambda_j=\frac{\nu_{\mathfrak{r}}(\lambda_j)}{L\rho(\lambda_j)}+\mathrm{O}(L^{-2})\ ,\qquad j=1,\dots,N. \tag{D.13}$$

Above, we have denoted the thermodynamic limit of the function $\nu_{\widehat{\mathfrak{r}}}$ by $\nu_{\mathfrak{r}}$. The linear integral equation (D.12) is readily solved as soon as one observes that the derivative $\partial_2\phi$ of the dressed phase (2.12) with respect to its second variable is actually related to the resolvent R of the Lieb kernel ($\partial_2\phi(t,\lambda)=-R(t,\lambda)$, with $(I-K/2\pi)(I+R)=I$):

$$\nu_{\mathfrak{r}}(t)=i\frac{\beta}{2\pi}+\frac{1}{2\pi}\sum_{s=1}^{n}[\theta(t-\lambda_{i_s})-\theta(t-y_s)]. \tag{D.14}$$

It also follows from (D.12) that the Jacobian of the transformation (see (C.11)) is simply given by

$$J=\det_{[-q,q]}\big[I-\partial_2\phi\big]=\det{}^{-1}_{[-q,q]}\Big[I-\frac{K}{2\pi}\Big]. \tag{D.15}$$

The expression (D.14) for the thermodynamic limit of $\nu_{\widehat{\mathfrak{r}}}(t)$ along with the estimations for the shift (D.13) of the parameters $\bar{\mu}_j\,[\nu_{\widehat{\mathfrak{r}}}]$ with respect to the λ_j's allows one to compute the thermodynamic limit of $\mathcal{U}^{(L)}_{\{i_a\}}(\{\lambda_a\},\{\bar{\mu}_a\},\{y_s\})$. Namely,

$$\begin{aligned}\mathcal{U}^{(L)}_{\{i_a\}}(\{\lambda_a\},\{\bar{\mu}_a\},\{y_s\}) \underset{N,L\to\infty}{\longrightarrow} -\int_{-q}^{q}u'(\lambda)\,\nu_{\mathfrak{r}}(\lambda)\,\mathrm{d}\lambda+\sum_{a=1}^{n}u(y_a)-u(\lambda_{i_a})\\ =-i\frac{\beta p_F}{\pi}+\sum_{a=1}^{n}[u_0(y_a)-u_0(\lambda_{i_a})].\end{aligned} \tag{D.16}$$

To obtain this limit, we have used the definitions (2.15)-(2.16) of ε and p so as to re-cast the expressions involving integrals of $u=p-t\varepsilon/x$ in terms of $u_0=p_0-t\varepsilon_0/x$.

The thermodynamic limit of $\mathcal{G}_N$ can be computed along the lines of [43]. Indeed, one has

$$F\left(\lambda\,\middle|\,\begin{array}{c}\{\lambda_j\}_1^N\cup\{y_s\}_{s=1}^n\\ \{\bar{\mu}_j\}_1^N\cup\{\lambda_{i_s}\}_{s=1}^n\end{array}\right)\underset{N,L\to\infty}{\longrightarrow}\nu_{\mathfrak{r}}(\lambda)\ . \tag{D.17}$$

The latter implies that

$$\prod_{j=1}^{N}\frac{\omega-\bar{\mu}_j+i\epsilon c}{\omega-\lambda_j+i\epsilon c}\cdot\mathrm{e}^{-2i\pi\mathcal{C}[F](\omega+i\epsilon c)}\underset{N,L\to\infty}{\longrightarrow}1\ ,\qquad\text{for}\quad\epsilon\in\{\pm1,0\}\ . \tag{D.18}$$

By applying identity (D.18) to the various products entering in the definition of $\mathcal{G}_N$, one can convince oneself that

$$\operatorname{det}_{[-q,q]}\big[I-K/2\pi\big]\prod_{s=1}^{N}4\sin^2[\pi\nu_{\mathfrak{r}}(\lambda_{i_s})]\ \mathcal{G}_N\begin{pmatrix}\{\lambda_j\}_1^N\setminus\{\lambda_{i_s}\}_{s=1}^n\cup\{y_s\}_{s=1}^n\\ \{\bar{\mu}_j\}_1^N\end{pmatrix}$$
$$\underset{N,L\to\infty}{\longrightarrow}\prod_{s=1}^{n}\left[1-\kappa\frac{V_-}{V_+}\left(\lambda_{i_s}\,\middle|\,\begin{matrix}\{\lambda_{i_a}\}_1^n\\ \{y_a\}_1^n\end{matrix}\right)\right]\cdot\mathcal{F}_n\begin{pmatrix}\{\lambda_{i_a}\}_1^n\\ \{y_s\}_1^n\end{pmatrix}. \quad (D.19)$$

Above, we have set

$$V_{\pm}\left(\mu\,\middle|\,\begin{matrix}\{\lambda_j\}_{j=1}^N\\ \{z_j\}_{j=1}^N\end{matrix}\right)=\prod_{a=1}^{N}\frac{ic\mp(\mu-\lambda_a)}{ic\mp(\mu-z_a)}, \quad (D.20)$$

and, agreeing that the auxiliary arguments of $V_{\pm}$ are undercurrent by those of $\mathcal{F}_n$,

$$\mathcal{F}_n\begin{pmatrix}\{\lambda_s\}_1^n\\ \{y_s\}_1^n\end{pmatrix}=\frac{(1-\kappa)^2}{\operatorname{det}[I-K/2\pi]}\prod_{s=1}^{n}\left[1-\frac{V_+(\lambda_s)}{\kappa V_-(\lambda_s)}\right]\prod_{j=1}^{2}\left\{\frac{V_-(\theta_j)\operatorname{det}_n\big[\delta_{jk}+\widehat{V}_{jk}^{(\theta_j)}\big]}{1-\kappa V_-(\theta_j)/V_+(\theta_j)}\right\}$$
$$\times\prod_{a,b=1}^{n}\frac{(y_a-\lambda_b-ic)\,(y_b-\lambda_a-ic)}{(y_a-y_b-ic)\,(\lambda_b-\lambda_a-ic)}, \quad (D.21)$$

with

$$\widehat{V}_{k\ell}^{(\theta)}=-\frac{\prod_{s=1}^{n}(\lambda_k-y_s)}{\prod_{\substack{s=1\\ \neq k}}^{n}(\lambda_k-\lambda_s)}\frac{K_\kappa(\lambda_k-\lambda_\ell)-K_\kappa(\theta-\lambda_\ell)}{V_-^{-1}(\lambda_k)-\kappa V_+^{-1}(\lambda_k)}\,. \quad (D.22)$$

It only remains to deal with the y-integrations over $\mathscr{C}^{(L)}$. In this limit, the integrals over $\mathscr{C}_{\uparrow/\downarrow}$ give vanishing contributions and hence the y-type integration contour boils down to the contour $\mathscr{C}=\mathscr{C}_E^{(\infty)}\cup\widetilde{\mathscr{C}_q}$ with a weight function that now reads

$$f(y,\nu_{\mathfrak{r}}(y))=\mathbf{1}_{\mathscr{C}_E^{(\infty)}}(y)-\frac{\mathbf{1}_{\widetilde{\mathscr{C}_q}}(y)}{\mathrm{e}^{-2i\pi\nu_{\mathfrak{r}}(y)}-1}\,. \quad (D.23)$$

The contour $\mathscr{C}_E^{(\infty)}$ corresponds to the $L\to+\infty$ limit of the contour $\mathscr{C}_E^{(L)}$ depicted on Fig. 2.

Therefore, in the thermodynamic limit, $\mathcal{Q}_N^{\kappa}(x,t)_{\text{eff}}\underset{N,L\to\infty}{\longrightarrow}\mathcal{Q}^{\kappa}(x,t)$, with

$$\mathcal{Q}^{\kappa}(x,t)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n!)^2}\int_{-q}^{q}\frac{\mathrm{d}^n\lambda}{(2i\pi)^n}\oint_{\mathscr{C}_q}\frac{\mathrm{d}^n z}{(2i\pi)^n}\int_{\mathscr{C}}\frac{\mathrm{d}^n y}{(2i\pi)^n}\prod_{a=1}^{n}\mathrm{e}^{ix[u_0(y_a)-u_0(\lambda_a)]}$$
$$\times\prod_{s=1}^{n}\left\{\frac{f(y_s,\nu_{\mathfrak{r}}(y_s))}{y_s-z_s}\left[1-\kappa\frac{V_+}{V_-}\left(\lambda_s\,\middle|\,\begin{matrix}\{\lambda_a\}\\ \{y_a\}\end{matrix}\right)\right]\right\}\operatorname{det}_n^2\left[\frac{1}{z_j-\lambda_k}\right]\mathcal{F}_n\begin{pmatrix}\{\lambda_a\}\\ \{y_a\}\end{pmatrix}. \quad (D.24)$$

Here, we remind that $\nu_{\mathfrak{r}}$ is given by (D.14) and is a function of $\{y_s\}$ and $\{\lambda_j\}$.

D.3 The equal-time case

We now show that in the equal-time case, one recovers the series obtained from the master equation in [45]. This stems from the fact that, at $t = 0$, the y-integrals over $\mathscr{C}_E^{(\infty)}$ do not contribute. Indeed, when $t = 0$, we can deform the contour $\mathscr{C}_E^{(\infty)}$ into $\mathbb{R} + i\delta$, with $\delta > 0$ and small but such that the line $\mathbb{R} + i\delta$ is above $\widetilde{\mathscr{C}}_q$.

In order to prove this assertion, we first build on the symmetries of the integrand in (D.24) so as to split the y-integrations into those along $\mathbb{R}+i\delta$ (y_a, $a = 1, \dots, k$) and those along $\widetilde{\mathscr{C}}_q$ (y_a, $a = k+1, \dots, n$), with $k = 1, \dots, n$. The integrals over $\widetilde{\mathscr{C}}_q$ can then be computed by residues. One eventually obtains

$$\mathcal{Q}^{\kappa}(x,0) = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{(-1)^k}{n!\,k!(n-k)!} \int_{-q}^{q} \frac{\mathrm{d}^n \lambda}{(2i\pi)^n} \oint_{\mathscr{C}_q} \frac{\mathrm{d}^n z}{(2i\pi)^n} \int_{\mathbb{R}+i\delta} \frac{\mathrm{d}^k y}{(2i\pi)^k} \prod_{a=k+1}^{n} \mathrm{e}^{ix[z_a - \lambda_a]} \prod_{a=1}^{k} \frac{\mathrm{e}^{ix[y_a-\lambda_a]}}{y_a - z_a}$$
$$\times \operatorname{det}_n^2 \left[\frac{1}{z_j - \lambda_k} \right] \frac{\prod_{s=1}^{n} \left[1 - \kappa \frac{V_-}{V_+}\left(\lambda_s \,\middle|\, \begin{matrix} \{\lambda_a\}_1^n \\ \{y_a\}_1^k \cup \{z_a\}_{k+1}^n \end{matrix} \right) \right]}{\prod_{a=k+1}^{n} \left[\kappa \frac{V_-}{V_+}\left(z_a \,\middle|\, \begin{matrix} \{\lambda_a\}_1^n \\ \{y_a\}_1^k \cup \{z_a\}_{k+1}^n \end{matrix} \right) - 1 \right]} \mathcal{F}_n \left(\begin{matrix} \{\lambda_a\}_1^n \\ \{y_a\}_1^k \cup \{z_a\}_{k+1}^n \end{matrix} \right) . \tag{D.25}$$

When considered as a function of $y_1, \dots, y_k$, the integrand in (D.25) has no poles (or even other singularities) in the half-planes $\Im(y_k) \geq \delta$. Indeed, no poles can arize from the θ_j dependent terms in $\mathcal{F}_n$ in as much as this function does not depend on θ_j, *cf* [45]. Also the potential singularities of the determinants at the zeroes of $1 - \kappa(V_-/V_+)(\lambda_a)$ are only apparent due to the presence of the pre-factors $\prod_s [1 - \kappa^{-1}(V_+/V_-)(\lambda_s)][1 - \kappa(V_-/V_+)(\lambda_s)]$ distributed in between (D.21) and (D.25). The potential poles at $y_a = \lambda_b \pm ic$ introduced by these pre-factors are cancelled by the zeroes of the double product in the last line of (D.21). In its turn, the poles in the upper-half plane at $y_a = z_s + ic$, with $a = 1, \dots, k$ and $s = k+1, \dots, n$, that are introduced by the double product in (D.21), are compensated[10] by the same poles present in $\prod_{s=k+1}^{n} [1 - \kappa(V_-/V_+)(z_s)]$. Therefore the only singularities in the y variables correspond to the zeroes of

$$\kappa \frac{V_+}{V_-}\left(z_s \,\middle|\, \begin{matrix} \{\lambda_a\}_1^n \\ \{y_a\}_1^k \cup \{z_a\}_{k+1}^n \end{matrix} \right) - 1 . \tag{D.26}$$

However, one can always squeeze the z-integration contours in (D.25) so that $\Im(z_a) = 0^{\pm}$, $a = 1, \dots, n$. In such a situation, it is readily seen that for $y_a \in \mathbb{R} + i\delta$, $\delta > 0$, one has

$$\left| \kappa \frac{V_+}{V_-}\left(\lambda_s \,\middle|\, \begin{matrix} \{\lambda_a\}_1^n \\ \{y_a\}_1^k \cup \{z_a\}_{k+1}^n \end{matrix} \right) \right| > 1 . \tag{D.27}$$

In other words, the aforementioned equation has no solutions in the upper half-plane. This lack of singularities allows one to simultaneously deform the y-integration contours from $\mathbb{R} + i\delta$ to $\mathbb{R} + i\Delta$, with $\Delta > 0$ and as large as desired. Due to the presence of the oscillatory factors e^{ixy_a}, these integrals will contribute as $\mathrm{e}^{-\Delta x}$. Therefore, by sending $\Delta \to +\infty$, we see that these contributions vanish.

[10] The poles at $y_a = z_s - ic$ lying in the lower half-plane are not explicitly compensated but this is irrelevant for our purposes

We have thus proven that, in the $t=0$ case, the series of multiple integral representation for $\mathcal{Q}^{\kappa}(x,0)$ boils down to

$$\mathcal{Q}^{\kappa}(x,0)=\sum_{n=0}^{N}\frac{(-1)^n}{(n!)^2}\int\limits_{-q}^{q}\frac{\mathrm{d}^n\lambda}{(2i\pi)^n}\oint\limits_{\mathscr{C}_q}\frac{\mathrm{d}^n z}{(2i\pi)^n}\prod_{s=1}^{n}\mathrm{e}^{ix(z_s-\lambda_s)}\ \det{}_n^2\left[\frac{1}{z_j-\lambda_k}\right]\ \mathcal{F}_n\begin{pmatrix}\{z_a\}\\\{\lambda_a\}\end{pmatrix}$$
$$\times\prod_{s=1}^{n}\frac{1-\kappa\dfrac{V_-}{V_+}\left(\lambda_s\,\middle|\,\begin{matrix}\{\lambda_a\}\\\{z_a\}\end{matrix}\right)}{1-\kappa\dfrac{V_-}{V_+}\left(z_s\,\middle|\,\begin{matrix}\{\lambda_a\}\\\{z_a\}\end{matrix}\right)}. \tag{D.28}$$

In order to prove our assertion, it remains to show that the last line in (D.28) does not contribute once that the z-integrals are taken. In virtue of the symmetry of the integrand, one can replace one of the Cauchy determinants by $n!$ times the products of its diagonal entries. By expanding the remaining Cauchy determinant into a sum over permutations, one sees that one has to compute double poles at $z_{p_k}=\lambda_{p_k}$ with $p_1<\cdots<p_\ell$ with $p_i\in\{1,\dots n\}$ and $1<\ell<n$. All other poles will be simple, leading to the equality between the two sets $\{z_a\}_{a\neq p_k}=\{\lambda_a\}_{a\neq p_k}$. These double poles will produce first order derivatives with respect to z_{p_k} at $z_{p_k}=\lambda_{p_k}$. Therefore, the effect of these double poles can be taken into account by setting[11] $z_a=\lambda_a$ for $a\neq p_k$, $k=1,\dots,\ell$, and $z_{p_k}=\lambda_{p_k}+\epsilon_{p_k}$, and then taking the first order ϵ_{p_k}-derivatives of the obtained expression at $\epsilon_{p_k}=0$. As a consequence, only the linear in each ϵ_{p_k} order of the integrand will contribute. However, under such a substitution, one can readily convince oneself that

$$\prod_{s=1}^{n}\frac{1-\kappa\dfrac{V_-}{V_+}\left(\lambda_s\,\middle|\,\begin{matrix}\{\lambda_a\}\\\{z_a\}\end{matrix}\right)}{1-\kappa\dfrac{V_-}{V_+}\left(z_s\,\middle|\,\begin{matrix}\{\lambda_a\}\\\{z_a\}\end{matrix}\right)}=\prod_{s=1}^{\ell}\left\{1+\frac{i\kappa}{\kappa-1}\sum_{k=1}^{\ell}\epsilon_{p_s}\epsilon_{p_k}K'(\lambda_{p_s}-\lambda_{p_k})+\mathrm{O}(\epsilon_{p_k}^2)\right\}=1+\mathrm{O}(\epsilon_{p_k}^2).$$

The linear order in the ϵ_{p_k}'s vanishes. Therefore, we are led to

$$\mathcal{Q}^{\kappa}(x,0)=\sum_{n=0}^{N}\frac{(-1)^n}{(n!)^2}\int\limits_{-q}^{q}\frac{\mathrm{d}^n\lambda}{(2i\pi)^n}\oint\limits_{\mathscr{C}_q}\frac{\mathrm{d}^n z}{(2i\pi)^n}\prod_{s=1}^{n}\mathrm{e}^{ix(z_s-\lambda_s)}\det{}_n^2\left[\frac{1}{z_j-\lambda_k}\right]\ \mathcal{F}_n\begin{pmatrix}\{z_a\}\\\{\lambda_a\}\end{pmatrix}. \tag{D.29}$$

Once upon taking the complex conjugate we recover, word for word, the series obtained in [45].

Remark D.1. Note that our conventions correspond to $x\mapsto -x$ with respect to the work [45].

E Controlling sub-leading corrections: the Natte series representation

The Natte series representation for the Fredholm determinant $\det[I+V]$ is obtained [60] from a specific representation of the solution to the Riemann–Hilbert problem associated with the integrable integral operator $I+V$. As this particular representation for the solution of the Riemann–Hilbert problem is obtained by a series of contour deformations on the so-called initial Riemann–Hilbert problem associated with the integral operator $I+V$, it can be seen that the

[11] The integrand is symmetric with respect to the integration variables z_a and λ_a.

Natte series stems from a certain number of *algebraic transformations*[12] carried out on the initial Fredholm series for the determinant. This fact allows one, at least on the formal ground, to use this Natte series representation in (6.5). The rigorous justification of the possibility to use the Natte series is given in [59].

E.1 The formula for the remainder

In fact, the Natte series corresponds to a representation of the remainder $\mathcal{R}_x[\nu, u, g]$. The latter is expressed as a series of multiple integrals. The integrands appearing in this series have good properties with respect to the large-x limit.

More precisely, the Natte series for the sine kernel (6.2) takes the following form:

$$\mathcal{R}_x[\nu, u, g] \;=\; \sum_{n\geq 1}\sum_{\mathcal{K}_n}\sum_{\mathcal{E}_n(\mathbf{k})} \int\limits_{\mathscr{C}_{\{\epsilon_{\mathsf{t}}\}}} \frac{\mathrm{d}^n z_{\mathsf{t}}}{(2i\pi)^n} H_{n;x}(\{\epsilon_{\mathsf{t}}\}, \{z_{\mathsf{t}}\})[\nu] \prod_{\mathsf{t}\in J(\mathbf{k})} \mathrm{e}^{\epsilon_{\mathsf{t}} g(z_{\mathsf{t}})} \;. \tag{E.1}$$

The second sum appearing above runs through all the n-tuples $\mathbf{k}$ belonging to

$$\mathcal{K}_n = \left\{ \mathbf{k} = (k_1, \dots, k_n) \;:\; k_a \in \mathbb{N}\;, a = 1, \dots, n \quad \text{and such that} \quad \sum_{s=1}^{n} s k_s = n \right\}. \tag{E.2}$$

For each element $\mathbf{k}$ of $\mathcal{K}_n$, one defines the associated set of triplets $J(\mathbf{k})$ (with cardinal n),

$$J(\mathbf{k}) = \left\{ \mathsf{t} = (t_1, t_2, t_3)\;,\; t_1 \in [\![\, 1\,;\, n \,]\!]\;,\; t_2 \in [\![\, 1\,;\, k_{t_1} \,]\!]\;,\; t_3 \in [\![\, 1\,;\, t_1 \,]\!] \right\}, \tag{E.3}$$

which provides a convenient way of labelling sets of n variables. The third sum runs through all the possible choices of elements belonging to the set

$$\mathcal{E}_n(\mathbf{k}) = \left\{ \{\epsilon_{\mathsf{t}}\}_{\mathsf{t}\in J(\mathbf{k})} \;:\; \epsilon_{\mathsf{t}} \in \{\pm 1, 0\} \quad \text{and} \quad \sum_{t_3=1}^{t_1} \epsilon_{\mathsf{t}} = 0 \quad \forall (t_1, t_2) \in [\![\, 1\,;\, n \,]\!] \times [\![\, 1\,;\, k_{t_1} \,]\!] \right\}.$$

In other words, $\mathcal{E}_n(\mathbf{k})$ consists of sets of n parameters $\epsilon_{\mathbf{t}} \in \{\pm 1, 0\}$, indexed by triplets $\mathsf{t} = (t_1, t_2, t_3) \in J(\mathbf{k})$ and subject to summation constraints. Finally, there is an n-fold integral appearing in the n^{th} summand of (E.1). The integration variables $z_{\boldsymbol{t}}$ are, again, indexed by the triplets $\mathsf{t} = (t_1, t_2, t_3)$ of $J(\mathbf{k})$. The contours of integration $\mathscr{C}_{\{\epsilon_{\mathsf{t}}\}}$ depend on the set $\{\epsilon_{\mathsf{t}}\} \in \mathcal{E}_n(\mathbf{k})$. They are realized as n-fold Cartesian products of one-dimensional compact curves corresponding to various deformations of $\mathbb{R}$.

The integrand $H_{n;x}(\{\epsilon_{\boldsymbol{t}}\}\,;\{z_{\boldsymbol{t}}\})[\nu]$ is a smooth functional of ν which is also a function of the integration variables z_{t}. This functional depends on the choice of the parameters $\{\epsilon_{\mathsf{t}}\}$ from $\mathcal{E}_n(\mathbf{k})$ and on x. It is a holomorphic function of the integration variables $\{z_{\mathsf{t}}\}$ belonging to some open neighborhood of the integration contour $\mathscr{C}_{\{\epsilon_{\mathsf{t}}\}}$.

The Natte series converges for x large enough in as much as, for n large,

$$\left| \int\limits_{\mathscr{C}_{\{\epsilon_{\boldsymbol{t}}\}}} \frac{\mathrm{d}^n z_{\mathsf{t}}}{(2i\pi)^n} H_{n;x}(\{\epsilon_{\mathsf{t}}\}\,, \{z_{\mathsf{t}}\})[\nu] \prod_{\mathsf{t}\in J(\mathbf{k})} \mathrm{e}^{\epsilon_{\mathsf{t}} g(z_{\mathsf{t}})} \right| \leq c_2 \left(\frac{c_1}{x}\right)^{n c_3}, \tag{E.4}$$

[12] Such as contour deformations or algebraic summations.

where c_1 and c_2 are some constants depending on the values taken by u and g in some small neighborhood of $\mathbb{R}$ and on the values taken by ν on a small neighborhood of $[-q,q]$. The constant $c_3>0$ only depends on ν.

Finally, one has $H_{1;x}=\mathrm{O}(x^{-\infty})$ and, for $n\geq 2$,

$$H_{n;x}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[\nu]=\mathrm{O}(x^{-\infty})+\sum_{b=0}^{[n/2]}\sum_{p=0}^{b}\sum_{m=b-\left[\frac{n}{2}\right]}^{[n/2]-b}\left(\frac{\mathrm{e}^{ix[u(\lambda_0)-u(-q)]}}{x^{2[\nu(-q)]}}\right)^{\alpha b}\left(\frac{\mathrm{e}^{ix[u(q)-u(-q)]}}{x^{2[\nu(q)+\nu(-q)]}}\right)^{m-\alpha p} \times\frac{1}{x^{n-\frac{b}{2}}}\cdot H_{n;x}^{m,p,b}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[\nu]\,, \quad (E.5)$$

where, for all admissible values of m,b,p,

$$H_{n;x}^{m,p,b}(\{\varepsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[\nu]=\mathrm{O}(x^{n\widetilde{w}-\delta_{n,2}}) \quad (E.6)$$

uniformly on the integration contour. Above, we have set

$$\widetilde{w}=2\sup\left\{|\Re\left[\nu(z)-\nu(\tau q)\right]|\;:\;|z-\tau q|\leq\delta\;,\;\tau=\pm\right\}, \quad (E.7)$$

where $\delta>0$ can be taken as small as desired. These functions $H_{n;x}^{m,p,b}\left(\{\varepsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\}\right)[\nu]$ are such that their asymptotic expansion into inverse powers of x possesses poles that are encircled by the part of the contour $\mathscr{C}_{\{\epsilon_{\mathbf{t}}\}}$ producing algebraic (in x large enough) contribution to the integral. As a consequence, the n-fold integration occurring in the n^{th} term of the series can produce z derivatives. It was shown in [60] that the total order of these derivative is at most n. Once that these derivatives are taken, the estimates in (E.6) change from $\mathrm{O}(x^{n\widetilde{w}-\delta_{n,2}})$ to $\mathrm{O}(x^{-\delta_{n,2}}\log^n x)$.

E.2 The Natte series for the determinant

We now use the Natte series representation (E.1) for the remainder $\mathcal{R}_x[\nu,u,\widehat{g}]$ to prove that the subleading corrections appearing in the large-x asymptotic behavior of the (non operator-dependent) Fredholm determinant and included in $\mathcal{R}_x[\nu,u,g]$ still remain corrections with respect to the leading terms once that one computes the action of the functional translation operators. By expanding the Fredholm determinant in (6.5) into its Natte series, we obtain

$$\begin{aligned}\mathcal{Q}^{\kappa}(x,t)=\mathrm{e}^{-\frac{\beta x p_F}{\pi}}\,\boldsymbol{:}\,\Bigg\{&\mathcal{B}_x[\nu_\tau,u]+\sum_{\epsilon=\pm1}\mathrm{e}^{i\epsilon x[u(q)-u(-q)]+\epsilon[\widehat{g}(q)-\widehat{g}(-q)]}\,\mathcal{B}_x[\nu_\tau+\epsilon,u]\\&+\frac{1}{x^{\frac{3}{2}}}\sum_{\epsilon=\pm1}\mathrm{e}^{i\alpha x[u(\lambda_0)-u(\epsilon q)]+\alpha[\widehat{g}(\lambda_0)-\widehat{g}(\epsilon q)]}\,b_1^{(\epsilon,\alpha)}[\nu_\tau,u]\,\mathcal{B}_x[\nu_\tau,u]\\&+\sum_{n\geq1}\sum_{\mathcal{K}_n}\sum_{\mathcal{E}_n(\mathbf{k})}\int_{\mathscr{C}_{\{\epsilon_{\mathbf{t}}\}}}\frac{\mathrm{d}^n z_{\mathrm{t}}}{(2i\pi)^n}\prod_{\mathrm{t}\in J(\mathbf{k})}\left[\mathrm{e}^{\epsilon_{\mathrm{t}}\widehat{g}(z_{\mathrm{t}})}\right]H_{n;x}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[\nu_\tau]\,\mathcal{B}_x[\nu_\tau,u]\Bigg\}\cdot\mathscr{G}[\omega]\,\boldsymbol{:}\,\Bigg|_{\substack{\omega=0\\ \tau=0}}.\end{aligned} \quad (E.8)$$

Note that in (E.8), we have already simplified the two exponents. Also, we remind that $\alpha=1$ in the space-like regime and $\alpha=-1$ in the time-like regime. The action of the translation operators occuring in the first two lines has already been computed in Section 6.3. In order to compute this action in the last line of (E.8), it is convenient to introduce the notations

$$a_{\{\epsilon_{\mathrm{t}}\}}=\#\{\mathrm{t}\;:\;\epsilon_{\mathrm{t}}=1\}\quad\text{and}\quad\{z_{\mathrm{t}}^{\pm}\}=\{z_{\mathrm{t}}\;:\mathrm{t}\in J(\mathbf{k})\;\text{ such that }\;\pm\,\epsilon_{\mathrm{t}}>0\}\;. \quad (E.9)$$

It then readily follows that

$$\prod_{\mathrm{t}\in J(\mathbf{k})}\left[\mathrm{e}^{\epsilon_{\mathrm{t}}\widehat{g}_{\omega}(z_{\mathrm{t}})}\right]\cdot\mathscr{G}[\omega]\Big|_{\omega=0}=\mathcal{G}_{a_{\{\epsilon_{\mathrm{t}}\}}}\begin{pmatrix}\{z_{\mathrm{t}}^{+}\}\\ \{z_{\mathrm{t}}^{-}\}\end{pmatrix}, \tag{E.10}$$

and

$$:\prod_{\mathrm{t}\in J(\mathbf{k})}\left[\mathrm{e}^{\epsilon_{\mathrm{t}}\widehat{g}_{\tau}(z_{\mathrm{t}})}\right]\cdot H_{n;x}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[\nu_{\tau}]\,\mathcal{B}_{x}[\nu_{\tau},u]:\Big|_{\tau=0}=H_{n;x}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[F_{\{\epsilon_{\mathrm{t}}\}}]\,\mathcal{B}_{x}[F_{\{\epsilon_{\mathrm{t}}\}},u]. \tag{E.11}$$

Note that, in order to obtain (E.10), we have used the fact that $\epsilon_{\mathrm{t}}\in\{\pm1,0\}$ for all $\mathrm{t}\in J(\mathbf{k})$ and that these parameters are subject to the condition $\sum_{\mathrm{t}\in J(\mathbf{k})}\epsilon_{\mathrm{t}}=0$. As a consequence, $\#\{z_{\mathrm{t}}^{+}\}=\#\{z_{\mathrm{t}}^{-}\}$. Also we have set

$$F_{\{\epsilon_{\mathrm{t}}\}}(\lambda;\{z_{\mathrm{t}}\})\equiv F_{\{\epsilon_{\mathrm{t}}\}}(\lambda)=\nu_{\epsilon_{\mathrm{t}}}(\lambda,\{z_{\mathrm{t}}\})=\frac{i\beta Z(\lambda)}{2\pi}-\sum_{\mathrm{t}\in J(\mathbf{k})}\epsilon_{\mathrm{t}}\,\phi(\lambda,z_{\mathrm{t}}). \tag{E.12}$$

This leads to the following representation for the generating function:

$$\mathcal{Q}^{\kappa}(x,t)=\mathcal{Q}^{\kappa;\alpha}_{\mathrm{asym}}(x,t)\;+\sum_{n\geq1}\sum_{\mathcal{K}_n}\sum_{\mathcal{E}_n(\mathbf{k})}\int\limits_{\mathscr{C}_{\{\epsilon_{\mathrm{t}}\}}}\frac{\mathrm{d}^{n}z_{\mathrm{t}}}{(2i\pi)^{n}}$$
$$\times\,H_{n;x}(\{\epsilon_{\mathrm{t}}\},\{z_{\mathrm{t}}\})[F_{\{\epsilon_{\mathrm{t}}\}}]\,\mathcal{B}_{x}[F_{\{\epsilon_{\mathrm{t}}\}},u]\,\mathcal{G}_{a_{\{\epsilon_{\mathrm{t}}\}}}\begin{pmatrix}\{z_{\mathrm{t}}^{+}\}\\ \{z_{\mathrm{t}}^{-}\}\end{pmatrix}. \tag{E.13}$$

Here, $\mathcal{Q}^{\kappa;\alpha}_{\mathrm{asym}}(x,t)$ denotes the leading asymptotic part that is given in (6.15) or by its similar expression in the time-like regime (without the $\mathrm{O}\big((\ln x)/x\big)$ remainders since these are now explicitly taken into account in the rest of the above equation).

Note that the n-fold integration occuring in the n^{th} term of the series can produce z derivatives whose total order is at most n. It thus follows from the representation for the functional $H_{n;x}$ (E.5) and from the form of the estimates (E.6) that, indeed, the remainder produces corrections of the form written in (6.15).

References

[1] I. Affleck, *Critical behavior of two-dimensional systems with continuous symmetries*, Phys. Rev. Lett. **55** (1985), no. 13, 1355–1358.

[2] M. Arikawa, M. Karbach, G. Muller, and K. Wiele, *Spinon excitations in the XX chain: spectra, transition rates, observability*, J. Phys. A: Math. Gen. **39** (2006), 10623–10640.

[3] E. Barouch and B. M. McCoy, *Statistical mechanics of XY-model .2. Spin-correlation functions*, Phys. Rev. A **3** (1971), no. 2, 786–804.

[4] H. Bethe, *Zür Theorie der Metalle I. Eigenwerte und Eigenfunktionen Atomkete*, Zeitschrift für Physik **71** (1931), 205–226.

[5] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, *Conformal invariance, the central charge, and universal finite-size amplitudes at criticality*, Phys. Rev. Lett. **56** (1986), no. 7, 742–745.

[6] H. Boos, M. Jimbo, T. Miwa, and F. Smirnov, *Completeness of a fermionic basis in the homogeneous XXZ model*, J. Math. Phys. **50** (2009), 095206.

[7] H. Boos, M. Jimbo, T. Miwa, and F. Smirnov, *Hidden Grassmann Structure in the XXZ Model IV: CFT Limit*, Comm. Math. Phys. **299** (2010), 825–866.

[8] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *Fermionic basis for space of operators in the XXZ model*, hep-th/0702086.

[9] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *A recursion formula for the correlation functions of an inhomogeneous XXX model*, hep-th/0405044, 2004.

[10] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *Algebraic representation of correlation functions in integrable spin chains*, Annales Henri Poincaré **7** (2006), 1395–1428.

[11] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *Density matrix of a finite sub-chain of the Heisenberg anti-ferromagnet*, Lett. Math. Phys. **75** (2006), 201–208.

[12] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *Hidden Grassmann structure in the XXZ model*, Comm. Math. Phys. **272** (2007), 263–281.

[13] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, *Hidden Grassmann Structure in the XXZ Model II: Creation Operators*, Comm. Math. Phys. **286** (2009), 875–932.

[14] J. L. Cardy, *Conformal invariance and universality in finite size scaling*, J. Phys. A: Math. Gen. **17** (1984), L385–L387.

[15] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. **B270** (1986), 186–204.

[16] V. V. Cheianov, T. Giamarchi, and M. R. Zvonarev, *Time-dependent correlation functions of the Jordan-Wigner operator as a Fredholm determinant*, J. Stat. Mech. (2009), P07035.

[17] F. Colomo, A. G. Izergin, V. E. Korepin, and V. Tognetti, *Correlators in the Heisenberg XX0 chain as Fredholm determinants*, Phys. Lett. A **169** (1992), 237–247.

[18] F. Colomo, A. G. Izergin, V. E. Korepin, and V. Tognetti, *Temperature correlation functions in the XX0 Heisenberg chain*, Teor. i Mat. Fiz. **94** (1993), 19–38.

[19] F. Colomo, A. G. Izergin, and V. Tognetti, *Correlation functions in the XX0 Heisenberg chain and their relations with spectral shapes*, J. Phys. A: Math. Gen. **30** (1997), 361–370.

[20] H. J. de Vega and F. Woynarovich, *Method for calculating finite size corrections in Bethe ansatz systems: Heisenberg chain and 6-vertex model*, Nucl. Phys. B (1985), 439–456.

[21] C. Destri and H. J. de Vega, *Integrable quantum field theories and conformal field theories from lattice models in the light-cone approach*, Phys. Lett. B **201** (1988), 261–268.

[22] C. Destri and H. J. de Vega, *Unified approach to thermodynamic bethe ansatz and finite size corrections for lattice models and field theories*, Nucl. Phys. B **438** (1995), 413–454.

[23] T. C. Dorlas, *Orthogonality and completeness of the Bethe Ansatz eigenstates of the nonlinear Schrödinger model*, Comm. Math. Phys. **154** (1993), 347–376.

[24] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, *Quantum inverse problem method I*, Theor. Math. Phys. **40** (1980), 688–706, Translated from Teor. Mat. Fiz. 40, 1979.

[25] M. Gaudin, *Preprints*, Centre d'études nucléaires de Saclay, CEA-N-1559, 1971.

[26] M. Gaudin, B. M. Mc Coy, and T. T. Wu, *Normalization sum for the Bethe's hypothesis wave functions of the Heisenberg-Ising model*, Phys. Rev. D **23** (1981), 417–419.

[27] F. Göhmann, A. Klümper, and A. Seel, *Integral representations for correlation functions of the XXZ chain at finite temperature*, J. Phys. A **37** (2004), 7625–7652.

[28] F. Göhmann, A. Klümper, and A. Seel, *Integral representation of the density matrix of the XXZ chain at finite temperatures*, J. Phys. A : Math. Gen. **38** (2005), 1833–1841.

[29] F. D. M. Haldane, *General relation of correlation exponents and spectral properties of one-dimensional Fermi systems: Application to the anisotropic s=1/2*, Phys. Rev. Lett. **45** (1980), 1358–1362.

[30] F. D. M. Haldane, *Demonstration of the "Luttinger liquid" character of Bethe-ansatz soluble models of 1-d quantum fluids*, Phys. Lett. A **81** (1981), 153–155,.

[31] F. D. M. Haldane, *Luttinger liquid theory of one-dimensional quantum fluids: I. Properties of the Luttinger model and their extension to the general 1d interacting spinless Fermi gas*, J. Phys. C: Solid State Phys. **14** (1981), 2585–2609.

[32] A. R. Its, A. G. Izergin, and V. E. Korepin, *Long-distance asymptotics of temperature correlators of the impenetrable bose gas*, Comm. Math. Phys. **130** (1990), 471–486.

[33] A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov, *Differential equations for quantum correlation functions*, Int. J. Phys. **B 4** (1990), 1003.

[34] A. R. Its, A. G. Izergin, V. E. Korepin, and G. G. Varzugin, *Large time and distance asymptotics of the correlator of the impenetrable bosons at finite temperature*, Physica D **54** (1991), 351.

[35] A. R. Its, A. G. Izergin, V. E. Korepin, and G. G. Varzugin, *Large time and distance asymptotics of field correlation function of impenetrable bosons at finite temperature*, Physica D **54** (1992), 351–395.

[36] A. G. Izergin and V. E. Korepin, *Lattice versions of quantum field theory models in two dimensions*, Nucl. Phys. B **205** (1982), 401–413.

[37] M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, *Correlation functions of the XXZ model for* $\Delta < -1$, Phys. Lett. A **168** (1992), 256–263.

[38] M. Jimbo and T. Miwa, *Algebraic analysis of solvable lattice models*, AMS, 1995.

[39] M. Jimbo and T. Miwa, *Quantum KZ equation with* $|q| = 1$ *and correlation functions of the XXZ model in the gapless regime*, J. Phys. A: Math. Gen. **29** (1996), 2923–2958.

[40] M. Jimbo, T. Miwa, Y. Mori, and M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, Physica D **1** (1980), 80–158.

[41] M. Jimbo, T. Miwa, and F. Smirnov, *Hidden Grassmann structure in the XXZ model III: introducing the Matsubara direction*, J. Phys. A : Math. Gen. **42** (2009), 304018.

[42] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *Form factor approach to the asymptotic behavior of correlation functions in critical models*, To appear.

[43] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *Thermodynamic limit of particle-hole form factors in the massless XXZ Heisenberg chain*, arXiv:1003.4557.

[44] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *On correlation functions of integrable models associated with the six-vertex R-matrix*, J. Stat. Mech. Theory Exp. (2007), P01022.

[45] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions*, J. Stat. Mech. Theory Exp. (2009), P04003.

[46] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain*, J. Math. Phys. **50** (2009), 095209.

[47] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, and V. Terras, *Riemann-Hilbert approach to a generalized sine kernel and applications*, Comm. Math. Phys. **291** (2009), 691–761.

[48] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Emptiness formation probability of the XXZ spin-1/2 Heisenberg chain at Delta=1/2*, J. Phys. A: Math. Gen. **35** (2002), L385–L391.

[49] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Large distance asymptotic behavior of the emptiness formation probability of the XXZ spin-1/2 Heisenberg chain*, J. Phys. A : Math. Gen. **35** (2002), L753–L758.

[50] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Spin-spin correlation functions of the XXZ-1/2 Heisenberg chain in a magnetic field*, Nucl. Phys. B **641** (2002), 487–518.

[51] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Dynamical correlation functions of the XXZ spin-1/2 chain*, Nucl. Phys. B **729** (2005), 558–580, hep-th/0407108.

[52] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *Master equation for spin-spin correlation functions of the XXZ chain*, Nucl. Phys. B **712** (2005), 600–622, hep-th/0406190.

[53] N. Kitanine, J. M. Maillet, and V. Terras, *Form factors of the XXZ Heisenberg spin-1/2 finite chain*, Nucl. Phys. B **554** (1999), 647–678.

[54] N. Kitanine, J. M. Maillet, and V. Terras, *Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field*, Nucl. Phys. B **567** (2000), 554–582.

[55] V. E. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. **86** (1982), 391–418.

[56] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993.

[57] V. E. Korepin and N. A. Slavnov, *The time-dependent correlation functions of an impenetrable Bose gas as a Fredholm minor*, Commun. Math. Phys. **129** (1990), 103–113.

[58] V. E. Korepin and N. A. Slavnov, *Form factors in the finite volume*, Int. J. Mod. Phys. B **13** (1999), 2933–2941.

[59] K. K. Kozlowski, *Long-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model*, to appear.

[60] K. K. Kozlowski, *Riemann-Hilbert approach to the time-dependent generalized sine kernel*, math-ph: 10115897.

[61] K. K. Kozlowski, *On the emptiness formation probability of the open XXZ spin 1/2 chain*, J. Stat. Mech.: Theory Exp. (2008), P02006.

[62] K. K. Kozlowski, *Fine structure of the asymptotic expansion of cyclic integrals*, J. Math. Phys. **50** (2009), 095205.

[63] K. K. Kozlowski, J. M. Maillet, and N. A. Slavnov, *Long-distance behavior of temperature correlation functions of the quantum one-dimensional Bose gas*, arXiv:1011.3149.

[64] A. Lenard, *Momentum distribution in the ground state of the one-dimensional system of impenetrable bosons*, J. Math. Phys. **5** (1964), 930–943.

[65] A. Lenard, *One-dimensional impenetrable bosons in thermal equilibrium*, J. Math. Phys. **7** (1966), 1268–72.

[66] E. Lieb, T. Schultz, and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Ann. Phys. **16** (1961), 407–466.

[67] E. H. Lieb, *Exact analysis of an interacting Bose gas II. The excitation spectrum*, Phys. Rev. **130** (1963), 1616–1624.

[68] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Phys. Rev. **130** (1963), no. 4, 1605–1616.

[69] A. Luther and I. Peschel, *Calculation of critical exponents in two dimensions from quantum field theory in one dimension*, Phys. Rev. B **12** (1975), 3908–3917.

[70] B. M. McCoy, *Spin correlation functions of the X-Y model*, Phys. Rev. **173** (1968), 531–541.

[71] B. M. McCoy, J. H. H. Perk, and R. E. Schrock, *Correlation functions of the transverse Ising chain at the critical field for large temporal and spatial separations*, Nucl. Phys. B **220** (1983), 269–282.

[72] B. M. McCoy, J. H. H. Perk, and R. E. Schrock, *Time-dependent correlation functions of the transverse Ising chain at the critical magnetic field*, Nucl. Phys. B **220** (1983), 35–47.

[73] B. M. McCoy, J. H. H. Perk, and T. T. Wu, *Ising field theory: quadratic difference equations for the n-point Green's functions on the lattice*, Phys. Rev. Lett. **46** (1981), 757–760.

[74] B. M. McCoy, C. A. Tracy, and T. T. Wu, *Two-dimensional Ising model as an exactly soluble relativistic quantum field theory: explicit formulas for the n-point functions*, Phys. Rev. Lett. **38** (1977), 793–796.

[75] B. M. McCoy and T. T. Wu, *Theory of Toeplitz determinants and the spin correlation functions of the two-dimensional Ising model IV*, Phys. Rev. **162** (1967), 436.

[76] B. M. McCoy and T. T. Wu, *The two-dimensional lsing model*, Harvard Univ. Press, Cambridge, Massachusetts, 1973.

[77] T. Oota, *Quantum projectors and local operators in lattice integrable models*, J. Phys. A: Math. Gen. **37** (2004), 441–452.

[78] R. Orbach, *Linear antiferromagnetic chain with anisotropic coupling*, Phys. Rev. **112** (1958), 309–316.

[79] A. M. Polyakov, *Conformal symmetry of critical fluctuations*, JETP Lett. **12** (1970), 381–383.

[80] K. Sakai, *Dynamical correlation functions of the XXZ model at finite temperature*, J. Phys. A: Math. Theor. **40** (2007), 7523–7542.

[81] N. A. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe Ansatz*, Theor. Math. Phys. **79** (1989), 502–508.

[82] N. A. Slavnov, *Nonequal-time current correlation-function in a one-dimensional Bose-gas*, Theor. Math. Phys. **82** (1990), 273–282.

[83] W. E. Thirring, *A soluble relativistic field theory*, Ann. Phys. **3** (1958), 91–112.

[84] A. K. Tsikh, *Multidimensional residues and their applications*, Translations of Mathematical Monographs, AMS, 1992.

[85] H. G. Vaidya and C. A. Tracy, *One-particle reduced density matrix of impenetrable bosons in one dimension at zero temperature*, J. Math. Phys. **20** (1979), 2291–2303.

[86] F. Woynarovich, *Excitation spectrum of the spin-1/2 Heisenberg chain and conformal invariance*, Phys. Rev. Lett. **59** (1987), 259–261.

[87] F. Woynarovich, H. P. Eckle, and T. T. Truong, *Non-analytic finite-size corrections in the one-dimensional Bose gas and Heisenberg chain*, J. Phys. A: Math. Gen. **22** (1989), 4027–4043.

[88] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, *The spin-spin correlation function of the two-dimensional Ising model: exact results in the scaling region*, Phys. Rev. B **13** (1976), 316–374.

[89] C. N. Yang and C. P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10** (1969), 1115–1122.