Quantization of Moduli Spaces of Flat Connections
Applications to Supersymmetric Gauge Theories

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Moduli spaces of flat connections on a Riemann surface $\mathcal{C}_{g,n}$ are relevant for 4d class $S$ gauge theories from compactification of 6d $\mathcal{N} = (2, 0)$ SCFT’s on $\mathcal{C}_{g,n}$

[Giayotto, Moore, Neitzke ’09]

**Context and motivation:**

- **AGT correspondence** [Alday, Giayotto, Tachikawa ’10]

  $\mathcal{N} = 2$ **SUSY 4d gauge theories**

  Wilson/’t Hooft operators

- Study moduli spaces of flat $SL_N(\mathbb{C})$-connections on $\mathcal{C}_{g,n}$

  $\mathcal{M}_{\text{flat}}(\mathcal{C}_{g,n}) \simeq \text{Hom}(\pi_1(\mathcal{C}_{g,n}), SL_N(\mathbb{C}))/SL_N(\mathbb{C})$

  $\dim \mathcal{M} = (2g - 2 + n)(N^2 - 1)$
Outline

- Study the algebra of functions $A_{g,n}$ on $M_{\text{flat}}(C_{g,n})$.
- Find a preferred set of generators w.r.t. a pair of pants decomposition.
  - Algebraic relations between functions on $M_{\text{flat}}$
- Describe $A_{g,n}^q \equiv$ a quantization of the algebra of functions on $M_{\text{flat}}$.
- Investigate the relation to the algebra of Verlinde operators in Toda CFT.
I. $\mathcal{A}_{g,n}$ and tinkertoys

Basis of algebraic generators:

- Construct generators of $\mathcal{A}_{g,n}$ from $SL_N(\mathbb{C})$ holonomy matrices
  - Trace functions for simple loops on $C_{g,n}$, from characteristic polynomial
  - Networks – contractions of $\prod$ holonomies by $SL_N(\mathbb{C})$-invariant tensors

**Fig. 1:** Loops and networks on $C_{1,3}$. 
I. $A_{g,n}$ and tinkertoys

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- Construct generators of $A_{g,n}$ from $SL_N(\mathbb{C})$ holonomy matrices
  - Trace functions for simple loops on $C_{g,n}$, from characteristic polynomial
  - Networks – contractions of $\prod$ holonomies by $SL_N(\mathbb{C})$-invariant tensors
- Nicely localized w.r.t. a pair of pants decomposition of $C_{g,n}$, as in CFT.
• **Skein relations** express the product of two functions as a sum over generators.

• Punctured torus:

  \[
  \begin{align*}
  \begin{array}{c}
  \text{Diagram 1} \\
  \text{Diagram 2} \\
  \text{Diagram 3}
  \end{array}
  \end{align*}
  \]

  Representations of crossing relations.

• **Poisson relations** - determined by applying the Goldman bracket:

  \[
  \{ L_{\gamma_1}, L_{\gamma_2} \} = L_{\{\gamma_1, \gamma_2\}} \quad \text{[Goldman '86].}
  \]

  \[
  \left\{ \begin{array}{c}
  \begin{array}{c}
  \text{Diagram 4} \\
  \text{Diagram 5}
  \end{array}
  \end{array} \right\} = \frac{N-1}{N} \begin{array}{c}
  \begin{array}{c}
  \text{Diagram 6} \\
  \text{Diagram 7}
  \end{array}
  \end{array} - \frac{1}{N} \begin{array}{c}
  \begin{array}{c}
  \text{Diagram 8} \\
  \text{Diagram 9}
  \end{array}
  \end{array} \]
Generators and relations

- Coordinates on a triangulation of \( C_{g,n} \)
  - Fock-Goncharov coordinates \( x_i \): attach to triangulation. [Fock, Goncharov '06]
  - Construct the holonomy matrices from elementary matrices: edge - crossing or moving through a face.

- Example: \( SL(4, \mathbb{C}) \) holonomy around puncture \( A \) on \( C_{0,3} \)
  \[
  A_1 = \text{tr} A = \prod_i \alpha_i^{-\frac{1}{\kappa_i}} (1 + \alpha_1 + \alpha_1 \alpha_2 + \alpha_1 \alpha_2 \alpha_3) .
  \]

- Networks expansion \( N_i = \sum_a c_a x_a \) for \( x_a \) monomials of \( x_i \) coordinates.

Fig. 2: \( C_{0,3} \) coordinates.
II. Quantization of $\mathcal{A}_{g,n}$

- **Algebra**: $q$-deformed algebraic relations between the generators of $\mathcal{A}_{g,n}^q$ using $q$-skein relations derived from quantum groups, with $q = e^{i\hbar}$.

- **Representation**: canonically quantize Fock-Goncharov coordinates. Construct the quantized generators in terms of $\hat{x}_i$ coordinates. $\hat{x}\hat{y} = q^{\epsilon_{xy}}\hat{y}\hat{x}$

The quantized $\mathcal{A}_{g,n}^q$ is provided by a 1-parameter family of skein non-commutative algebras of links in an oriented 3-manifold. [Turaev '91]

One can define networks in terms of $\mathcal{U}_q(sl(N))$ invariant tensors. [Sikora '05]

$$[\hat{A}, \hat{B}] \xrightarrow{q \rightarrow 1} \hbar\{A, B\}$$
**q-deformed relations**

**Examples:**

- $\mathcal{U}_q(sl(N))$ quantum crossing relation: by *rescaling* networks constructed from tensor contractions.

- **Representation:** $\mathcal{U}_q(sl(4))$ quantum skein relation

\[ \text{Diagram showing quantum skein relations} \]
Quantized generators

Fock-Goncharov coordinates satisfy
\[ \hat{x}_\alpha \hat{x}_\beta = q^{\epsilon_{\alpha \beta}} \hat{x}_\beta \hat{x}_\alpha . \]

Monomials:
\[ \chi_a = \exp \sum_\alpha a_\alpha \hat{X}_\alpha \rightarrow \hat{\chi}_a = \exp \sum_\alpha a_\alpha \hat{\chi}_\alpha \]

We construct quantized networks \( \hat{N}_i = \sum_a c^a \hat{\chi}_a \) and trace functions.

Example: quantum skein relation for \( \mathcal{U}_q(\mathfrak{sl}(3)) \)
\[ \hat{N}_{AC} \hat{N}_{BC} = q^{1/2} \hat{W}_1 + \hat{W}_1 \hat{W}_1 + A_1 B_1 C_1 + A_1 A_2 B_1 B_2 + C_1 C_2 + [3] \]

General expanded form of quantized functions
\[ \hat{F} = \sum_a c^a \hat{\chi}_a \]
So far...

- **Classically**: studied the algebra of functions $A_{g,n}$ on $M_{\text{flat}}(C_{g,n})$.

  Tinkertoys – preferred set of generators w.r.t. pair of pants decomposition.

- **Quantization**:

  Described a quantization $A_{g,n}^q$ of the algebra of functions on $M_{\text{flat}}$.

  $A_{g,n}^q$ in FG-coordinates $\leftrightarrow$ representation of quantum skein relations.

  Examples of quantization of BPS indices for higher rank.

- **Claim**:

  The algebra of Verlinde line and network operators in Toda CFT on $C_{g,n}$ provides a representation of $A_{g,n}^q$. 
III. Toda field theory and Verlinde operators

- Verlinde loop and network operators describe the monodromy acquired by a vertex operator as it moves along a path.

- Fusion/braiding on conformal blocks.

- Braiding matrix $B(\alpha) \rightarrow \tilde{R} \in \mathcal{U}_q(sl_N)^{\otimes 2}$

- Drinfeld twist: standard $R \rightarrow J^{-1} \tilde{R} J$

- $VO_m \simeq M_{\lambda_m} \otimes \ldots \otimes M_{\lambda_1}$
Thank you for your attention!

GATIS

Gauge Theories as Integrable Systems