Dissecting CFT Correlators and String Amplitudes: Conformal Blocks and On-Shell Recursion for General Tensor Fields

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Abstract

This thesis covers two main topics: the tensorial structure of quantum field theory correlators in general spacetime dimensions and a method for computing string theory scattering amplitudes directly in target space.

In the first part tensor structures in generic bosonic CFT correlators and scattering amplitudes are studied. To this end arbitrary irreducible tensor representations of $SO(d)$ (traceless mixed-symmetry tensors) are encoded in group invariant polynomials, by contracting with sets of commuting and anticommuting polarization vectors which implement the index symmetries of the tensors. The tensor structures appearing in CFT_d correlators can then be inferred by studying these polynomials in a $d + 2$ dimensional embedding space. It is shown with an example how these correlators can be used to compute general conformal blocks describing the exchange of mixed-symmetry tensors in four-point functions, which are crucial for advancing the conformal bootstrap program to correlators of operators with spin.

Bosonic string theory lends itself as an ideal example for applying the same methods to scattering amplitudes, due to its particle spectrum of arbitrary mixed-symmetry tensors. This allows in principle the definition of on-shell recursion relations for string theory amplitudes. A further chapter introduces a different target space definition of string scattering amplitudes. As in the case of on-shell recursion relations, the amplitudes are expressed in terms of their residues via BCFW shifts. The new idea here is that the residues are determined by use of the monodromy relations for open string theory, avoiding the infinite sums over the spectrum arising in on-shell recursion relations. Several checks of the method are presented, including a derivation of the Koba-Nielsen amplitude in the bosonic string. It is argued that this method provides a target space definition of the complete S-matrix of string theory at tree-level in a flat background in terms of a small set of conditions, without relying on any worldsheet computation.

Zusammenfassung

Diese Arbeit behandelt zwei Themen: die effiziente Beschreibung von Tensorstrukturen in Quantenfeldtheorie-Korrelatoren in beliebigen Raumzeit-Dimensionen sowie eine Methode zur Berechnung von Stringtheorie-Amplituden direkt im Zielraum.

Im ersten Teil werden Tensorstrukturen in allgemeinen bosonischen CFT-Korrelatoren und Streuamplituden studiert. Hierzu werden beliebige irreduzible Tensordarstellungen von $SO(d)$ mit kommutierenden und antikommutierenden Polarisationsvektoren kontrahiert und so als Polynome beschrieben. Die in CFT_d-Korrelatoren vorkommenden Tensorstrukturen können in einem $(d + 2)$-dimensionalen einbettenden Raum als solche Polynome identifiziert werden. Anhand eines Beispiels wird gezeigt, wie mit diesen Korrelatoren allgemeine Konforme Blöcke für den Austausch von Tensoren gemischter Symmetrie berechnet werden können. Solche Konformen Blöcke werden benötigt, um den Konformen Bootstrap auf Korrelatoren von Operatoren mit Spin anzuwenden.

Die gleichen Techniken finden bei der Bestimmung von Rekursionsrelationen für Streuamplituden Anwendung, was anhand von bosonischer Stringtheorie demonstriert wird. Diese Theorie bietet aufgrund ihres Teilchenspektrums mit beliebigen Tensordarstellungen gemischter Symmetrie ein gelegenes Beispiel. Ein weiterer Abschnitt behandelt eine alternative Beschreibung von Stringtheorie-Amplituden im Zielraum. Diese Konstruktion beruht auf der Berechnung der Residuen von Amplituden unter Ausnutzung von Monodromie-Relationen, die von String-Amplituden erfüllt werden. Die Amplituden werden dann mithilfe von BCFW-Verschiebungen anhand der Residuen bestimmt. Verschiedene Tests dieser Methode werden präsentiert, einschließlich einer Herleitung der Koba-Nielsen-Amplituden im bosonischen String. Schließlich wird argumentiert, dass hierdurch eine Definition der tree-level Stringtheorie-S-Matrix für flache Hintergründe direkt im Zielraum gegeben ist, d.h. ohne auf die Weltflächen-Beschreibung zurückgreifen zu müssen.

This thesis is based on the following publications

- M. S. Costa and T. Hansen, *Conformal correlators of mixed-symmetry tensors*, *JHEP* **1502** (2015) 151, [arXiv:1411.7351].
- R. H. Boels and T. Hansen, *String theory in target space*, *JHEP* **1406** (2014) 054, [arXiv:1402.6356].

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 28. April 2015 Tobias Hansen

Contents

Chapter 1

Introduction

One of the biggest quests of theoretical physics is the search for quantum gravity, the last missing building block for a consistent theory of everything. The Standard Model of particle physics provides a unified quantum description of three of the four forces appearing in nature: the electromagnetic, the weak and the strong force. The fourth force, gravity, is by far the weakest among the four, with the consequence that quantum corrections to gravity have never been measured in experiments. In contrast, quantum effects occurring in the Standard Model have been measured to high precision in colliders such as the LHC where particles are accelerated to high energies and then scattered in head-on collisions. Gravity has instead only been observed on macroscopic scales when acting between large objects such as astronomical bodies which can be described classically. Such observations are in agreement with Einsteins theory of general relativity as the classical theory of gravity, however there is little hope that experiments will help in the search for quantum gravity.

Instead, the aim of many physicists is to find a theory for quantum gravity on theoretical grounds. This is a hard problem because in the usual perturbative approach for quantizing field theories gravity is a non-renormalizable theory. For such theories perturbation theory fails to produce meaningful results in the limit of high energies due to an infinite amount of divergences occurring in calculations. Hence one approach to advance regarding quantum gravity (and also strongly coupled field theories which cannot be described by perturbation theory at all) would be to look for non-perturbative methods to study field theories. A different idea is to look for a renormalizable theory of quantum gravity. Here the most popular proposal is to describe particles as tiny strings. String theory does not produce the problematic UV divergences. Furthermore string theory automatically contains a graviton and therefore describes a form of quantum gravity. This thesis will neither find quantum gravity nor overcome the limits of perturbation theory, however the two topics studied are a non-perturbative method for constraining quantum field theories (the conformal bootstrap) and string theory.

A guiding theme of this thesis is the idea to exploit the restrictions that are posed on a theory by symmetry. Symmetry plays a central role in theoretical physics. For example, the three forces described by the Standard Model are all described by so-called gauge theories and essentially distinguished by the type of their internal gauge symmetry, given by the groups $U(1)$, $SU(2)$ and $SU(3)$. Special relativity can be rephrased as the observation that d dimensional spacetime is Minkowski space $\mathbb{R}^{d-1,1}$, a real vector space with metric of signature $(d-1,1)$. This implies that equations must be covariant under rotations in this space, which are described by its isometry group $SO(d-1,1)$. Another way to phrase this is that the equations governing physics must not depend on the choice of inertial coordinate system. General relativity arises when this concept of coordinate independence is generalized to curved spacetimes, combined with the assumption that spacetime is curved by the energy and momentum present at any given point of spacetime, for instance by the presence of massive objects.

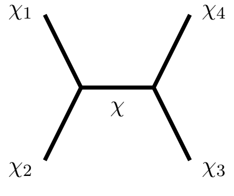
A more recent example is the celebrated AdS/CFT correspondence [1], a conjectured duality stating that correlators of strings on a curved Anti-de Sitter (AdS) space can equivalently be described by a quantum field theory on the boundary of this space. Again a major role is played by symmetry: The isometry group of $(d+1)$-dimensional AdS is $SO(d,2)$ and its d-dimensional boundary is a Minkowski space $\mathbb{R}^{d-1,1}$. A duality between objects on these two spaces implies that the symmetry under $SO(d,2)$ must hold for objects on the boundary as well. This is more symmetry than required by the isometry group of $\mathbb{R}^{d-1,1}$ and implies that the quantum field theory on the boundary must be conformally invariant, a conformal field theory (CFT).

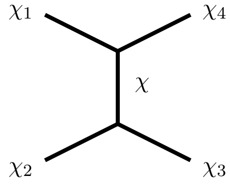
CFTs appear also in many other contexts in theoretical physics, be it condensed matter physics, particle physics, critical phenomena or in the worldsheet description of string theory. From the theoretical standpoint the remarkable property about CFTs is that the amount of symmetry, combined with the usual assumptions about quantum field theories, severely restricts the space of consistent CFTs. This can be turned into a non-perturbative method to study CFTs, known as the conformal bootstrap [2, 3]. In two dimensions the conformal symmetry has an infinite dimensional extension given by the Virasoro algebra leading to many known models where the conformal bootstrap fixes the *CFT data* (spectrum and couplings) exactly. For higher dimensions the method becomes more involved, however it was recently reinitiated [4] and used successfully to study the 3D Ising model at the critical temperature [5–7]. This model is a strongly coupled CFT describing ferromagnetism. Using the conformal bootstrap has led to the most accurate computation of its CFT data to date .

Up to now, the conformal bootstrap in more than two dimensions has been performed only for scalar operators. The logical next step is to study correlators of currents or stress-tensors. Because the stress-tensor is related to the graviton in AdS via the AdS/CFT correspondence, this could lead to qualitative statements about the strength of (quantum) gravity in AdS, potentially explaining the weak gravity conjecture, made in [8]. Another reason to be interested in the stress-tensor is that it exists in every CFT - results that are based only on consistency of a CFT containing a stress-tensor would be truly universal.

1.1 Conformal bootstrap for spinning operators

In this thesis we will consider CFTs in d dimensional Euclidean space $\mathbb{R}^d$ (see [9] for an introduction). If necessary, all equations can be Wick-rotated to Minkowski space $\mathbb{R}^{d-1,1}$. In any CFT in $\mathbb{R}^d$, primary operators $\mathcal{O}^\chi$ are labelled by an irreducible representation (irrep) $\chi = [\Delta, \lambda]$ of the conformal group $SO(d+1,1)$, which in turn is specified by the conformal dimension Δ and an $SO(d)$ irrep λ. The conformal bootstrap exploits the consistency condition that arises when performing operator product expansions (OPEs) in a four-point function in two different ways

$\sum_\chi$ 

$= \sum_\chi$ 

. (1.1)

The corresponding equation is

$$\begin{aligned}&\left\langle \mathcal{O}^{\chi_1}(x_1)\mathcal{O}^{\chi_2}(x_2)\mathcal{O}^{\chi_3}(x_3)\mathcal{O}^{\chi_4}(x_4)\right\rangle \\ &= \sum_\chi c_{\chi_1\chi_2\chi} c_{\chi_3\chi_4\chi} W^{\chi_i}_\chi(x_1,x_2,x_3,x_4) = \sum_\chi c_{\chi_2\chi_3\chi} c_{\chi_4\chi_1\chi} W^{\chi_i}_\chi(x_2,x_3,x_4,x_1)\,.\end{aligned} \tag{1.2}$$

Here $c_{\chi_i\chi_j\chi_k}$ are the OPE coefficients and $W^{\chi_i}_\chi(x_1,x_2,x_3,x_4)$ are functions known as the conformal partial waves. Conformal invariance fixes the form of conformal correlators and partial waves up to the choice of CFT data, given by the conformal dimensions Δ_i and OPE coefficients $c_{\chi_i\chi_j\chi_k}$. This CFT data can be thought of as parametrizing the space of possible CFTs. The values that can be chosen for Δ_i and $c_{\chi_i\chi_j\chi_k}$ are restricted by the requirement that the resulting CFT should be consistent. The space of CFTs is restricted for instance by the bootstrap equation (1.2). Although the sums on both sides are in general infinite, it is possible to truncate the sums and still prove for certain regions in the parameter space that the condition cannot be satisfied, using numerical techniques [4–7]. This method of constraining the space of CFTs is known as the conformal bootstrap.

In order to do this, the knowledge of the conformal partial waves is required. These functions can in principle be uniquely determined based only on conformal invariance, however due to the complexity of known computational techniques explicit expressions are only known for the case that the exchanged representation χ is a fully symmetric tensor [10–13]. This is sufficient for performing the bootstrap in the case that the external states $\chi_1, \chi_2, \chi_3, \chi_4$ are scalars, since the OPE of scalars contains only symmetric tensors. This simply follows from the fact that the OPE between two scalar (denoted $\bullet$) operators must transform as a scalar under conformal transformations. This can only be achieved by contraction of the operators to the only possible vector which is compatible with translation invariance, the difference of the coordinates $x_{12} \equiv x_1 - x_2$

$$\mathcal{O}^{[\Delta_1,\bullet]}(x_1)\mathcal{O}^{[\Delta_2,\bullet]}(x_2) \sim \sum_\chi x_{12}^{a_1}\dots x_{12}^{a_l}\mathcal{O}^\chi_{a_1\dots a_l}(x_1)\,. \tag{1.3}$$

For bootstrap with external operators with spin, the conformal blocks for exchange of mixed-symmetry tensors are necessary and have not been worked out yet. These can be computed using a general method presented in [14], however this requires a better understanding of the conformal correlators of mixed-symmetry tensors.

A study of the tensor structures that can appear in conformal correlators of fully symmetric tensors was performed in [15]. The strategy there was to systematically combine two ideas:

1. Use coordinates on an embedding space $\mathbb{R}^{d+1,1}$ to describe a CFT on $\mathbb{R}^d$, using that the conformal group $SO(d+1,1)$ coincides with the isometry group of the embedding space. In this way conformally invariant functions are trivially identified as the Lorentz invariants of this space.

2. Contract each tensor with a polarization that is by construction in the same irrep, turning correlators from covariant objects into invariants (polynomials), which can be identified with the help of 1.

In Chapter 2 of this thesis this approach is extended to general tensor irreps of $SO(d)$, the traceless mixed-symmetry tensors. In this case the identification of linearly independent tensor structures becomes less obvious, however using the $SO(d)$ tensor product it is possible to count tensor structures in an independent and general way. Using the resulting correlators, a conformal partial wave for exchange of a mixed-symmetry tensor is computed for the first time.

1.2 String theory scattering amplitudes in target space

Of course the power of symmetry can not only be used to tackle big unsolved problems, but also to obtain a better understanding of areas that already have a viable description. In high-energy physics central objects of interest are perturbative scattering amplitudes. The standard method for computing amplitudes (using Feynman diagrams) is very intuitive, however when increasing the number of legs and loops in the diagrams such computations quickly become unfeasible, even using modern computers. Furthermore, the final result is often much simpler than the expressions appearing in intermediate steps of the calculations. For this reason it became an important paradigm to avoid using Feynman diagrams and instead derive amplitudes as functions constrained by symmetries or other universal properties (e.g. analyticity and unitarity). This approach is known as the analytic S-matrix program and has its roots in the 1960s (see [16]). In recent years this program has gained traction again, resulting in many new methods and insights regarding field theory amplitudes.

The birth of string theory is strongly linked to the analytic S-matrix program, as can be already seen from the title of Veneziano's paper [17]: "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories". The search for an amplitude with these properties (which were dictated by experiments at the time) led to the first string theory amplitude. Later it was found that string amplitudes are governed by a CFT on the worldsheet, the two-dimensional surface spanned by a string moving through spacetime. Although the standard worldsheet method for computing string amplitudes is relatively elegant, the generalization to curved backgrounds is complicated and progress in this area has been slow. Therefore it would be interesting to revisit the analytic approach to string theory and investigate whether recent techniques developed for field theory can be applied here as well. Apart from a possible application to curved backgrounds, a complementary target space based view on string amplitudes on flat spacetime would also be interesting in its own right. The idea to calculate string theory amplitudes while avoiding the worldsheet picture is very much in the spirit of avoiding Feynman diagrams in field theory.

As a starting point we will consider the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [18,19], which can be used to express any tree-level amplitude in terms of lower-point amplitudes. These relations were proven in [20, 21] to hold also in string theory. The idea is to turn the amplitude into a function of a single complex variable and relate it to its residues using Cauchy's residue theorem. In the next step the residues are derived by exploiting unitarity of the S-matrix, which is required for any physical theory. The non-trivial part of the S-matrix is captured by the T matrix $S = 1 + \mathrm{i}T$ which must satisfy the so-called optical theorem, following directly from unitarity of S

$$S^\dagger S = 1 \qquad \Leftrightarrow \qquad -\mathrm{i}\left(T - T^\dagger\right) = T^\dagger T. \tag{1.4}$$

To relate this equation to amplitudes, it is inserted between multi-particle states

$$-2\,\mathrm{Im}\,\big\langle \chi_1 \dots \chi_i | T | \chi_{i+1} \dots \chi_n \big\rangle = \sum_\chi \big\langle \chi_1 \dots \chi_i | T^\dagger | \chi \big\rangle \big\langle \chi | T | \chi_{i+1} \dots \chi_n \big\rangle \,. \tag{1.5}$$

Each particle in $\mathbb{R}^{D-1,1}$ is specified by $\chi = [m, \lambda]$, the mass m and an irrep λ of the little group $SO(D-1)$ for massive or $SO(D-2)$ for massless particles. Furthermore a complete set of states was inserted, which at tree-level involves only single particle states $|\chi\rangle$ with momentum fixed by momentum conservation. Delta functions and integrals are omitted here for simplicity. This relation directly turns into a formula expressing any residue at a physical pole in terms of lower point amplitudes[1]

$$\text{Res}_{(k_1+\ldots+k_i)^2\to m^2}\mathcal{A}_{\chi_1\cdots\chi_n} = \sum_{\lambda} \mathcal{A}_{\chi_1\cdots\chi_i[m,\lambda]}\mathcal{A}_{[m,\lambda]\chi_{i+1}\cdots\chi_n} , \tag{1.7}$$

where the sum on the right hand side is now over the spectrum of the theory at the mass specified by the position of the pole. Hence an amplitude is now expressed in terms of a sum over products of lower-point on-shell amplitudes and the on-shell recursion relation is complete.

In string theory, the spectrum includes an infinite tower of masses and for each mass an increasing number of tensor states. This makes it hard to use the on-shell recursion relation in practice, however it implies that the mathematical problem is essentially the same as the decomposition of conformal correlators into partial waves discussed above (1.2). The general tensor structures that can appear on the right hand side of (1.7) will be constructed as an example for the application of the formalism in the context of scattering amplitudes. The on-shell recursion relations, which were previously used only for the typical field content of field theories with spin up to 2, are made explicit for any bosonic field content, however using them in general is still prohibitively complicated.

The key to a more efficient way to compute the residues is to pay attention to the zeros of the amplitudes. For example, consider the Veneziano amplitude, describing the scattering of four tachyons in open bosonic string theory

$$\mathcal{A}(s_{12}, s_{23}) \propto \frac{\Gamma[\alpha(s_{12})]\Gamma[\alpha(s_{23})]}{\Gamma[\alpha(s_{12}) + \alpha(s_{23})]} . \tag{1.8}$$

In addition to the poles generated by the gamma functions in the numerator which directly determine a part of the mass spectrum of the theory, there is also a gamma function in the denominator generating zeros of the amplitude. If the position of those zeros could be predicted this would be an important input for a target space derivation of the amplitudes. It turns out that this question was studied a long time ago [22] with the result that some of the zeros can be predicted based on the monodromy relations found in [23]. A modern CFT based derivation of these relations appeared in [21]. Here it will be shown that the monodromy relations predict the location of the zeros of the residues of amplitudes at kinematic poles and (at least in the case of the Koba-Nielsen amplitudes) the complete residues.

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[1] An imaginary part can only come from the $i\epsilon$ in the propagators

$$\lim_{\epsilon\to 0} \frac{1}{(k_1+\ldots+k_i)^2 - m^2 + i\epsilon} = \frac{1}{(k_1+\ldots+k_i)^2 - m^2} - i\pi\delta\left((k_1+\ldots+k_i)^2 - m^2\right) , \tag{1.6}$$

whenever an internal particle goes on-shell. This relates (1.5) and (1.7).

1.3 Organization of the thesis

Chapter 2 introduces a formalism that allows efficient computations with irreducible tensor representations of orthogonal groups, including mixed-symmetry tensors. These methods can be used for computations involving either CTF correlators or scattering amplitudes in general spacetime dimensions. Crucial ingredients are the encoding of mixed-symmetry tensors as polynomials, the counting and construction of independent tensor structures and the embedding of the $SO(d)$ irreps into Minkowski space $\mathbb{R}^{d,1}$ or $\mathbb{R}^{d+1,1}$ for describing massive or massless particles in spacetime or conformal primaries in the embedding space formalism. This chapter is mainly based on [24].

In Chapter 3 the methods of the previous chapter are applied to the case of CFT correlators to construct the tensor structures for any CFT correlator of bosonic primaries in d dimensions. A simple derivation of the unitarity bound for such primaries is obtained by considering the lift of the conservation condition into embedding space. Furthermore we present how the constructed conformal correlators can be combined with the shadow formalism to compute arbitrary conformal blocks, which is illustrated with an example. This chapter is based on [24].

Chapter 4 applies the methods of Chapter 2 to the case of scattering amplitudes of open bosonic string states. As an example involving general tensor representations, residues of Koba-Nielsen amplitudes are related via unitarity cuts to general three-point amplitudes. Results include the derivation of the three-point amplitudes for the coupling of two tachyons to any other state in the spectrum, an (almost complete) proof of the unitarity of the Veneziano amplitude and an alternative derivation of the parameters that are traditionally computed by the no-ghost theorem. This chapter is based on [25].

Chapter 5 presents an alternative way to compute the residues of string amplitudes which serve as input for computing amplitudes using the BCFW residue formula. This method does not use unitarity cuts to compute the residues, but instead exploits that all open string amplitudes obey monodromy relations. This chapter is based on [25].

The discussion and outlook regarding further research directions is located in Chapter 6. The appendices contain some of the more technical computations and a code sample for the convenient computation of $SO(d)$ tensor products.

Chapter 2

Irreducible tensor representations of SO(d)

This chapter introduces a formalism allowing efficient and clear computations involving general irreducible tensor representations of $SO(d)$. The formalism is presented in a form general enough to allow its use both for CFT correlators and scattering amplitudes. Examples of such general representations appearing in both contexts were shown above in (1.2) and (1.7). The main idea is to construct irreducible polarization tensors out of vectors, which leads to short formulae without free indices. The construction of tensor structures, the linearly independent terms in functions involving multiple $SO(d)$ irreps, is discussed as well as the embedding of $SO(d)$ irreps into $d+2$ or $d+1$ dimensional Minkowski space, which is relevant for the description of CFT correlators in the embedding space formalism as well as scattering amplitudes of massless or massive particles.

2.1 Parametrizing Young diagrams

The traceless irreducible tensor representations of $SO(d)$ are enumerated mostly[1] by Young diagrams, which encode the (anti-)symmetry of the tensors under permutation of their indices, for example

$$\lambda = \ydiagram{4,2,1} = [2,1,1]\,, \qquad l^\lambda = (4,2,1)\,, \qquad h^\lambda = (3,2,1,1)\,. \tag{2.1}$$

There are two different ways to parametrize the shape of a Young diagram λ. The first is by giving a partition $l^\lambda = \left(l_1^\lambda, l_2^\lambda, \ldots\right)$ containing the lengths of the rows, l_i^λ being the length of the i-th row. The diagram that is obtained from λ by exchanging rows and columns is called the transpose λ^t. The partition h^λ describes the column heights of λ and is the conjugate partition to l^λ, $l^{\lambda^t} \equiv h^\lambda = \left(h_1^\lambda, h_2^\lambda, \ldots\right)$. A second way to describe the shape of a Young diagram is by its Dynkin label $\lambda = \left[\lambda_1, \lambda_2, \ldots, \lambda_{h_1^\lambda}\right]$, which lists the numbers λ_i of columns with i boxes. Apart from the exception mentioned in the footnote, the Young diagram λ

[1] There is a one-to-one correspondence between traceless irreducible tensor representations of $SO(d)$ and the Young diagrams satisfying (2.2) except for the case $d = 2n, h_1^\lambda = n$ [26]. In this case the representation with the symmetry corresponding to λ can be decomposed further using the Levi-Civita tensor and is therefore not irreducible. This restriction can be ignored under the assumption that the considered theory is parity invariant, which we will do throughout.

labels an irrep of $SO(d)$ if and only if its overall height h_1^λ does not exceed the rank of the Lie algebra corresponding to $SO(d)$,

$$h_1^\lambda \leq \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} \frac{d}{2}\,, & d \text{ even}\,, \\ \frac{d-1}{2}\,, & d \text{ odd}\,. \end{cases} \tag{2.2}$$

The total number of boxes is denoted by $|\lambda|$

$$|\lambda| = \sum_i i\lambda_i = \sum_i l_i^\lambda = \sum_i h_i^\lambda\,. \tag{2.3}$$

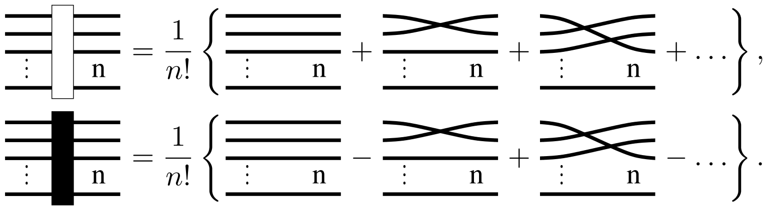
The λ on l^λ and h^λ will frequently be omitted or replaced by i if the Young diagram is of shape λ_i.

2.2 Birdtracks and Grassmann variables

Probably the best way to think about mixed-symmetry tensors is in terms of birdtrack notation[2] where index contractions are simply drawn as lines

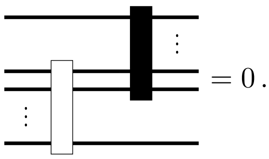
$$= \delta^{a_1 b_1}\delta^{a_2 b_2}\,. \tag{2.4}$$

Symmetrization and antisymmetrization are indicated by the symbols



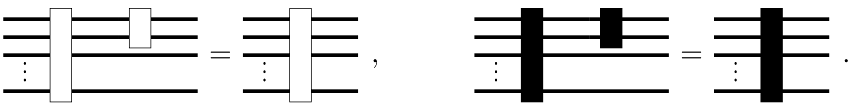
(2.5)

This notation has the advantage that it makes it immediately visible when terms are vanishing because two or more symmetric indices are antisymmetrized or vice versa



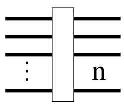
(2.6)

Furthermore, birdtracks can be diagrammatically transformed, for example using that repeated (anti)symmetrizations of subsets of indices have no effect



(2.7)

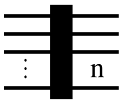
A symmetrized contraction of n indices is generated by the n-th derivative of n components of an auxiliary vector z,



$$= \frac{1}{n!}\,\partial_{z_{a_1}} \ldots \partial_{z_{a_n}} z_{b_1} \ldots z_{b_n}\,. \tag{2.8}$$

[2] See [27] for a beautiful group theory book entirely in terms of birdtracks.

Antisymmetrization works analogously with an auxiliary vector in Grassmann variables θ,



$$= \frac{1}{n!}\,\partial_{\theta_{a_1}} \dots \partial_{\theta_{a_n}} \theta_{b_1} \dots \theta_{b_n}\,. \tag{2.9}$$

The Grassmann variables are anticommuting in the sense that

$$\theta_a^{(p)} \theta_b^{(q)} = (-1)^{\delta^{pq}} \theta_b^{(q)} \theta_a^{(p)}\,. \tag{2.10}$$

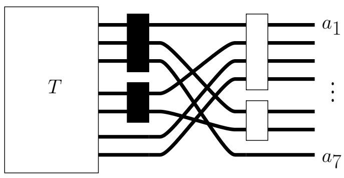
Here an additional label (p) was introduced to allow for several independent antisymmetrizations at the same time. Derivatives with respect to Grassmann variables are implied to be right derivatives,

$$\partial_{\theta_c^{(r)}} \theta_a^{(p)} \theta_b^{(q)} = \delta^{rq} \delta^{cb} \theta_a^{(p)} + (-1)^{\delta^{pq}} \delta^{rp} \delta^{ca} \theta_b^{(q)}\,. \tag{2.11}$$

2.3 Young symmetrization and antisymmetric basis

The symmetry of a Young diagram is imposed on a tensor via Young symmetrizers. Each row of the diagram corresponds to a symmetrization and each column corresponds to an antisymmetrization. This can be nicely illustrated by an example, following [27]. To actually write down components of mixed-symmetry tensors it is necessary to choose a basis for the irreducible representation at hand. This requires an assignment between the boxes of the Young diagram and the indices of the tensor. Therefore the bases of the irreps under consideration are labeled by Young tableaux. A symmetrizer given by the Young tableau $\mathbf{YT}$ creates the tensor $T^{\mathbf{YT}}$ with appropriate symmetry from a generic tensor T,

$$\mathbf{YT} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \rightarrow \qquad T^{\mathbf{YT}} = \tag{2.12}$$

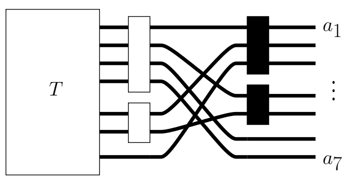


This tensor has the manifest symmetry properties

$$T^{\mathbf{YT}}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7} = T^{\mathbf{YT}}_{(a_1 a_2 a_3 a_4)(a_5 a_6) a_7}\,, \tag{2.13}$$

but there are also less obvious symmetries caused by the antisymmetrizations. Due to the manifest symmetries, $T^{\mathbf{YT}}$ is said to belong to the symmetric basis. The antisymmetric basis is obtained by changing the order of symmetrization and antisymmetrization

$$\mathbf{YT}' = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad \rightarrow \qquad T^{\mathbf{YT}'} = \tag{2.14}$$



Here we have manifest antisymmetry

$$T^{\mathbf{YT}'}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7} = T^{\mathbf{YT}'}_{[a_1 a_2 a_3][a_4 a_5](a_6 a_7)}\,. \tag{2.15}$$

The only reason we used a different Young tableau for this second example is to spare us from having to cross lines on the right hand side of the birdtrack diagram. We will in this thesis work only in antisymmetric bases with Young tableaux where the boxes are enumerated column by column, as in (2.14). The tensors corresponding to different bases (different tableaux) can be obtained simply by commutation of indices.

It may also be instructive to see how the non-explicit index symmetries manifest themselves on the components of the tensors, again in the antisymmetric basis with boxes labeled column by column. To this end assign different labels to each anticommuting group of indices

$$f_{a_1\ldots a_{h_1}b_1\ldots b_{h_2}c_1\ldots c_{h_3}\ldots g_1\ldots g_{h_{l_1}}} = f_{[a_1\ldots a_{h_1}][b_1\ldots b_{h_2}][c_1\ldots c_{h_3}]\ldots[g_1\ldots g_{h_{l_1}}]} \,. \tag{2.16}$$

Apart from the antisymmetry, the Young symmetrization implies that the antisymmetrization of any of the indices b with all the a vanishes, as well as the antisymmetrization of any of the c with all indices a or all b and so forth [28]. Explicitly this means that

$$\begin{aligned}
& f_{[a_1\ldots a_{h_1}][b_1\ldots b_{h_2}][c_1\ldots c_{h_3}]\ldots[g_1\ldots g_{h_{l_1}}]} \qquad (2.17)\\
&= f_{[b_1a_2\ldots a_{h_1}][a_1b_2\ldots b_{h_2}][c_1\ldots} + f_{[a_1b_1a_3\ldots a_{h_1}][a_2b_2\ldots b_{h_2}][c_1\ldots} + \cdots + f_{[a_1\ldots a_{h_1-1}b_1][a_{h_1}b_2\ldots b_{h_2}][c_1\ldots}\\
&= f_{[c_1a_2\ldots a_{h_1}][b_1\ldots b_{h_2}][a_1c_2\ldots} + f_{[a_1c_1a_3\ldots a_{h_1}][b_1\ldots b_{h_2}][a_2c_2\ldots} + \cdots + f_{[a_1\ldots a_{h_1-1}c_1][b_1\ldots b_{h_2}][a_{h_1}c_2\ldots} \,.
\end{aligned}$$

There are also more general relations that arise from exchanging k indices from one column with all possible k-element subsets of a column to its left. Here the order of the two sets of indices is kept, so that the right hand side of the general equation has $\binom{h_l}{k}$ terms if the left column has height h_l. As a special case of these relations the tensors are symmetric under exchange of complete groups of antisymmetric indices if the corresponding columns in the Young tableau are of equal height, e.g. for $h_2 = h_3$,

$$f_{[a_1\ldots a_{h_1}][b_1\ldots b_{h_2}][c_1\ldots c_{h_3}]\ldots[g_1\ldots g_{h_{l_1}}]} = f_{[a_1\ldots a_{h_1}][c_1\ldots c_{h_3}][b_1\ldots b_{h_2}]\ldots[g_1\ldots g_{h_{l_1}}]} \,. \tag{2.18}$$

Since it will be needed in Section 3.2 we also state the equation analogous to (2.17) for a tensor

$$f_{a_1\ldots a_{l_1}b_1\ldots b_{l_2}c_1\ldots c_{l_3}\ldots g_1\ldots g_{l_{h_1}}} = f_{(a_1\ldots a_{l_1})(b_1\ldots b_{l_2})(c_1\ldots c_{l_3})\ldots(g_1\ldots g_{l_{h_1}})} \,, \tag{2.19}$$

in the symmetric basis with boxes enumerated row by row as in (2.12),

$$\begin{aligned}
& -f_{(a_1\ldots a_{l_1})(b_1\ldots b_{l_2})(c_1\ldots c_{l_3})\ldots(g_1\ldots g_{l_{h_1}})} \qquad (2.20)\\
&= f_{(b_1a_2\ldots a_{l_1})(a_1b_2\ldots b_{l_2})(c_1\ldots} + f_{(a_1b_1a_3\ldots a_{l_1})(a_2b_2\ldots b_{l_2})(c_1\ldots} + \cdots + f_{(a_1\ldots a_{l_1-1}b_1)(a_{l_1}b_2\ldots b_{l_2})(c_1\ldots}\\
&= f_{(c_1a_2\ldots a_{l_1})(b_1\ldots b_{l_2})(a_1c_2\ldots} + f_{(a_1c_1a_3\ldots a_{l_1})(b_1\ldots b_{l_2})(a_2c_2\ldots} + \cdots + f_{(a_1\ldots a_{l_1-1}c_1)(b_1\ldots b_{l_2})(a_{l_1}c_2\ldots} \,.
\end{aligned}$$

2.4 Encoding mixed-symmetry tensors by polynomials

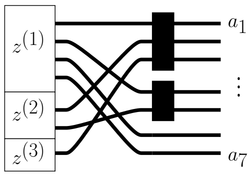
In general, to encode a traceless mixed-symmetry tensor by a polynomial, the strategy is to contract it with a polarization tensor which is by construction in the same irrep. This constrains the original tensor to be in the same irrep as the polarization, because the full contraction of two tensors of given irreps vanishes unless the irreps are the same. Another way to state this is that the tensor product of two $SO(d)$ irreps λ_1 and λ_2 contains the scalar representation if and only if $\lambda_1 = \lambda_2$. This will be discussed in more detail in Section 2.6 below.

Consider as an example the contraction of a tensor $f^{a_1 \dots a_{|\lambda|}}$ in the irrep λ with $|\lambda|$ copies of the same auxiliary vector z_a. The result is a homogeneous polynomial of degree $|\lambda|$ if $f^{a_1 \dots a_{|\lambda|}}$ is a fully symmetric tensor and vanishes if it is a mixed-symmetry tensor

$$z_{a_1} \dots z_{a_{|\lambda|}} f^{a_1 \dots a_{|\lambda|}} = \begin{cases} f(z)\,, & h_1^\lambda = 1\,, \\ 0\,, & h_1^\lambda > 1\,. \end{cases} \tag{2.21}$$

If the polarization is in the same irrep as the original tensor, the polynomial contains the full information about the tensor. The polarization tensor $z_{a_1} \dots z_{a_{|\lambda|}}$ is traceless if the vector z_a satisfies $z^2 = 0$. Next the general case of mixed-symmetry tensors will be discussed, ignoring tracelessness at first and coming back to this at the very end of the section.

To construct a Young symmetrized tensor in the antisymmetric basis out of auxiliary vectors, one can start with a set of polarizations that is already symmetrized so that only the antisymmetrization is left to do. For the example (2.14), the following tensor depending on the auxiliary vectors $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$ is appropriately symmetrized



$$. \tag{2.22}$$

Using (2.9) to encode the antisymmetrization, (2.22) can be written as

$$\begin{aligned} &\frac{1}{3!2!} \left(z^{(1)} \cdot \partial_{\theta^{(1)}}\right) \left(z^{(1)} \cdot \partial_{\theta^{(2)}}\right) \left(z^{(1)} \cdot \partial_{\theta^{(3)}}\right) \left(z^{(1)} \cdot \partial_{\theta^{(4)}}\right) \\ &\left(z^{(2)} \cdot \partial_{\theta^{(1)}}\right) \left(z^{(2)} \cdot \partial_{\theta^{(2)}}\right) \left(z^{(3)} \cdot \partial_{\theta^{(1)}}\right) \theta^{(1)}_{a_1} \theta^{(1)}_{a_2} \theta^{(1)}_{a_3} \theta^{(2)}_{a_4} \theta^{(2)}_{a_5} \theta^{(3)}_{a_6} \theta^{(4)}_{a_7}\,. \end{aligned} \tag{2.23}$$

This can be shortened by avoiding the introduction of polarizations that appear only once and hence do not cause any (anti-)symmetrization, i.e. doing explicitly the derivatives in the polarizations $\theta^{(3)}$ and $\theta^{(4)}$,

$$\left(z^{(1)} \cdot \partial_{\theta^{(3)}}\right) \left(z^{(1)} \cdot \partial_{\theta^{(4)}}\right) \theta^{(3)}_{a_6} \theta^{(4)}_{a_7} = z^{(1)}_{a_6} z^{(1)}_{a_7}\,. \tag{2.24}$$

After this step the symmetry in the indices a_6 and a_7 is manifest. Likewise, $z^{(3)}$ that appears only once in this example through the derivative $\left(z^{(3)} \cdot \partial_{\theta^{(1)}}\right)$, does not encode any symmetry. More generally, for diagrams with more than one row of length one, the action of such derivatives hides antisymmetry. We shall therefore omit these derivative terms, with the result that the encoding polynomial will depend not only on symmetric polarizations, but also on $\theta^{(1)}$, therefore making antisymmetrization explicit on the indices corresponding to all rows of length one.

Thus, the slightly less elegant, but more pragmatic Young symmetric polarization we use for the example at hand will be the polynomial in $\mathbf{z} \equiv \left(z^{(1)}, z^{(2)}, \theta^{(1)}\right)$ given by

$$\left(z^{(1)} \cdot \partial_{\theta^{(1)}}\right) \left(z^{(1)} \cdot \partial_{\theta^{(2)}}\right) \left(z^{(2)} \cdot \partial_{\theta^{(1)}}\right) \left(z^{(2)} \cdot \partial_{\theta^{(2)}}\right) \theta^{(1)}_{a_1} \theta^{(1)}_{a_2} \theta^{(1)}_{a_3} \theta^{(2)}_{a_4} \theta^{(2)}_{a_5} z^{(1)}_{a_6} z^{(1)}_{a_7}\,, \tag{2.25}$$

which is quartic in $z^{(1)}$, quadratic in $z^{(2)}$ and linear in $\theta^{(1)}$, as appropriate for a Young diagram with lengths of rows given by $l^\lambda = (4, 2, 1)$. This Young symmetric polarization is obtained by acting with derivatives of the type $\left(z^{(p)} \cdot \partial_{\theta^{(q)}}\right)$ on a polynomial in $\boldsymbol{\theta} \equiv \left(\theta^{(1)}, \theta^{(2)}, z^{(1)}\right)$, cubic in $\theta^{(1)}$, quadratic in $\theta^{(2)}$ and quadratic in $z^{(1)}$, as appropriate for a Young diagram with lengths of columns given by $h^\lambda = (3,2,1,1)$. A tensor with components $f^{a_1 \dots a_7}$ in the irrep of this example will then be encoded by the polynomial

$$f(\mathbf{z}) \equiv \left(z^{(1)} \cdot \partial_{\theta^{(1)}}\right)\left(z^{(1)} \cdot \partial_{\theta^{(2)}}\right)\left(z^{(2)} \cdot \partial_{\theta^{(1)}}\right)\left(z^{(2)} \cdot \partial_{\theta^{(2)}}\right) \bar{f}(\boldsymbol{\theta}) , \tag{2.26}$$

where

$$\bar{f}(\boldsymbol{\theta}) \equiv \theta^{(1)}_{a_1}\theta^{(1)}_{a_2}\theta^{(1)}_{a_3}\theta^{(2)}_{a_4}\theta^{(2)}_{a_5} z^{(1)}_{a_6} z^{(1)}_{a_7} f^{a_1 \dots a_7} . \tag{2.27}$$

Notice that, in this example, the assignment of the polarization vectors in $\mathbf{z}$ and in $\boldsymbol{\theta}$ to the boxes of the Young diagram is done according to

$$\begin{array}{|c|c|c|c|} \hline z^{(1)} & z^{(1)} & z^{(1)} & z^{(1)} \\ \hline z^{(2)} & z^{(2)} \\ \cline{1-2} \theta^{(1)} \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|c|} \hline \theta^{(1)} & \theta^{(2)} & z^{(1)} & z^{(1)} \\ \hline \theta^{(1)} & \theta^{(2)} \\ \cline{1-2} \theta^{(1)} \\ \cline{1-1} \end{array} \quad , \tag{2.28}$$

respectively.

In general we shall consider n_Z commuting and n_Θ anticommuting polarization vectors for a given tensor operator, where n_Z is the number of rows with more than one box and n_Θ is the number of columns with more than one box

$$n_Z^\lambda = \sum_{i=2}^{l_1^\lambda} \lambda_i^t , \qquad n_\Theta^\lambda = \sum_{i=2}^{h_1^\lambda} \lambda_i . \tag{2.29}$$

A convenient notation for the mostly anticommuting polarizations which are first contracted to the tensor is

$$\boldsymbol{\theta} \equiv \left(\theta^{(1)}, \theta^{(2)} \dots, \theta^{(n_\Theta)}, z^{(1)}\right) . \tag{2.30}$$

In cases where there are no columns with one box the last entry is absent and $\boldsymbol{\theta}$ contains only anti-commuting polarizations. Similarly, we will write for the mostly commuting polarizations on which the final encoding polynomial depends

$$\mathbf{z} \equiv \left(z^{(1)}, z^{(2)}, \dots, z^{(n_Z)}, \theta^{(1)}\right) . \tag{2.31}$$

Again, in cases where there are no rows with one box the last entry is absent and $\mathbf{z}$ contains only commuting polarizations. Generalizing the previous example, we have that a tensor $f^{a_1 \dots a_{|\lambda|}}$ in the irrep λ is encoded by the polynomial

$$f(\mathbf{z}) \equiv \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_\Theta)} \left(z^{(p)} \cdot \partial_{\theta^{(q)}}\right) \bar{f}(\boldsymbol{\theta}) , \tag{2.32}$$

where

$$\begin{aligned} \bar{f}(\boldsymbol{\theta}) \equiv \theta^{(1)}_{a_1} \dots \theta^{(1)}_{a_{h_1}} \theta^{(2)}_{a_{h_1+1}} \dots \theta^{(2)}_{a_{h_1+h_2}} \\ \dots \theta^{(n_\Theta)}_{a_{h_1+\dots+h_{n_\Theta-1}+1}} \dots \theta^{(n_\Theta)}_{a_{h_1+\dots+h_{n_\Theta}}} z^{(1)}_{a_{|\lambda|-\lambda_1+1}} \dots z^{(1)}_{a_{|\lambda|}} f^{a_1 \dots a_{|\lambda|}} . \end{aligned} \tag{2.33}$$

When there are more than one row with one box, the dependence of $f(\mathbf{z})$ on $\theta^{(1)}$ makes manifest the antisymmetry of the indices corresponding to such boxes. Likewise, when there are more than one column with one box, the dependence of $\bar{f}(\boldsymbol{\theta})$ on $z^{(1)}$ makes manifest the symmetry of the indices corresponding to such boxes.

The constructed polarization tensors (e.g. (2.25)) can easily be made traceless by choosing all the auxiliary vectors to have vanishing products. In this way the requirement that the polarization tensor is in an $SO(d)$ irrep is fulfilled. As an additional bonus, this prescription makes for shorter encoding polynomials

$$
\begin{aligned}
f^{a_1\dots a_{|\lambda|}} \text{ traceless } &\leftrightarrow \bar{f}(\boldsymbol{\theta})\big|_{\theta^{(p)}\cdot\theta^{(q)}=\theta^{(p)}\cdot z^{(1)}=z^{(1)2}=0}\,, \\
&\leftrightarrow f\left(\mathbf{z}\right)\big|_{z^{(p)}\cdot z^{(q)}=z^{(p)}\cdot\theta^{(1)}=0}\,.
\end{aligned}
\tag{2.34}
$$

This means that all terms in the tensor $f^{a_1\dots a_{|\lambda|}}$ proportional to Kronecker deltas $\delta^{a_i a_j}$ are discarded from the encoding polynomials. They have to be restored by projection to traceless tensors if one wishes to extract the tensor from the polynomial.

2.5 Projectors to traceless tensors

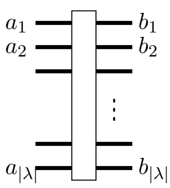
To extract the tensor $f^{a_1\dots a_{|\lambda|}}$ back from the polynomials one can simply restore the indices by acting with $|\lambda|$ derivatives on the polarizations and then project to the irreducible representation λ with the projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$,

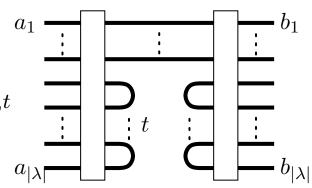
$$
\begin{aligned}
f^{a_1\dots a_{|\lambda|}} &= \pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}\,\frac{1}{h_1!}\,\partial_{\theta^{(1)}_{b_1}}\dots\partial_{\theta^{(1)}_{b_{h_1}}}\,\frac{1}{h_2!}\,\partial_{\theta^{(2)}_{b_{h_1+1}}}\dots\partial_{\theta^{(2)}_{b_{h_1+h_2}}} \\
&\quad \dots\frac{1}{h_{n_\Theta}!}\,\partial_{\theta^{(n_\Theta)}_{b_{h_1+\dots+h_{n_\Theta-1}+1}}}\dots\partial_{\theta^{(n_\Theta)}_{b_{h_1+\dots+h_{n_\Theta}}}}\,\frac{1}{\lambda_1!}\,\partial_{z^{(1)}_{b_{|\lambda|-\lambda_1+1}}}\dots\partial_{z^{(1)}_{b_{|\lambda|}}}\,\bar{f}(\boldsymbol{\theta}) \qquad (2.35)\\
&= \pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}\,\frac{1}{H(\lambda)}\,\partial_{z^{(1)}_{b_1}}\dots\partial_{z^{(1)}_{b_{l_1}}}\partial_{z^{(1)}_{b_{l_1+1}}}\dots\partial_{z^{(1)}_{b_{l_1+l_2}}} \\
&\quad \dots\partial_{z^{(n_Z)}_{b_{l_1+\dots+l_{n_Z-1}+1}}}\dots\partial_{z^{(n_Z)}_{b_{l_1+\dots+l_{n_Z}}}}\,\partial_{\theta^{(1)}_{b_{|\lambda|-\lambda_1^t+1}}}\dots\partial_{\theta^{(1)}_{b_{|\lambda|}}}\,f(\mathbf{z})\,. \qquad (2.36)
\end{aligned}
$$

The normalizations can be explained as follows. When extracting the components $f^{a_1\dots a_{|\lambda|}}$ from the polynomial $\bar{f}(\boldsymbol{\theta})$ all that happens is the antisymmetrization of a tensor which is already in the antisymmetric basis. For each set of antisymmetric indices every generated term is the same and the normalization factor only has to cancel the number of terms. Going from $f(\mathbf{z})$ to $f^{a_1\dots a_{|\lambda|}}$ involves a Young projection of a tensor that is already Young symmetrized. Therefore the normalization $H(\lambda)$ is that of the Young projectors, which are given in [27]. It is computed from the shape of λ by a hook rule. Write into each box of a Young diagram the number of boxes to its right and below, including the box itself. The product of all numbers is $H(\lambda)$. For example,

$$
H\left(\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\right) = H\left(\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\right) = 6\cdot 4\cdot 3\cdot 2\,. \tag{2.37}
$$

As far as we are aware an explicit general formula for the projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$ is only known for symmetric tensors

$$
\pi_{\square\square\square\cdots\square}^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}} =
$$


$$
+ \sum_{t=1}^{\lfloor\frac{|\lambda|}{2}\rfloor} W_{|\lambda|,t}
$$


$$, \tag{2.38}$$

with

$$W_{|\lambda|,0} = 1, \qquad W_{|\lambda|,t} = (-1)^t \frac{|\lambda|!}{(|\lambda|-2t)!2^t t!} \prod_{j=1}^{t} \frac{1}{d+2|\lambda|-2j-2} \,. \tag{2.39}$$

A derivation of these coefficients $W_{|\lambda|,t}$ is provided in an appendix of our paper [25]. These projectors can also be efficiently implemented using the differential operators [29]

$$D_z^a = \left(\frac{1}{|\lambda|!(\frac{d}{2}-1)^{(|\lambda|)}}\right)^{\frac{1}{|\lambda|}} \left(\left(\frac{d}{2}-1+z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z_a} - \frac{1}{2}\, z^a \frac{\partial^2}{\partial z\cdot\partial z}\right), \tag{2.40}$$

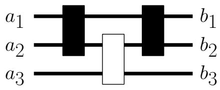
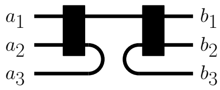
where $a^{(b)} = a(a+1)\dots(a+b-1)$ is the rising factorial. These operators satisfy

$$\pi^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}_{\Box\Box\Box\cdots\Box} = D_z^{a_1}\dots D_z^{a_{|\lambda|}} z^{b_1}\dots z^{b_{|\lambda|}} \,. \tag{2.41}$$

and can be used to express a contraction of two traceless symmetric tensors $f^{a_1\dots a_{|\lambda|}}$, $g^{a_1\dots a_{|\lambda|}}$ in terms of their encoding polynomials

$$f^{a_1\dots a_{|\lambda|}} g_{a_1\dots a_{|\lambda|}} = f(D_z)g(z) \,. \tag{2.42}$$

For the simplest mixed-symmetry tensor $\yng(2,1)$ the projector is [27]

$$\pi^{a_1a_2a_3,b_1b_2b_3}_{\yng(2,1)} = \frac{4}{3}$$  $$-\frac{2}{d-1}$$  $$. \tag{2.43}$$

Let $f^{a_1a_2a_3}$ and $g^{b_1b_2b_3}$ be two tensors in the irrep $\yng(2,1)$ and

$$\begin{aligned} f(\mathbf{z}) = f(z,\theta) &= (z\cdot\partial_\theta)\,\theta_{a_1}\theta_{a_2}z_{a_3}f^{a_1a_2a_3} = (\theta_{a_1}z_{a_2}z_{a_3} - \theta_{a_2}z_{a_1}z_{a_3})\,f^{a_1a_2a_3}\big|_{z^2=z\cdot\theta=0}\,,\\ g(\mathbf{z}) = g(z,\theta) &= (z\cdot\partial_\theta)\,\theta_{a_1}\theta_{a_2}z_{a_3}g^{a_1a_2a_3} = (\theta_{a_1}z_{a_2}z_{a_3} - \theta_{a_2}z_{a_1}z_{a_3})\,g^{a_1a_2a_3}\big|_{z^2=z\cdot\theta=0}\,, \end{aligned} \tag{2.44}$$

their encoding polynomials. We would like to know how to contract these tensors using directly the polynomials. The antisymmetrization in the projector (2.43) is already done in the construction of the polynomials, only the symmetrization and subtraction of the trace is left to do. This can be done by introducing a differential operator D_z^a that satisfies

$$D_z^{a_1}D_z^{a_2}z^{b_1}z^{b_2} = \frac{1}{4}\left(\frac{2}{3}\left(\delta^{a_1b_1}\delta^{a_2b_2} + \delta^{a_1b_2}\delta^{a_2b_1}\right) - \frac{2}{d-1}\,\delta^{a_1a_2}\delta^{b_1b_2}\right), \tag{2.45}$$

where the factor $\frac{1}{4}$ normalizes the antisymmetrizations. D_z^a can be found to be

$$D_z^a = \frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial z_a} - \frac{3}{2(d-1)}\, z^a \frac{\partial^2}{\partial z\cdot\partial z}\right). \tag{2.46}$$

The contraction of the two traceless tensors can then be expressed in terms of the encoding polynomials as

$$f^{a_1a_2a_3}g_{a_1a_2a_3} = f(D_z,\partial_\theta)\,g(z,\theta)\,. \tag{2.47}$$

This is entirely analogous to the situation of symmetric traceless tensors, but now the explicit form of the projector and corresponding differential operator acting on the polarization vectors is not known in general. We will assume that there exists for every irrep λ a set of differential operators

$$\mathbf{D_z} = \left(D^{(1)}_{z^{(1)}},\dots,D^{(n_Z)}_{z^{(n_Z)}},D^{(n_Z+1)}_{\theta^{(1)}}\right), \tag{2.48}$$

that reproduces the projector in this way. We have no proof that every projector can be expressed like this. If nothing else it is a notation that allows us to write any contraction as

$$f^{a_1\ldots a_{|\lambda|}} g_{a_1\ldots a_{|\lambda|}} = f(\mathbf{D_z})g(\mathbf{z})\,. \tag{2.49}$$

We postpone a more general treatment of the projectors to traceless mixed-symmetry tensors to a subsequent publication. See the discussion (Chapter 6) for a comment on this point.

2.6 Tensor-product coefficients

The main problem in constructing correlators involving mixed-symmetry tensors is finding all the possible ways a given set of such tensors can be contracted. A more mathematical way to pose this question is to ask for the multiplicity of the scalar representation in the tensor product of the tensors in question. Fortunately, this problem is already solved. Here we shall review the relevant results for our purposes; for a comprehensive introduction to the general properties of tensor-product coefficients see [9].

Let G be $SU(n)$, $SO(n)$ or $Sp(n)$ and λ, μ, ν irreducible G-modules which are enumerated by Young diagrams. These are the vector spaces of tensors with the index symmetries described in Section 2.4. They will often be called representations instead of modules in the following. λ^* denotes the vector space dual to λ, i.e. if λ contains tensors with lower indices, λ^* contains tensors with upper indices. Upper and lower indices can be contracted and the result will then transform under G as indicated by the remaining indices.

Let $\mathcal{N}_{\lambda\mu}{}^{\nu}$ be the tensor-product coefficients of G. They count the multiplicity with which the irrep ν appears in the tensor product of λ and μ

$$\lambda \otimes \mu = \bigoplus_{\nu} \mathcal{N}_{\lambda\mu}{}^{\nu}\, \nu\,, \tag{2.50}$$

and satisfy

$$\mathcal{N}_{\lambda\bullet}{}^{\nu} = \delta_{\lambda}^{\nu}\,, \qquad \mathcal{N}_{\lambda\lambda^*}{}^{\bullet} = 1\,, \qquad \mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda\nu^*}{}^{\mu^*}\,, \tag{2.51}$$

where $\bullet$ denotes the scalar representation. Let us also denote by $\mathcal{N}_{\lambda\mu\nu}$ the multiplicity of the scalar representation in the triple product

$$\lambda \otimes \mu \otimes \nu = \mathcal{N}_{\lambda\mu\nu}\, \bullet\, \oplus \text{ other irreps}\,. \tag{2.52}$$

This notation has the advantage of being symmetric in its three labels and contains the same information due to

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda\mu\nu^*}\,. \tag{2.53}$$

The multiplicity of a given representation μ in products of more than two tensors will be denoted by $\mathcal{N}_{\lambda_1\ldots\lambda_n}{}^{\mu}$

$$\lambda_1 \otimes \ldots \otimes \lambda_n = \bigoplus_{\mu} \mathcal{N}_{\lambda_1\ldots\lambda_n}{}^{\mu}\, \mu\,, \tag{2.54}$$

and can be calculated by recursively using (2.50)

$$\mathcal{N}_{\lambda_1\ldots\lambda_n}{}^{\mu} = \sum_{\nu_3,\ldots,\nu_n} \mathcal{N}_{\lambda_1\lambda_2}{}^{\nu_3} \prod_{i=3}^{n-1} \left(\mathcal{N}_{\nu_i\lambda_i}{}^{\nu_{i+1}} \right) \mathcal{N}_{\nu_n\lambda_n}{}^{\mu}\,. \tag{2.55}$$

This also computes the multiplicity of the scalar representation in the product $\lambda_1 \otimes \ldots \otimes \lambda_n \otimes \mu^*$.

2.6.1 Unitary groups

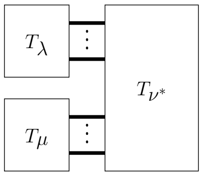
When specializing to $G = SU(n)$ the tensor-product coefficients are the famous Littlewood-Richardson coefficients ${c_{\lambda\mu}}^{\nu}$,

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = c_{\lambda\mu}{}^{\nu} \quad \text{for } G = SU(n)\,. \tag{2.56}$$

The only allowed contraction in this group is between upper and lower indices, so the number of indices adds up when the tensor product between two tensors with lower indices is formed

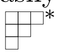
$$c_{\lambda\mu}{}^{\nu} = 0 \quad \text{for } |\lambda| + |\mu| \neq |\nu|\,. \tag{2.57}$$

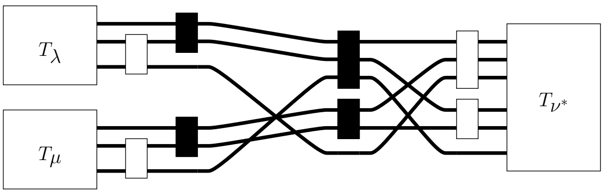
This implies that the product of three tensors can only contain the scalar representation if one of them is in a dual representation relative to the other two. This can be illustrated by the following schematic contraction of tensor indices



$$\propto \; c_{\lambda\mu}{}^{\nu} \bullet\,. \tag{2.58}$$

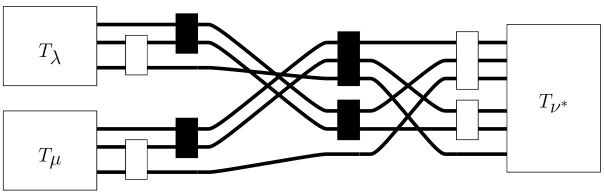
The coefficients $c_{\lambda\mu}{}^{\nu}$ can be calculated using the Littlewood-Richardson rule [30].[3]

For simple examples one can often find the possible contractions for a given tensor product quickly using birdtracks. For example, one can easily convince oneself that the only two inequivalent ways to contract $\lambda = \mu =$  and $\nu^* =$ * are



$$, \tag{2.59}$$

and



$$. \tag{2.60}$$

The Littlewood-Richardson coefficient is thus $c_{\lambda\mu}{}^{\nu} = 2$.

2.6.2 Orthogonal and symplectic groups

Following the reasoning of [27], the orthogonal and symplectic groups can be obtained from the unitary groups by taking into account the fact that these groups have by definition additional group invariants. For $SO(d)$ this is a symmetric quadratic form g_{ab} and its inverse g^{ab}, while for $Sp(d)$ the invariant is skew symmetric $f_{ab} = -f_{ba}$. In both cases these invariants can be used to raise and lower indices, which implies that the distinction between the two becomes unnecessary, the representations are self-dual $\lambda^* = \lambda$.[4] Any two indices can be

[3] The algorithm has been implemented for instance in Anders Skovsted Buch's lrcalc program. Code and instructions for doing the tensor product calculations in this thesis on the basis of lrcalc are given in Appendix A.1.

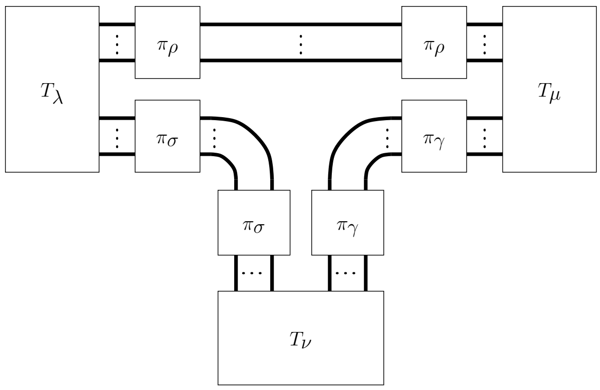
[4] Another way to explain self-duality for $SO(d)$ is to note that the group acts on the dual representation by the transpose of the inverse, which is the same as the original orthogonal matrix.

contracted and this leads to different tensor-product coefficients

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda\mu\nu} = b_{\lambda\mu\nu} \quad \text{for } G \in \left\{SO(2n), SO(2n+1), Sp(2n)\right\}. \tag{2.61}$$

Because of the self-duality of the representations the position of the indices of the tensor-product coefficients becomes meaningless, so these coefficients are always written with only lower indices. It is not hard to convince oneself that the counting of tensor structures here can be broken down to the counting that was relevant in the $SU(n)$ case where the restriction $|\lambda|+|\mu| = |\nu|$ applied. The following figure shows how three sets of indices can be contracted with each other, by first dividing each set of indices into two,

$$\sum_{\rho,\sigma,\gamma}$$



$$\propto b_{\lambda\mu\nu}\,\bullet\,, \tag{2.62}$$

where π_λ is a projector to the irrep λ, as introduced above. The number of tensor structures obtained in such a way is

$$b_{\lambda\mu\nu} = \sum_{\rho,\sigma,\gamma} c_{\sigma\rho}{}^{\lambda}\, c_{\rho\gamma}{}^{\mu}\, c_{\gamma\sigma}{}^{\nu}. \tag{2.63}$$

This formula is known as the Newell-Littlewood formula [31, 32] and holds if the sum of the heights of two of the three irreps λ, μ and ν does not exceed n, i.e. for

$$h_1^\lambda + h_1^\mu + h_1^\nu - \max\left(h_1^\lambda, h_1^\mu, h_1^\nu\right) \leq n = \left\lfloor \frac{d}{2} \right\rfloor. \tag{2.64}$$

Otherwise even the tensor product of the two irreps with the smallest h_1 contains Young diagrams that violate (2.2) and hence do not correspond to irreps of $SO(d)$ or $Sp(d)$. In this case (2.63) can be used anyway by transforming these Young diagrams into diagrams that correspond to irreps using modification rules [26] and taking the additional contributions that arise in this way into account. Then also the statement (2.61) that the tensor-product coefficients are the same for $SO(2n)$, $SO(2n+1)$ and $Sp(2n)$ does not hold true anymore. For simplicity, we will assume (2.64) to be satisfied throughout the thesis. Note that this implies that explicit examples in this thesis hold only for d sufficiently large.

The coefficients describing the decomposition of the tensor product of more than two irreps are given by (2.55)

$$b_{\lambda_1\dots\lambda_n} = \sum_{\nu_3,\dots,\nu_n} b_{\lambda_1\lambda_2\nu_3} \prod_{i=3}^{n-2} \left(b_{\nu_i\lambda_i}{}^{\nu_{i+1}}\right) b_{\nu_{n-1}\lambda_{n-1}\lambda_n}. \tag{2.65}$$

For $SO(d)$ or $Sp(d)$ the same coefficients also count the multiplicity of the scalar representation in the tensor product $\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n$.

A notation that will be used below is the restriction of a tensor product to irreps that have the same number of indices as both irreps in the product. This operation will be denoted with square brackets and amounts to using the $SU(n)$ Littlewood-Richardson coefficients as tensor-product coefficients,

$$[\lambda \otimes \mu] \equiv \bigoplus_{\nu} b_{\lambda\mu\nu}\, \nu \Big|_{|\nu|=|\lambda|+|\mu|} = \bigoplus_{\nu} c_{\lambda\mu}{}^{\nu}\, \nu\,. \tag{2.66}$$

The second equality can be found for instance in [26]. To wrap up this section consider the following example

$$[\lambda \otimes \mu \otimes \nu] \otimes \rho \otimes \sigma = \left(\sum_{\gamma,\kappa} c_{\lambda\mu}{}^{\gamma}\, c_{\gamma\nu}{}^{\kappa}\, b_{\kappa\rho\sigma}\, \bullet \right) \oplus \text{ other irreps}\,. \tag{2.67}$$

2.7 Counting tensor structures

This section will demonstrate the construction of $SO(d)$ invariant functions involving $SO(d)$ irreps without any further physical restrictions such as transversality, scaling behavior or momentum conservation. In this way it will be clear how the tensor product coefficients introduced in the previous section count the number of tensor structures in such functions. Later on we show that the constraints necessary for CFT correlators or scattering amplitudes can be taken into account by replacing the building blocks appearing in the construction of the functions while leaving the tensorial structure untouched. Hence the counting of tensor structures presented here will directly apply to CFT correlators and scattering amplitudes.

Let us start by counting the independent tensor structures $f^{k}_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1,\ldots,\mathbf{z}_n)$ in a function $f_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1,\ldots,\mathbf{z}_n)$ constructed from polarizations of n $SO(d)$ irreps λ_i

$$f_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1,\ldots,\mathbf{z}_n) = \sum_k c_k f^{k}_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1,\ldots,\mathbf{z}_n)\,, \tag{2.68}$$

where c_k are constants. Each tensor structure can be specified by giving an expression for $\bar{f}^{k}_{\lambda_1\ldots\lambda_n}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n)$ which is related to the tensor structure by a generalization of (2.32)

$$f^{k}_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1,\ldots,\mathbf{z}_n) = \prod_{j=1}^{n} \prod_{p=1}^{n^j_Z} \prod_{q=1}^{\min(l^j_p, n^j_\Theta)} \left(z^{(p)}_j \cdot \partial_{\theta^{(q)}_j} \right) \bar{f}^{k}_{\lambda_1\ldots\lambda_n}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n)\,. \tag{2.69}$$

These have to satisfy the condition that each set of polarization appears linearly

$$\begin{aligned} &\bar{f}^{k}_{\lambda_1\ldots\lambda_n}(\boldsymbol{\beta}_1\boldsymbol{\theta}_1,\ldots,\boldsymbol{\beta}_n\boldsymbol{\theta}_n) = \\ &= \bar{f}^{k}_{\lambda_1\ldots\lambda_n}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n) \prod_{i=1}^{n} \left(\beta^{(1)}_i\right)^{h^i_1} \cdots \left(\beta^{(n^i_\Theta)}_i\right)^{h^i_{n^i_\Theta}} \left(\beta^{(z)}_i\right)^{(\lambda_i)_1}, \end{aligned} \tag{2.70}$$

where we defined

$$\boldsymbol{\beta}_i\boldsymbol{\theta}_i = \left(\beta^{(1)}_i \theta^{(1)}_i, \ldots, \beta^{(n_\Theta)}_i \theta^{(n_\Theta)}_i, \beta^{(z)}_i z^{(1)}_i \right), \tag{2.71}$$

for arbitrary constants $\beta^{(p)}_i$. Since we still deal with traceless tensors we can construct $\bar{f}^{k}_{\lambda_1\ldots\lambda_n}(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n)$ without contractions between two polarizations belonging to the same irrep such as $\theta_i^{(p)} \cdot z_i^{(1)}$ or ${z_i^{(1)}}^2$, as explained above (2.34). Hence $\bar{f}^k_{\lambda_1\ldots\lambda_n}\big(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n\big)$ can be entirely constructed from the building blocks

$$H_{ij}^{(p,q)} = \theta_i^{(p)} \cdot \theta_j^{(q)}, \quad i \neq j, \quad p,q \in \{1,\ldots,n_\Theta, z\} \quad \text{with } \theta_i^{(z)} \equiv z^{(1)}, \tag{2.72}$$

where $\theta_i^{(z)} \equiv z^{(1)}$ was defined for convenience.

The number of independent tensor structures in the function $f_{\lambda_1\ldots\lambda_n}\big(\mathbf{z}_1,\ldots,\mathbf{z}_n\big)$ is given by the multiplicity of the scalar representation in the tensor product

$$\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n, \tag{2.73}$$

which is given by the tensor-product coefficient defined in (2.65)

$$b_{\lambda_1\lambda_2\ldots\lambda_n}. \tag{2.74}$$

It is important to have an independent and efficient way to compute this, because it is not always clear if two expressions $\bar{f}^k_{\lambda_1\ldots\lambda_n}\big(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n\big)$ lead to independent tensor structures after performing the derivatives in (2.69) . To illustrate this, consider the example $n = 2$, $\lambda_1 = \lambda_2 = \ydiagram{2,2}$ where the following combinations of building blocks could contribute to $\bar{f}_{\lambda_1\ldots\lambda_n}\big(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n\big)$

$$\left(H_{12}^{(1,1)}H_{12}^{(2,2)}\right)^2, \quad H_{12}^{(1,1)}H_{12}^{(2,2)}H_{12}^{(1,2)}H_{12}^{(2,1)} \quad \text{and} \quad \left(H_{12}^{(1,2)}H_{12}^{(2,1)}\right)^2. \tag{2.75}$$

However, the tensor product of two copies of an irrep contains the scalar representation with multiplicity one, as written in (2.51), so there can be only one tensor structure for each two-point function. Indeed, all possible ways to distribute the polarizations among the H_{12}'s lead to the same result after Young symmetrization. This can be checked explicitly by considering (2.69). Indeed, (2.51) states that $\lambda_1 = \lambda_2$ must hold for any two-point function and it is always enough to consider the single tensor structure

$$\bar{f}_\lambda(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) = \prod_{r=1}^{n_\Theta}\left(H_{12}^{(r,r)}\right)^{h_r}\left(H_{12}^{(z,z)}\right)^{\lambda_1}. \tag{2.76}$$

Note that here λ_1 is the number of columns of height 1 in the Young diagram λ.

As a further example consider one of the first combinations of irreps where the tensor product coefficient is larger than one, $\lambda_1 = \lambda_2 = \ydiagram{2,1}$, $\lambda_3 = \ydiagram{2}$. The corresponding tensor product coefficient is $b_{\ydiagram{2,1}\,\ydiagram{2,1}\,\ydiagram{2}} = 2$. Indeed, there are two combinations of the $H_{ij}^{(p,q)}$ that lead to linearly independent tensor structures

$$\begin{aligned}
&(z_1\cdot\partial_{\theta_1})\,(z_2\cdot\partial_{\theta_2})\left(H_{12}^{(\theta,\theta)}\right)^2 H_{13}^{(z,z)}H_{23}^{(z,z)} = \\
&\qquad\qquad = 2\left(H_{12}^{(z,z)}H_{12}^{(\theta,\theta)} - H_{12}^{(\theta,z)}H_{12}^{(z,\theta)}\right)H_{13}^{(z,z)}H_{23}^{(z,z)}, \\
&(z_1\cdot\partial_{\theta_1})\,(z_2\cdot\partial_{\theta_2})H_{12}^{(z,z)}H_{12}^{(\theta,\theta)}H_{13}^{(\theta,z)}H_{23}^{(\theta,z)} = \\
&\qquad\qquad = H_{12}^{(z,z)}\Big[\left(H_{12}^{(\theta,\theta)}H_{13}^{(z,z)} - H_{12}^{(z,\theta)}H_{13}^{(\theta,z)}\right)H_{23}^{(z,z)} \\
&\qquad\qquad\qquad + \left(H_{12}^{(z,z)}H_{13}^{(\theta,z)} - H_{12}^{(\theta,z)}H_{13}^{(z,z)}\right)H_{23}^{(\theta,z)}\Big].
\end{aligned} \tag{2.77}$$

Knowing that $b_{\ydiagram{2,1}\,\ydiagram{2,1}\,\ydiagram{2}} = 2$ one can stop looking for more independent tensor structures and conclude that for this example the tensor structures can be specified by

$$
\begin{aligned}
\bar{f}^1_{\ydiagram{2,1}\,\ydiagram{2,1}\,\ydiagram{2}}(\theta_1, z_1, \theta_2, z_2, z_3) &= \left(H_{12}^{(\theta,\theta)}\right)^2 H_{13}^{(z,z)} H_{23}^{(z,z)}\,, \\
\bar{f}^2_{\ydiagram{2,1}\,\ydiagram{2,1}\,\ydiagram{2}}(\theta_1, z_1, \theta_2, z_2, z_3) &= H_{12}^{(z,z)} H_{12}^{(\theta,\theta)} H_{13}^{(\theta,z)} H_{23}^{(\theta,z)}\,.
\end{aligned}
\tag{2.78}
$$

Next we want to construct tensor structures that can also depend on w additional vectors $x_j^a \in \mathbb{R}^d$ ($j \in \{1, \ldots, w\}$) that can be contracted into the polarizations, i.e. we want to count the number of linearly independent tensor structures of the form

$$
\begin{aligned}
&f^k_{\lambda_1\ldots\lambda_n}(\mathbf{z}_1, \ldots, \mathbf{z}_n, x_1, \ldots, x_w) = \\
&\qquad \prod_{i=1}^{n} \prod_{p=1}^{n_Z^i} \prod_{q=1}^{\min(l_p^i, n_\Theta^i)} \left(z_i^{(p)} \cdot \partial_{\theta_i^{(q)}} \right) \bar{f}^k_{\lambda_1\ldots\lambda_n}(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n, x_1, \ldots, x_w)\,.
\end{aligned}
\tag{2.79}
$$

In the physical applications considered below these vectors correspond to transverse combinations of spacetime coordinates or momenta. We will not allow contractions between the vectors x_i such as $x_i \cdot x_j$, such terms correspond to conformal cross-ratios, Mandelstam invariants or masses which are not considered part of the tensor structures. This means we allow additional building blocks

$$
\mathcal{V}_{ij}^{(p)} = \theta_i^{(p)} \cdot x_j\,, \quad i \in \{1, \ldots, n\},\ j \in \{1, \ldots, w\}. \tag{2.80}
$$

Considering a tensor structure where a given x_j appears q_j times, these x_j form the polarization for a fully symmetric rank q_j tensor, labelled by a one row Young diagram with q_j boxes or the Dynkin label $[q_j]$. The number of independent tensor structures in (2.79) with each x_j appearing q_j times is thus given by the multiplicity of the scalar representation in

$$
\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \Big[[q_1] \otimes \ldots \otimes [q_w] \Big]\,, \tag{2.81}
$$

where the restricted tensor product introduced in (2.66) was used because we do not want to count terms containing contractions between the x_i. The total number of tensor structures is hence given by

$$
n^{\lambda_1\ldots\lambda_n}_{\text{structures}}(w) = \sum_{q_1,\ldots,q_w} \sum_{\mu} c_{[q_1]\ldots[q_w]}{}^{\mu}\, b_{\mu\lambda_1\ldots\lambda_n} \equiv \sum_{q_1,\ldots,q_w} d(q_1, \ldots, q_w)\,, \tag{2.82}
$$

where defining the numbers $d(q_1, \ldots, q_w)$ is useful for the examples below and we define for the cases $w = 0, 1$

$$
c^{\mu} \equiv c_{\bullet\bullet}{}^{\mu} = \delta^{\mu}_{\bullet}\,, \qquad c_{\nu}{}^{\mu} \equiv c_{\bullet\nu}{}^{\mu} = \delta^{\mu}_{\nu}\,. \tag{2.83}
$$

The sums over q_j in (2.82) run over all non-negative integers which lead to non-zero contributions. A safe number for truncating the sums is at

$$
q_j \leq \sum_{i=1}^{n} l_1^i\,, \tag{2.84}
$$

because this is the maximal number of symmetrized indices that can come from all polarizations together, so there are no possible group invariant functions with more copies of the same x_j.

To use these techniques for conformal correlators or scattering amplitudes, the $SO(d)$ irreps appearing here have to be embedded into a higher dimensional Minkowski space. The building blocks $H_{ij}^{(p,q)}$ and $\mathcal{V}_{ij}^{(p)}$ will be replaced by building blocks living in that space and additional constraints like transversality have to be taken into account. The actual tensor structures for these cases will be introduced in Sections 3.1.1 and 4.2. The embedding into Minkowski space, which works in the same way for CFT correlators and scattering amplitudes, will be treated in Section 2.8, after a number of detailed examples illustrating the construction of tensor structures.

2.7.1 Examples of 3-point tensor structures

This section presents some examples for the construction of tensor structures as described above. As will be shown below, the number w of vectors that can appear in addition to the n polarizations in an n-point function is $w = n - 2$ for conformal correlators of non-conserved operators or scattering amplitudes of massive particles. For this reason we will make this choice for all the examples presented here. This also means that (2.76) is the only tensor structure needed for 2-point functions. For 3-point functions, we set $w = 1$ and will use the notation

$$\mathcal{V}_i^{(p)} \equiv \mathcal{V}_{i1}^{(p)}. \tag{2.85}$$

The tensor products in this section have been computed using the code of Appendix A.1.

Example: (Two-form)-Vector-Scalar

We start with the simple example of a two-form, a vector and a scalar, $\lambda_1 = \ydiagram{1,1}, \lambda_2 = \ydiagram{1}$, $\lambda_3 = \bullet$. There is no need to introduce commuting polarizations for the two-form. Also, for the vector, there is obviously no need to introduce any symmetrization or antisymmetrization. It has $n_{Z_2} = n_{\Theta_2} = 0$, therefore one can freely choose whether to use z_2 or θ_2 as polarization. In this case the only possible tensor structure has $q = 1$, hence there is one $\mathcal{V}_i$ building block. This is simple to see, since

$$\ydiagram{1,1} \otimes \ydiagram{1} \otimes \bullet = \ydiagram{1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}\,, \tag{2.86}$$

whose product with $[q]$ has a scalar representation only for $q = 1$. The corresponding tensor structure is

$$f_{\ydiagram{1,1}\,\ydiagram{1}\,\bullet}\left(\theta_1^{(1)}, z_2^{(1)}, x\right) = \mathcal{V}_1^{(1)} H_{12}^{(1,z)} = \left(\theta_1^{(1)} \cdot x\right)\left(\theta_1^{(1)} \cdot z_2^{(1)}\right). \tag{2.87}$$

Example: Two-form-Vector-Vector

Next we consider the three-point function of a two-form and two vectors, $\lambda_1 = \ydiagram{1,1}$, $\lambda_2 = \lambda_3 = \ydiagram{1}$. In this case there are three possible tensor structures,

$$\begin{aligned} q = 0 \qquad &\to \qquad H_{12}^{(1,z)} H_{13}^{(1,z)}\,, \\ q = 2 \qquad &\to \qquad \mathcal{V}_1^{(1)} \mathcal{V}_2^{(z)} H_{13}^{(1,z)} \text{ and } \mathcal{V}_1^{(1)} \mathcal{V}_3^{(z)} H_{12}^{(1,z)}\,. \end{aligned} \tag{2.88}$$

This can be seen from the product

$$\ydiagram{1,1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \bullet \oplus 2\,\ydiagram{2} \oplus 3\,\ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus 2\,\ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1}\,, \tag{2.89}$$

which contains the scalar and $\ydiagram{2}$ representations with multiplicities one and two, respectively.

Example: Hook-Scalar-Vector

The polynomial that encodes the correlator of a small hook diagram $\lambda_1 = \ydiagram{2,1}$, a scalar $\lambda_2 = \bullet$ and a vector $\lambda_3 = \ydiagram{1}$ consists of a single tensor structure, as can easily seen by considering the product

$$\ydiagram{2,1} \otimes \bullet \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \,. \tag{2.90}$$

Recall that for the small hook diagram we have $n_{Z_1} = 1$ and $n_{\Theta_1} = 1$, with polarization vectors $\mathbf{z}_1 = (z_1, \theta_1)$ and $\boldsymbol{\theta}_1 = (\theta_1, z_1)$, so the tensor structure is obtained by acting with a derivative $z_1 \cdot \partial_{\theta_1}$ on a polynomial of the $\mathcal{V}_i^{(p)}$'s and $H_{ij}^{(p,q)}$'s. In this case the single tensor structure has the form

$$f_{\ydiagram{2,1}\bullet\ydiagram{1}}\big(z_1, \theta_1, z_3, x\big) = \big(z_1 \cdot \partial_{\theta_1}\big)\mathcal{V}_1^{(1)}\mathcal{V}_1^{(z)}H_{13}^{(1,z)} = \mathcal{V}_1^{(1)}\mathcal{V}_1^{(z)}H_{13}^{(z,z)} - \Big(\mathcal{V}_1^{(z)}\Big)^2 H_{13}^{(1,z)} \,. \tag{2.91}$$

Example: Hook-Spin 2-Vector

Let us finally consider the example $\lambda_1 = \ydiagram{2,1}$, $\lambda_2 = \ydiagram{2}$, $\lambda_3 = \ydiagram{1}$. Table 2.1 contains all independent tensor structures for this case.

q	$b_{\lambda_1\lambda_2\lambda_3[q]}$	tensor structures
0	1	$H_{12}^{(z,z)}H_{12}^{(1,z)}H_{13}^{(1,z)}$
2	4	$\mathcal{V}_1^{(1)}\mathcal{V}_1^{(z)}H_{12}^{(1,z)}H_{23}^{(z,z)}$, $\mathcal{V}_1^{(1)}\mathcal{V}_3^{(z)}H_{12}^{(z,z)}H_{12}^{(1,z)}$,
		$\mathcal{V}_1^{(1)}\mathcal{V}_2^{(z)}H_{12}^{(z,z)}H_{13}^{(1,z)}$, $\mathcal{V}_1^{(1)}\mathcal{V}_2^{(z)}H_{12}^{(1,z)}H_{13}^{(z,z)}$
4	2	$\mathcal{V}_1^{(1)}\mathcal{V}_1^{(z)}\mathcal{V}_2^{(z)}\mathcal{V}_3^{(z)}H_{12}^{(1,z)}$, $\mathcal{V}_1^{(1)}\mathcal{V}_1^{(z)}(\mathcal{V}_2^{(z)})^2H_{13}^{(1,z)}$

Table 2.1: All seven tensor structures appearing in a three-point function of irreps $\ydiagram{2,1}$, $\ydiagram{2}$ and $\ydiagram{1}$.

Notice that for $q = 2$ there is another tensor structure constructed from $\mathcal{V}_1^{(z)}\mathcal{V}_2^{(z)}H_{12}^{(1,z)}H_{13}^{(1,z)}$, but this is not linear independent since

$$\begin{aligned} &(z_1 \cdot \partial_{\theta_1})\mathcal{V}_1^{(z)}\mathcal{V}_2^{(z)}H_{12}^{(1,z)}H_{13}^{(1,z)} \\ &\qquad = (z_1 \cdot \partial_{\theta_1})\left(\mathcal{V}_1^{(1)}\mathcal{V}_2^{(z)}H_{12}^{(z,z)}H_{13}^{(1,z)} - \mathcal{V}_1^{(1)}\mathcal{V}_2^{(z)}H_{12}^{(1,z)}H_{13}^{(z,z)}\right) \,. \end{aligned} \tag{2.92}$$

In this case the product of the three representations λ_1, λ_2 and λ_3 contains the following representations consisting of a single row

$$\ydiagram{2,1} \otimes \ydiagram{2} \otimes \ydiagram{1} = \bullet \oplus 4\,\ydiagram{2} \oplus 2\,\ydiagram{4} \oplus \dots \,, \tag{2.93}$$

in agreement with Table 2.1.

2.7.2 Examples of 4-point tensor structures

The next examples consider four-point tensor structures $f^k\big(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, x_1, x_2\big)$, again choosing $w = n - 2$. The tensor product (2.81) to consider here is

$$\lambda_1 \otimes \lambda_2 \otimes \lambda_3 \otimes \lambda_4 \Big[[q_1] \otimes [q_2]\Big] \,. \tag{2.94}$$

Example: Scalar-Vector-Scalar-Vector

As an example, Table 2.2 lists the five tensor structures in a four-point function of irreps $\lambda_1 = \lambda_3 = \bullet$ and $\lambda_2 = \lambda_4 = \square$, which were already given in [15, 33].

q_1	q_2	$d(q_1, q_2)$	tensor structures
0	0	1	$H_{24}^{(z,z)}$
2	0	1	$\mathcal{V}_{21}^{(z)}\mathcal{V}_{41}^{(z)}$
1	1	2	$\mathcal{V}_{21}^{(z)}\mathcal{V}_{42}^{(z)}$, $\mathcal{V}_{22}^{(z)}\mathcal{V}_{41}^{(z)}$
0	2	1	$\mathcal{V}_{22}^{(z)}\mathcal{V}_{42}^{(z)}$

Table 2.2: All five tensor structures in a four-point function of irreps $\bullet$, $\square$, $\bullet$, $\square$.

Let us see explicitly how these structures arise from the scalar degeneracy in the tensor product (2.94). The tensor product of two scalars and two vectors decomposes as

$$\bullet \otimes \ydiagram{1} \otimes \bullet \otimes \ydiagram{1} = \ydiagram{1} \otimes \ydiagram{1} = \bullet \oplus \ydiagram{2} \oplus \ydiagram{1,1}\,. \tag{2.95}$$

That the scalar representation appears with multiplicity one here means that there is one tensor structure for $(q_1, q_2) = (0, 0)$, i.e. $d(0, 0) = 1$. For $(q_1, q_2) = (2, 0)$ or $(q_1, q_2) = (0, 2)$ we have to consider the tensor product

$$\bullet \otimes \ydiagram{1} \otimes \bullet \otimes \ydiagram{1} \otimes \ydiagram{2} = \left(\bullet \oplus \ydiagram{2} \oplus \ydiagram{1,1}\right) \otimes \ydiagram{2} = \bullet \oplus \text{other irreps}\,. \tag{2.96}$$

Thus $d(2, 0) = d(0, 2) = 1$. Finally, for $(q_1, q_2) = (1, 1)$ one needs to consider

$$\bullet \otimes \ydiagram{1} \otimes \bullet \otimes \ydiagram{1} \otimes \left[\ydiagram{1} \otimes \ydiagram{1}\right] = \left(\bullet \oplus \ydiagram{2} \oplus \ydiagram{1,1}\right) \otimes \left(\ydiagram{2} \oplus \ydiagram{1,1}\right) = 2\bullet \oplus \text{other irreps}\,, \tag{2.97}$$

so that $d(1, 1) = 2$.

Example: Hook-Vector-Scalar-Scalar

For this example we consider the irreps $\lambda_1 = \ydiagram{2,1}$, $\lambda_2 = \square$, $\lambda_3 = \lambda_4 = \bullet$. Table 2.3 shows all the eight tensor structures for this correlator. Let us see again explicitly how these structures arise from the scalar degeneracy in the tensor product (2.94). First we consider the product

$$\ydiagram{2,1} \otimes \ydiagram{1} \otimes \bullet \otimes \bullet = \ydiagram{2} \oplus \ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}\,. \tag{2.98}$$

Since there is no scalar irrep in this sum we have $d(0, 0) = 0$. For the other values of (q_1, q_2), (2.98) must be multiplied by $\big[[q_1] \otimes [q_2]\big]$, and then $d(q_1, q_2)$ is just the multiplicity of the scalar irrep in the overall product. Table 2.4 shows the different possibilities.

Example: Vector-Vector-Vector-Vector

Finally, the correlation function of four vectors illustrates how the tensor product also generates the number of possible contractions between H's, i.e. those corresponding to $q_1 = q_2 = 0$.

q_1	q_2	tensor structures
2	0	$\mathcal{V}_{11}^{(1)}\mathcal{V}_{11}^{(z)}H_{12}^{(1,z)}$
1	1	$\mathcal{V}_{11}^{(z)}\mathcal{V}_{12}^{(1)}H_{12}^{(1,z)}$, $\mathcal{V}_{11}^{(1)}\mathcal{V}_{12}^{(z)}H_{12}^{(1,z)}$
0	2	$\mathcal{V}_{12}^{(1)}\mathcal{V}_{12}^{(z)}H_{12}^{(1,z)}$
3	1	$\mathcal{V}_{11}^{(1)}\mathcal{V}_{11}^{(z)}\mathcal{V}_{21}^{(z)}\mathcal{V}_{12}^{(1)}$
2	2	$\mathcal{V}_{11}^{(1)}\mathcal{V}_{11}^{(z)}\mathcal{V}_{12}^{(1)}\mathcal{V}_{22}^{(z)}$, $\mathcal{V}_{11}^{(1)}\mathcal{V}_{21}^{(z)}\mathcal{V}_{12}^{(1)}\mathcal{V}_{12}^{(z)}$
1	3	$\mathcal{V}_{11}^{(1)}\mathcal{V}_{12}^{(1)}\mathcal{V}_{12}^{(z)}\mathcal{V}_{22}^{(z)}$

Table 2.3: All eight tensor structures in a four-point function of irreps $\ydiagram{2,1}$, $\ydiagram{1}$, $\bullet$, $\bullet$.

q_1	q_2	$\big[[q_1]\otimes[q_2]\big]$	$d(q_1,q_2)$
2	0	$\ydiagram{2}$	1
1	1	$\ydiagram{2}\oplus\ydiagram{1,1}$	2
0	2	$\ydiagram{2}$	1
3	1	$\ydiagram{4}\oplus\ydiagram{3,1}$	1
2	2	$\ydiagram{4}\oplus\ydiagram{3,1}\oplus\ydiagram{2,2}$	2
1	3	$\ydiagram{4}\oplus\ydiagram{3,1}$	1

Table 2.4: From the product of $\big[[q_1]\otimes[q_2]\big]$ with (2.98) it is straightforward to extract the scalar multiplicity $d(q_1,q_2)$, which counts the independent tensor structures given in Table 2.3.

The number of such tensor structures is calculated using the $SO(d)$ tensor product

$$\ydiagram{1}\otimes\ydiagram{1}\otimes\ydiagram{1}\otimes\ydiagram{1}=3\bullet\oplus 6\,\ydiagram{2}\oplus 6\,\ydiagram{1,1}\oplus\ydiagram{4}\oplus 3\,\ydiagram{3,1}\oplus 2\,\ydiagram{2,2}\oplus 3\,\ydiagram{2,1,1}\oplus\ydiagram{1,1,1,1}\,. \tag{2.99}$$

Correspondingly, there are three tensor structures that can be built out of H's, namely

$$H_{12}^{(z,z)}H_{34}^{(z,z)},\quad H_{13}^{(z,z)}H_{24}^{(z,z)}\quad\text{and}\quad H_{14}^{(z,z)}H_{23}^{(z,z)}\,. \tag{2.100}$$

There are $3!2^2$ other structures with two $\mathcal{V}$'s and one H and 2^4 other structures with four $\mathcal{V}$'s. Thus, in total for this case there are 43 independent tensor structures. As in the previous example, this counting is done by considering the scalar multiplicity in the product of $\big[[q_1]\otimes[q_2]\big]$ with (2.99). Table 2.5 shows the different possibilities to which it is trivial to assign the independent tensor structures.

q_1	q_2	$[[q_1] \otimes [q_2]]$	$d(q_1, q_2)$
0	0	$\bullet$	3
2	0	$\yng(2)$	6
1	1	$\yng(2) \oplus \yng(1,1)$	12
0	2	$\yng(2)$	6
4	0	$\yng(4)$	1
3	1	$\yng(4) \oplus \yng(3,1)$	4
2	2	$\yng(4) \oplus \yng(3,1) \oplus \yng(2,2)$	6
1	3	$\yng(4) \oplus \yng(3,1)$	4
0	4	$\yng(4)$	1

Table 2.5: Multiplicity $d(q_1, q_2)$ counting tensor structures for the correlation function of four vectors.

2.8 Embedding of $SO(d)$ irreps into Minkowski space

2.8.1 Embedding into $d+1$ dimensional Minkowski space

Massive particles in $d+1$ dimensional Minkowski space $\mathbb{R}^{d,1}$ are defined as irreps of $SO(d)$, so we need to know how these representations can be embedded into $\mathbb{R}^{d,1}$. The metric will have signature $(- + + \ldots +)$, which implies that the momentum of a particle $P^A \in \mathbb{R}^{d,1}$ is constrained by the on-shell condition

$$P^2 = -m^2 < 0. \tag{2.101}$$

We wish to encode a tensor field $F^{A_1 \ldots A_{|\lambda|}}(P)$ which embeds a $SO(d)$ irrep λ into $\mathbb{R}^{d,1}$ by a polynomial. The embedding is governed by the transversality condition

$$P^{A_i} F_{A_1 \ldots A_i \ldots A_{|\lambda|}} = 0\,. \tag{2.102}$$

The discussion is entirely analogous to that of Section 2.4, only that now the tensor will be a polynomial $F(P, \mathbf{Z})$ in polarization vectors on Minkowski space, denoted by capital letters

$$\mathbf{Z} \equiv \left(Z^{(1)}, Z^{(2)}, \ldots, Z^{(n_Z)}, \Theta^{(1)}\right). \tag{2.103}$$

Explicitly, the polynomial $F(P, \mathbf{Z})$ is given by

$$F(P, \mathbf{Z}) \equiv \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_\Theta)} \left(Z^{(p)} \cdot \partial_{\Theta^{(q)}}\right) \bar{F}(P, \boldsymbol{\Theta})\,, \tag{2.104}$$

where

$$\begin{aligned} \bar{F}(P, \boldsymbol{\Theta}) \equiv \Theta^{(1)}_{A_1} \ldots \Theta^{(1)}_{A_{h_1}} \Theta^{(2)}_{A_{h_1+1}} \ldots \Theta^{(2)}_{A_{h_1+h_2}} \\ \ldots \Theta^{(n_\Theta)}_{A_{h_1+\ldots+h_{n_\Theta-1}+1}} \ldots \Theta^{(n_\Theta)}_{A_{h_1+\ldots+h_{n_\Theta}}} Z^{(1)}_{A_{|\lambda|-\lambda_1+1}} \ldots Z^{(1)}_{A_{|\lambda|}} F^{A_1 \ldots A_{|\lambda|}}(P)\,, \end{aligned} \tag{2.105}$$

with

$$\mathbf{\Theta} \equiv \left(\Theta^{(1)}, \Theta^{(2)} \dots, \Theta^{(n_\Theta)}, Z^{(1)}\right). \tag{2.106}$$

The transversality condition (2.102) translates into the following condition, which defines transverse polynomials

$$F(P, \mathbf{Z} + \mathbf{c}P) = F(P, \mathbf{Z}), \tag{2.107}$$

for any set $\mathbf{c} = (c_1, \dots, c_{n_Z}, \gamma)$ of n_Z commuting numbers c_i and one anti-commuting number γ. This means that only the part of the polarizations which is orthogonal to P appears in transverse polynomials, hence in addition to scalar products of any two polarizations (as in (2.34)) also products of one polarization and the corresponding momentum can be dropped, i.e.

$$\begin{aligned} F^{A_1\dots A_{|\lambda|}}(P) \text{ traceless \& transverse } &\leftrightarrow \bar{F}(P, \mathbf{\Theta})\Big|_{\substack{\Theta^{(p)}\cdot\Theta^{(q)}=\Theta^{(p)}\cdot Z^{(1)}=Z^{(1)2}=0 \\ \Theta^{(p)}\cdot P=Z^{(1)}\cdot P=0}}, \\ &\leftrightarrow F(P, \mathbf{Z})\Big|_{\substack{Z^{(p)}\cdot Z^{(q)}=Z^{(p)}\cdot\Theta^{(1)}=0 \\ Z^{(p)}\cdot P=\Theta^{(1)}\cdot P=0}}. \end{aligned} \tag{2.108}$$

One can always find an orthonormal basis of polarizations ξ_b^A, $b = 1, \dots, d$ spanning the space of possible polarization vectors orthogonal to P. ξ_b^A is in the fundamental representation of $SO(d)$ with respect to the b index. Orthonormality is expressed by

$$\eta_{AB}\xi_e^A\xi_f^B = \delta_{ef}, \tag{2.109}$$

and summing over the basis of polarizations yields a completeness relation, which can be considered a projector $\mathbf{P}_{\perp P}$ to the subspace transverse to P

$$\sum_{c=1}^{d} \xi_c^A\xi_c^B = \eta^{AB} - \frac{P^AP^B}{P\cdot P} \equiv \mathbf{P}_{\perp P}^{AB}. \tag{2.110}$$

This basis connects tensors in $\mathbb{R}^d$ to tensors in $\mathbb{R}^{d,1}$

$$f_{a_1\dots a_{|\lambda|}} = \xi_{a_1}^{A_1}\dots\xi_{a_{|\lambda|}}^{A_{|\lambda|}} F_{A_1\dots A_{|\lambda|}}. \tag{2.111}$$

An arbitrary transverse polarization in $\mathbb{R}^{d,1}$ can be parametrized by a $\mathbb{R}^d$ vector $z^{(p)}$

$$Z^{(p)A} = z^{(p)a}\xi_a^A. \tag{2.112}$$

Let us finally show how the projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$ to the irrep λ lifts to the projector $\Pi_\lambda^{A_1\dots A_{|\lambda|},B_1\dots B_{|\lambda|}}$ in $\mathbb{R}^{d,1}$. It simply has to be lifted by use of ξ_a^A[5]

$$\begin{aligned} \Pi_\lambda^{A_1\dots A_{|\lambda|},B_1\dots B_{|\lambda|}} &= \xi_{a_1}^{A_1}\xi_{a_2}^{A_2}\dots\xi_{a_{|\lambda|}}^{A_{|\lambda|}}\ \pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}\ \xi_{b_1}^{B_1}\xi_{b_2}^{B_2}\dots\xi_{b_{|\lambda|}}^{B_{|\lambda|}} \\ &\propto \xi_{a_1}^{A_1}\dots\xi_{a_{|\lambda|}}^{A_{|\lambda|}}\left(\delta^{a_1b_1}\delta^{a_2b_2}\dots + \delta^{a_1a_2}\delta^{b_1b_2}\dots + \dots\right)\xi_{b_1}^{B_1}\dots\xi_{b_{|\lambda|}}^{B_{|\lambda|}} \\ &= \mathbf{P}_{\perp P}^{A_1B_1}\mathbf{P}_{\perp P}^{A_2B_2}\dots + \mathbf{P}_{\perp P}^{A_1A_2}\mathbf{P}_{\perp P}^{B_1B_2}\dots + \dots \end{aligned} \tag{2.113}$$

Here in the last step the completeness relation (2.110) was used to show that $\Pi_\lambda^{A_1\dots A_{|\lambda|},B_1\dots B_{|\lambda|}}$ can be obtained by replacing each Kronecker delta in $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$ by a $\mathbf{P}_{\perp P}^{AB}$. If the projector

[5]Note that any projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$ consists only of Kronecker deltas and constant coefficients.

is inserted between two transverse tensors (satisfying (2.102)) the second term in (2.110) drops out and it is enough to replace each Kronecker delta by the Minkowski metric η^{AB}. The same thing happens when working with polynomials that are transverse (i.e. that satisfy (2.107)). This implies that the operators $\mathbf{D_z}$ introduced above in (2.49) can also be carried over to the Minkowski space by replacing $\mathbf{z}$ by $\mathbf{Z}$, when they are used between two transverse polynomials. The contraction of two traceless transverse tensors $F^{A_1A_2A_3}$ and $G^{B_1B_2B_3}$ in the irrep $\tiny\yng(2,1)$ is, as in the example (2.47), given by

$$F^{A_1A_2A_3}G_{A_1A_2A_3} = F(D_Z, \partial_\Theta)G(Z,\Theta)\,, \tag{2.114}$$

with

$$D_Z^A = \frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial Z_A} - \frac{3}{2(d-1)}\,Z^A\frac{\partial^2}{\partial Z\cdot\partial Z}\right). \tag{2.115}$$

In general, contractions will be written as

$$F^{A_1\ldots A_{|\lambda|}}G_{A_1\ldots A_{|\lambda|}} = F(\mathbf{D_Z})\,G(\mathbf{Z})\,. \tag{2.116}$$

Note that given the properties of the encoding polynomials (manifestly transverse but tracelessness has to be restored), this encodes the contribution of the irrep λ when inserting a complete set of states into a correlator, as in (1.5). Such an insertion includes a sum over an orthonormal basis of polarizations, which turns all the polarizations into contractions via (2.110).

2.8.2 Embedding into $d+2$ dimensional Minkowski space

The embedding of $SO(d)$ irreps into $\mathbb{R}^{d+1,1}$ to form $SO(d+1,1)$ invariant functions has two important applications. One of them is, similar to the previous section, that massless particles in physical $\mathbb{R}^{d+1,1}$ are restricted by the on-shell condition of the momentum and transversality of the polarization to be $SO(d)$ irreps.

Another application of the same embedding is the embedding formalism for conformal field theories. Here we consider $\mathbb{R}^d$ to be the physical space, which implies that primary operators have to be in irreps of $SO(d)$. The conformal group is $SO(d+1,1)$, which makes it favorable to consider an embedding into $\mathbb{R}^{d+1,1}$ where this group acts linearly as standard Lorentz transformations. This idea dates back at least to Dirac [34]. A more thorough discussion of the embedding formalism can be seen in [15, 33] whose notation we follow here.

In both cases we use coordinates $P \in \mathbb{R}^{d+1,1}$ to describe light-rays, i.e. null vectors in $\mathbb{R}^{d+1,1}$ up to rescalings

$$P^2 = 0\,, \qquad P \sim \alpha P \quad (\alpha > 0)\,. \tag{2.117}$$

We use light-cone coordinates

$$P^A = \left(P^+, P^-, P^a\right), \tag{2.118}$$

with the metric

$$P_1\cdot P_2 = \eta_{AB}P_1^AP_2^B = -\frac{1}{2}\left(P_1^+P_2^- + P_1^-P_2^+\right) + \delta_{ab}P_1^aP_2^b\,. \tag{2.119}$$

The embedding of $SO(d)$ irreps into $\mathbb{R}^{d+1,1}$ works mostly analogously to the previous section, i.e. equations (2.102-2.108) hold by replacing coordinates and the metric by their lightcone versions for $\mathbb{R}^{d+1,1}$. However, for polarization vectors to have d degrees of freedom, they must be taken to be orthogonal to a second vector apart from P^A, say $\bar{P}^A$, satisfying

$$Z^{(p)} \cdot \bar{P} = \Theta^{(p)} \cdot \bar{P} = \bar{P}^2 = 0, \quad P \cdot \bar{P} \neq 0. \tag{2.120}$$

In contrast to the embedding into $\mathbb{R}^{d,1}$, $\bar{P}$ can be freely chosen within these constraints. This can be considered as a gauge choice.

Analogous to the previous section, one can again use a d element basis of polarizations $\xi_b^A, b = 1, \ldots, d$ which is orthonormal

$$\eta_{AB} \xi_e^A \xi_f^B = \delta_{ef}, \tag{2.121}$$

and satisfies a completeness relation

$$\sum_{c=1}^{d} \xi_c^A \xi_c^B = \eta^{AB} - \frac{P^A \bar{P}^B + \bar{P}^A P^B}{P \cdot \bar{P}} \equiv \mathbf{P}_{\perp P, \bar{P}}^{AB}. \tag{2.122}$$

This basis connects tensors in $\mathbb{R}^d$ to tensors in $\mathbb{R}^{d+1,1}$

$$f_{a_1 \ldots a_{|\lambda|}} = \xi_{a_1}^{A_1} \ldots \xi_{a_{|\lambda|}}^{A_{|\lambda|}} F_{A_1 \ldots A_{|\lambda|}}\,. \tag{2.123}$$

Poincaré section

In practice it is convenient to describe light rays by coordinates on a specific section of the light cone. In particular, for a CFT on d-dimensional Euclidean space $\mathbb{R}^d$, we consider the Poincaré section

$$P^A = \big(P^+, P^-, P^a\big) = \big(1, x^2, x^a\big)\,. \tag{2.124}$$

It is simple to see that the Euclidean distance between two points in $\mathbb{R}^d$ is written in the above coordinates on the Poincaré section as $-2P_1 \cdot P_2 = (x_1 - x_2)^2$. It will later be abbreviated by $P_{ij} \equiv -2P_i \cdot P_j$. In general, $SO(d+1,1)$ Lorentz transformations map the light-cone into itself and, by the identification (2.124), define the action of the conformal group in physical space.

The basis of transverse polarizations can then be chosen as

$$\xi_b^A = \frac{\partial P^A}{\partial x^b} = \big(0, 2x_b, \delta_b^a\big). \tag{2.125}$$

This fixes the $\bar{P}^A$ in the completeness relation (2.122) to be

$$\bar{P}^A = (0, 2, 0, \ldots, 0). \tag{2.126}$$

It is also possible to relate the polynomial $f(x, \mathbf{z})$ to the embedding polynomial $F(P, \mathbf{Z})$, as well as $\bar{f}(x, \boldsymbol{\theta})$ to $\bar{F}(P, \boldsymbol{\Theta})$. The procedure is entirely analogous to that described in [15]: in the case of the Poincaré patch where $P_x = (1, x^2, x)$, each embedding polarization can be written as

$$Z_{z,x}^{(p)A} = \xi_b^A z^{(p)b} = \Big(0, 2x \cdot z^{(p)}, z^{(p)a}\Big) \quad \text{and} \quad \Theta_{\theta,x}^{(p)A} = \xi_b^A \theta^{(p)b} = \Big(0, 2x \cdot \theta^{(p)}, \theta^{(p)a}\Big), \tag{2.127}$$

so that the relation between the polynomials is simply

$$f(x, \mathbf{z}) = F\big(P_x, \mathbf{Z}_{z,x}\big) \quad \text{and} \quad \bar{f}(x, \boldsymbol{\theta}) = \bar{F}\big(P_x, \boldsymbol{\Theta}_{\theta,x}\big)\,. \tag{2.128}$$

As in the previous case (2.113) the projectors to irreducible representations are obtained by replacing Kronecker deltas by the projectors $\mathbf{P}_{\perp P, \bar{P}}^{AB}$, or when contracting transverse tensors, just by the inverse lightcone metric η^{AB}.

Chapter 3

Conformal correlators of mixed-symmetry tensors

3.1 Correlation functions

In this section we address the main kinematic problem that is to be solved when thinking about correlation functions of arbitrary tensor irreps: to count and to construct all independent tensor structures. This is achieved by replacing the building blocks appearing in the tensor structures of Section 2.7. The new building blocks implement the additional constraints for conformal correlators, mainly by using coordinates on an embedding space as introduced in Section 2.8.2, and generalize the building blocks for conformal correlators of fully symmetric tensors introduced in [15].

3.1.1 Tensor structures in conformal correlators

We wish to encode conformal primary operators by polynomials. Such operators are labeled by the unitary irreducible representations of the conformal group $SO(d+1,1)$ which will be labeled by $\chi \equiv [\Delta, \lambda]$, where Δ is the conformal dimension and λ an irreducible representation of $SO(d)$. Let us consider such an operator $\mathcal{O}^{\Delta,\lambda}_{a_1 \dots a_{|\lambda|}}(x)$ which has indices with symmetries given by the Young diagram λ. We wish to express it in terms of an operator on the embedding space. This new tensor operator should be defined on the light cone $P^2 = 0$ and it should be homogeneous of degree $-\Delta$,

$$\mathcal{O}^{\Delta,\lambda}_{A_1 \dots A_{|\lambda|}}(\alpha P) = \alpha^{-\Delta} \mathcal{O}^{\Delta,\lambda}_{A_1 \dots A_{|\lambda|}}(P)\,, \quad \alpha > 0\,. \tag{3.1}$$

It can be encoded by a polynomial on embedding space as described above in (2.104)

$$\mathcal{O}^{\Delta,\lambda}(P, \mathbf{Z}) = \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_\Theta)} \left(Z^{(p)} \cdot \partial_{\Theta^{(q)}} \right) \bar{\mathcal{O}}^{\Delta,\lambda}(P, \mathbf{\Theta})\,, \tag{3.2}$$

where

$$\bar{\mathcal{O}}^{\Delta,\lambda}(P, \mathbf{\Theta}) = \Theta^{(1)}_{A_1} \dots \Theta^{(n_\Theta)}_{A_{|\lambda|-\lambda_1}} \dots Z^{(1)}_{A_{|\lambda|}} \bar{\mathcal{O}}^{\Delta,\lambda}_{A_1 \dots A_{|\lambda|}}(P)\,. \tag{3.3}$$

Here our focus is not on the operators themselves but on the tensor structures that can appear in correlators

$$\begin{aligned}&\left\langle \mathcal{O}^{\chi_1}(P_1,\mathbf{Z}_1)\ldots\mathcal{O}^{\chi_n}(P_n,\mathbf{Z}_n)\right\rangle\\ &\equiv G_{\chi_1\ldots\chi_n}\big(\{P_i;\mathbf{Z}_i\}\big)=\prod_{j=1}^{n}\prod_{p=1}^{n_Z^j}\prod_{q=1}^{\min(l_p^j,n_\Theta^j)}\left(Z_j^{(p)}\cdot\partial_{\Theta_j^{(q)}}\right)\bar{G}_{\chi_1\ldots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\,.\end{aligned} \tag{3.4}$$

To construct this function one has to find $\bar{G}_\chi\big(\{P_i;\mathbf{\Theta}_i\}\big)$, which is subject to the following conditions. Firstly, it is homogeneous of degrees $-\Delta_i$ in the embedding space coordinates

$$\bar{G}_{\chi_1\cdots\chi_n}\big(\{\alpha_i P_i;\mathbf{\Theta}_i\}\big)=\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\prod_{i=1}^{n}\alpha_i^{-\Delta_i}\,, \tag{3.5}$$

for α_i arbitrary positive constants. Secondly, it is a polynomial in the polarizations with degrees given by the shape of the Young diagram λ_i,

$$\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\boldsymbol{\beta}_i\mathbf{\Theta}_i)\}\big)=\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\prod_{i=1}^{n}\left(\beta_i^{(1)}\right)^{h_1^i}\ldots\left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}\,, \tag{3.6}$$

where we defined

$$\boldsymbol{\beta}_i\mathbf{\Theta}_i=\left(\beta_i^{(1)}\Theta_i^{(1)},\ldots,\beta_i^{(n_\Theta)}\Theta_i^{(n_\Theta)},\beta_i^{(Z)}Z_i^{(1)}\right), \tag{3.7}$$

for arbitrary (commuting) constants $\beta_i^{(p)}$. Furthermore it should be transverse

$$\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i+\boldsymbol{\gamma}_i P_i\}\big)=\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\,, \tag{3.8}$$

where

$$\boldsymbol{\gamma}_i=\left(\gamma_i^{(1)},\ldots,\gamma_i^{(n_Z)},c_i\right), \tag{3.9}$$

is a set of n_Z anticommuting numbers and one commuting number. This last condition has to be satisfied modulo $O\big(P^2\big)$ terms. An identically transverse function $\bar{G}_\chi$ can be obtained by dropping terms proportional to $\Theta^{(p)}\cdot\Theta^{(q)}$ and $\Theta^{(p)}\cdot P$, where $p=1,\ldots,n_\Theta,Z$.

Generically one can write,

$$\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)=S(\{P_i\})\sum_k f_k(u_a)\,\bar{F}^k_{\lambda_1\ldots\lambda_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\,. \tag{3.10}$$

Here k enumerates the different tensor structures $F^k_{\lambda_1\ldots\lambda_n}$, which are each multiplied by a function $f_k(u_a)$ of the $n(n-3)/2$ independent conformally invariant cross-ratios u_a (for $n\geq 4$) or a constant (for the cases where no cross-ratios exist $n=2,3$). Furthermore we define the function $S(\{P_i\})$ of the coordinates to make the scaling of the tensor structures independent of the Δ_i. We take this function to scale as

$$S(\{\alpha_i P_i\})=S(\{P_i\})\prod_{i=1}^{n}\alpha_i^{-\Delta_i-|\lambda_i|}\,, \tag{3.11}$$

for example we can choose

$$S(P_1,P_2,\ldots,P_n)=\prod_{g<h}^{n}P_{gh}^{-\kappa_{gh}}, \tag{3.12}$$

with

$$\begin{aligned}
\kappa_{gh} &= \tau_g \equiv \Delta_i + |\lambda_i|, && n=2\,, \\
\kappa_{gh} &= \frac{\tau_g+\tau_h}{n-2} - \frac{1}{(n-1)(n-2)}\sum_{i=1}^{n}\tau_i, && n\geq 3\,.
\end{aligned} \tag{3.13}$$

This implies that the three conditions on $\bar{G}_{\chi_1\cdots\chi_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)$ (3.5-3.8) are satisfied if the tensor structures obey

$$\bar{F}^k_{\lambda_1\ldots\lambda_n}\big(\{\alpha_i P_i;\boldsymbol{\beta}_i(\mathbf{\Theta}_i+\boldsymbol{\gamma}_i P_i)\}\big) = \bar{F}^k_{\lambda_1\ldots\lambda_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)\prod_i \alpha_i^{|\lambda_i|}\left(\beta_i^{(1)}\right)^{h_1^i}\cdots\left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}. \tag{3.14}$$

Such tensor structures can be constructed much in the same way as in Section 2.7 by replacing the building blocks defined in (2.72) and (2.80) by alternatives which are transverse and have an appropriate homogeneity in all the embedding space coordinates P_i

$$\begin{aligned}
H^{(p,q)}_{ij}\big(\alpha_i P_i,\alpha_j P_j,\beta_i(\Theta^{(p)}_i+\gamma_i P_i),\beta_j(\Theta^{(q)}_j+\gamma_j P_j)\big) &= \alpha_i\alpha_j\beta_i\beta_j\, H^{(p,q)}_{ij}\big(P_i,P_j,\Theta^{(p)}_i,\Theta^{(q)}_j\big) \\
\mathcal{V}^{(p)}_{ij}\big(\alpha P_i,\beta(\Theta^{(p)}_i+\gamma P_i),\{\beta_k P_{k\neq i}\}\big) &= \alpha\beta\,\mathcal{V}^{(p)}_{ij}\big(P_i,\Theta^{(p)}_i,\{P_{k\neq i}\}\big)\,.
\end{aligned} \tag{3.15}$$

Notice that we are using the notation $\Theta^{(Z)} = Z^{(1)}$ to make equations more compact. To obtain elementary transverse building blocks one inserts the projector to transverse functions (2.122) into the simple building blocks introduced before in (2.72)

$$H^{(p,q)}_{ij} \propto \Theta^{(p)}_{i\,A}\left(\eta^{AB} - \frac{P_i^A P_j^B + P_j^A P_i^B}{P_i\cdot P_j}\right)\Theta^{(q)}_{j\,B} = \left(\Theta^{(p)}_i\cdot\Theta^{(q)}_j\right) - \frac{\left(P_j\cdot\Theta^{(p)}_i\right)\left(P_i\cdot\Theta^{(q)}_j\right)}{P_i\cdot P_j}\,. \tag{3.16}$$

This has to be multiplied by a polarization independent factor to achieve the scaling specified in (3.15) and to match the conventions of [15]

$$H^{(p,q)}_{ij} = 2\bigg(\left(P_j\cdot\Theta^{(p)}_i\right)\left(P_i\cdot\Theta^{(q)}_j\right) - \left(\Theta^{(p)}_i\cdot\Theta^{(q)}_j\right)\left(P_i\cdot P_j\right)\bigg). \tag{3.17}$$

The building blocks $\mathcal{V}^{(p)}_{ij}$ are constructed in a similar way, however since the projector demands the choice of a second coordinate despite P_i to be projected out, the constructed building blocks depend on the choice of two embedding space coordinates

$$V^{(p)}_{i,jk} \propto \Theta^{(p)}_{i\,A}\left(\eta^{AB} - \frac{P_i^A P_k^B + P_k^A P_i^B}{P_i\cdot P_k}\right)P_{j\,B} = \left(\Theta^{(p)}_i\cdot P_j\right) - \frac{\left(\Theta^{(p)}_i\cdot P_k\right)\left(P_i\cdot P_j\right)}{P_i\cdot P_k} \tag{3.18}$$

To achieve the scaling specified in (3.15) we multiply by a factor

$$V^{(p)}_{i,jk} = \frac{\left(\Theta^{(p)}_i\cdot P_j\right)\left(P_i\cdot P_k\right) - \left(\Theta^{(p)}_i\cdot P_k\right)\left(P_i\cdot P_j\right)}{P_j\cdot P_k}\,. \tag{3.19}$$

These building blocks satisfy $V^{(p)}_{i,jk} = -V^{(p)}_{i,kj}$ and on first sight there are $\binom{n-1}{2}$ possibilities to choose labels $j\neq k$ from the set $\{1,\ldots,n\}\setminus\{i\}$, however only $n-2$ of them are linearly independent. This was already observed in [15] and can be seen for instance by considering the following $n-2$ terms which are clearly independent

$$\mathcal{V}^{(p)}_{ij} \equiv V^{(p)}_{i,(i+1)(i+1+j)}\,,\qquad j\in\{1,2,\ldots,n-2\}\,, \tag{3.20}$$

where the external point labels i etc. are meant to be interpreted modulo n. Then the remaining $\binom{n-2}{2}$ terms $V^{(p)}_{i,kl}$ with $k,l \in \{1,\dots,n\}\setminus\{i,i+1\}$ can be expressed in terms of this basis using the relation

$$\big(P_k\cdot P_l\big)\big(P_i\cdot P_{(i+1)}\big)V^{(p)}_{i,kl} = \big(P_l\cdot P_{(i+1)}\big)\big(P_i\cdot P_k\big)V^{(p)}_{i,(i+1)l} - \big(P_{(i+1)}\cdot P_k\big)\big(P_i\cdot P_l\big)V^{(p)}_{i,(i+1)k}\,. \quad (3.21)$$

The building blocks constructed here in (3.17) and (3.20) provide an embedding of the $\mathbb{R}^d$ tensor structures constructed in Section 2.7. The tensor structure $\bar{F}^k_{\lambda_1\dots\lambda_n}\big(\{P_i;\mathbf{\Theta}_i\}\big)$ is given by (2.79) with $H^{(p,q)}_{ij}$ and $\mathcal{V}^{(p)}_{ij}$ replaced by their embedding space variants and, as we have just shown, the number of independent building blocks $\mathcal{V}^{(p)}_{ij}$ for each i is $w = n-2$. In this way all the examples of Section 2.7 apply directly to CFT correlators. The number of tensor structures in a general CFT correlator is thus $n^{\lambda_1\dots\lambda_n}_{\text{structures}}(n-2)$ as given in (2.82).

An independent proof that all parity even transverse functions can be constructed from building blocks of the types H_{ij} and $\mathcal{V}_{ij}$ is given in [15]. While the proof is done in the context of symmetric tensors, it also applies directly to the more general functions encoding mixed-symmetry tensors. Furthermore, an OPE based argument that this counting of tensor structures in the embedding space is correct is given below in Section 3.1.5.

3.1.2 Two-point functions

It is pretty simple to read off the two-point function from (3.10). A natural choice for the function $\bar{G}_\chi$ that generates the only possible tensor structure is given in (2.76), and there exist no cross-ratios. The factor $S(P_1,P_2)$ is given in (3.12). Hence the two-point function $G_\chi\big(P_1,P_2;\mathbf{Z}_1,\mathbf{Z}_2\big)$ is obtained in the usual way (3.4) from

$$\bar{G}_\chi\big(P_1,P_2;\mathbf{\Theta}_1,\mathbf{\Theta}_2\big) = \frac{1}{(P_{12})^{\Delta+|\lambda|}}\prod_{r=1}^{n_\Theta}\left(H^{(r,r)}_{12}\right)^{h_r}\left(H^{(Z,Z)}_{12}\right)^{\lambda_1}. \quad (3.22)$$

Example: p-form field

As an example, let us write explicitly the two-point function of a p-form field. The Young diagram of a p-form field consists of one column of p boxes, therefore $|\lambda| = p$, $n_Z = 0$ and $n_\Theta = 1$. Since there are no rows with more than one box and hence there are no indices to symmetrize, there is no need to introduce commuting polarizations. There is a single anti-commuting polarization vector, which we denote by Θ. The correlation function can be read off from (3.22) to be

$$\bar{G}_\chi\big(P_1,P_2;\Theta_1,\Theta_2\big) = \frac{\left(H^{(\Theta,\Theta)}_{12}\right)^p}{(P_{12})^{\Delta+p}} = \frac{1}{(P_{12})^{\Delta}}\left(\big(\Theta_1\cdot\Theta_2\big) - \frac{\big(P_2\cdot\Theta_1\big)\big(P_1\cdot\Theta_2\big)}{P_1\cdot P_2}\right)^p. \quad (3.23)$$

Then using the maps (2.127) and (2.128) it is simple to find the polynomial $\bar{g}_\chi(x_1,x_2;\theta_1,\theta_2)$ that describes this tensor structure in physical space.

Note also that acting with the Θ derivatives $\partial_{\Theta_1^{A_1}}\dots\partial_{\Theta_1^{A_p}}\partial_{\Theta_2^{B_1}}\dots\partial_{\Theta_2^{B_p}}$ one can write explicitly the components of the tensor in the embedding space as

$$G_\chi^{A_1\dots A_pB_1\dots B_p} = \frac{1}{(P_{12})^{\Delta}}\,\delta^{A_1}_{[C_1}\dots\delta^{A_p}_{C_p]}\,\delta^{B_1}_{[D_1}\dots\delta^{B_p}_{D_p]}\prod_{k=1}^{p}\left(\eta^{C_kD_k} - \frac{P_2^{C_k}P_1^{D_k}}{P_1\cdot P_2}\right), \quad (3.24)$$

whose projection to physical space gives the components

$$g_\chi^{a_1\dots a_p b_1\dots b_p} = \frac{1}{(x_{12}^2)^\Delta}\,\delta^{a_1}_{[c_1}\dots\delta^{a_p}_{c_p]}\,\delta^{b_1}_{[d_1}\dots\delta^{b_p}_{d_p]}\prod_{k=1}^{p}\left(\delta^{c_k d_k} - 2\,\frac{(x_{12})^{c_k}(x_{12})^{d_k}}{x_{12}^2}\right), \tag{3.25}$$

where $x_{12} = x_1 - x_2$.

Example: Smallest hook diagram

As another example let us consider the irrep corresponding to the diagram $\yng(2,1)$. This is the simplest example where the Young symmetrization operator appears. Here we have $n_Z = 1$ and $n_\Theta = 1$, with polarization vectors $\mathbf{Z} = (Z, \Theta)$ and $\mathbf{\Theta} = (\Theta, Z)$. Thus, the polynomials encoding the tensor structure for the two-point function of these operators are

$$\bar{G}_\chi\big(P_1, P_2; \mathbf{\Theta}_1, \mathbf{\Theta}_2\big) = \frac{1}{(P_{12})^{\Delta+3}}\,\left(H_{12}^{(\Theta,\Theta)}\right)^2 H_{12}^{(Z,Z)}\,, \tag{3.26}$$

and

$$\begin{aligned} G_\chi\big(P_1, P_2; \mathbf{Z}_1, \mathbf{Z}_2\big) &= (Z_1\cdot\partial_{\Theta_1})\,(Z_2\cdot\partial_{\Theta_2})\,\bar{G}_\chi\big(P_1, P_2; \mathbf{\Theta}_1, \mathbf{\Theta}_2\big) \\ &= \frac{2}{(P_{12})^{\Delta+3}}\left(H_{12}^{(\Theta,\Theta)}H_{12}^{(Z,Z)} - H_{12}^{(\Theta,Z)}H_{12}^{(Z,\Theta)}\right)H_{12}^{(Z,Z)}\,. \end{aligned} \tag{3.27}$$

Using the differential operator (2.115), it is a simple exercise to derive the components of the physical tensor associated to this polynomial, which were already derived in [35, 36] for all hook shaped Young diagrams. We shall not pursue this here, and work instead with polynomials on embedding space.

3.1.3 Three-point functions

For three-point functions the function $S(P_1, P_2, P_3)$ defined in (3.12) reads

$$S(P_1, P_2, P_3) = \frac{1}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}}(P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}(P_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}}, \qquad \tau_i = \Delta_i + |\lambda_i|\,. \tag{3.28}$$

There are again no possible cross-ratios, so any three-point function is determined up to constants c_k

$$\bar{G}_{\chi_1\chi_2\chi_3}\big(\{P_i;\mathbf{\Theta}_i\}\big) = \frac{\sum_k c_k \bar{F}^k_{\lambda_1\lambda_2\lambda_3}\big(P_1, P_2, P_3; \mathbf{\Theta}_1, \mathbf{\Theta}_2, \mathbf{\Theta}_3\big)}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}}(P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}(P_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}}\,. \tag{3.29}$$

Next we state some explicit examples, using the tensor structures constructed in Section 2.7.1.

Example: (Two-form)-Vector-Scalar

The tensor structure for a correlator of a two-form, a vector and a scalar, $\lambda_1 = \yng(1,1)$, $\lambda_2 = \square$, $\lambda_3 = \bullet$ was constructed in (2.87) and leads to the three-point function

$$\begin{aligned} G_{\chi_1\chi_2\chi_3}\big(\{P_i\};\Theta_1, Z_2\big) &= \frac{V_1^{(\Theta)}H_{12}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+3}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+1}{2}}} \\ &= \frac{-4\Big((P_2\cdot\Theta_1)(P_1\cdot P_3) - (P_2\cdot P_1)(\Theta_1\cdot P_3)\Big)\Big((P_2\cdot\Theta_1)(P_1\cdot Z_2) - (\Theta_1\cdot Z_2)(P_1\cdot P_2)\Big)}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+3}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1+1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+1}{2}}}\,. \end{aligned} \tag{3.30}$$

It is a mechanical computation to act on this polynomial with the derivatives $\partial_{\Theta_1^A}\partial_{\Theta_1^B}\partial_{Z_2^C}$ to obtain the components of the corresponding tensor in the embedding space.

Example: Two-form-Vector-Vector

For a two-form and two vectors, $\lambda_1 = \yng(1,1)$, $\lambda_2 = \lambda_3 = \square$ the tensor structures were given in (2.88), leading to the three-point function

$$G_{\chi_1\chi_2\chi_3}\big(\{P_i\};\Theta_1,Z_2,Z_3\big) = \frac{c_1 H_{12}^{(\Theta,Z)} H_{13}^{(\Theta,Z)} + c_2 V_1^{(\Theta)} V_2^{(Z)} H_{13}^{(\Theta,Z)} + c_3 V_1^{(\Theta)} V_3^{(Z)} H_{12}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+2}{2}}}\,, \tag{3.31}$$

with c_1, c_2 and c_3 constants.

Example: Hook-Scalar-Vector

The tensor structure for a correlator of a small hook diagram $\lambda_1 = \yng(2,1)$, a scalar $\lambda_2 = \bullet$ and a vector $\lambda_3 = \square$ was constructed in (2.91) and the resulting three-point function is

$$\begin{aligned} G_{\chi_1\chi_2\chi_3}\big(\{P_i\};\mathbf{Z}_1,Z_3\big) &= \frac{(Z_1\cdot\partial_{\Theta_1})\,V_1^{(\Theta)}V_1^{(Z)}H_{13}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-2}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+4}{2}}} \\ &= \frac{V_1^{(\Theta)}V_1^{(Z)}H_{13}^{(Z,Z)} - \left(V_1^{(Z)}\right)^2 H_{13}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-2}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+4}{2}}}\,. \end{aligned} \tag{3.32}$$

3.1.4 Four-point functions

Starting from four-point functions, correlation functions can depend on functions of the conformally invariant cross-rations. For four points there are two cross-ratios that can be defined to be

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}}\,, \qquad v = \frac{P_{14}P_{23}}{P_{13}P_{24}}\,. \tag{3.33}$$

Then a generic four-point function can be written as

$$\bar{G}_{\chi_1\chi_2\chi_3\chi_4}\big(\{P_i;\mathbf{\Theta}_i\}\big) = \prod_{g<h}^{4} P_{gh}^{-\kappa_{gh}} \sum_k f_k(u,v)\,\bar{F}^k_{\lambda_1\lambda_2\lambda_3\lambda_4}\big(\{P_i;\mathbf{\Theta}_i\}\big)\,. \tag{3.34}$$

The examples of Section 2.7.2 immediately apply here by inserting the tensor structures for $\bar{F}^k_{\lambda_1\lambda_2\lambda_3\lambda_4}\big(\{P_i;\mathbf{\Theta}_i\}\big)$, exactly as in the previous section for three-point functions.

3.1.5 Counting of tensor structures and the OPE

As a consistency check that the lift of tensor structures to transverse functions in embedding space correctly counts the tensor structures, one can consider the correspondence between three-point functions and leading OPE coefficients established in [37,38] and discussed in the context of the embedding formalism in [15]. We start with the leading terms in the OPE of operators $\mathcal{O}_i$ in arbitrary irreps labeled by $[\Delta_i,\lambda_i]$ using physical space coordinates x_i^a and polarizations $\mathbf{z}_i^a$, following the discussion in [15]

$$\mathcal{O}_1(x_1,\mathbf{z}_1)\,\mathcal{O}_2(x_2,\mathbf{z}_2) \sim \sum_k \mathcal{O}_k(x_1,\mathbf{D}_{\mathbf{z_k}})\,t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_k)\left(x_{12}^2\right)^{-\frac{\Delta_1+\Delta_2-\Delta_k+|\lambda_1|+|\lambda_2|+|\lambda_k|}{2}}\,. \tag{3.35}$$

When this is inserted into a three-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$, only $\mathcal{O}_k = \mathcal{O}_3$ contributes. $t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)$ is a rotationally invariant polynomial which scales as

$$t(\alpha x_{12},\boldsymbol{\beta}_1\mathbf{z}_1,\boldsymbol{\beta}_2\mathbf{z}_2,\boldsymbol{\beta}_3\mathbf{z}_3) = t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)\prod_{i=1}^{3}\alpha^{|\lambda_i|}\left(\beta_i^{(1)}\right)^{l_1^i}\cdots\left(\beta_i^{(n_Z)}\right)^{l_{n_Z}^i}\left(\beta_i^{(\theta)}\right)^{(\lambda_i^t)_1}, \tag{3.36}$$

where we defined

$$\boldsymbol{\beta}_i\mathbf{z}_i = \left(\beta_i^{(1)}z_i^{(1)},\ldots,\beta_i^{(n_Z)}z_i^{(n_Z)},\beta_i^{(\theta)}\theta_i^{(1)}\right), \tag{3.37}$$

for arbitrary constants $\beta_i^{(p)}$. The number of independent tensor structures in $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ is now equal to the number of structures in $t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)$, which is clearly given by

$$\sum_{q=0}^{l_1^1+l_1^2+l_1^3} b_{\lambda_1\lambda_2\lambda_3[q]}\,, \tag{3.38}$$

where the contribution for any q counts the terms where q of the x_{12} are contracted with polarizations and the remaining $|\lambda_1|+|\lambda_2|+|\lambda_3|-q$ powers contribute to a factor $\left(x_{12}^2\right)^{\frac{|\lambda_1|+|\lambda_2|+|\lambda_3|-q}{2}}$. This is in agreement with the previous statement that the number of tensor structures is given by (2.82) with $w = n-2 = 1$.

The tensor structures of four-point functions can be counted similarly by inserting the OPE (3.35) twice into the four-point function

$$\begin{aligned}
&\mathcal{O}_1(x_1,\mathbf{z}_1)\,\mathcal{O}_2(x_2,\mathbf{z}_2)\,\mathcal{O}_3(x_3,\mathbf{z}_3)\\
&\sim \mathcal{O}_1(x_1,\mathbf{z}_1)\sum_k \mathcal{O}_k(x_2,\mathbf{D}_{\mathbf{z_k}})\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)\left(x_{23}^2\right)^{-\frac{\Delta_2+\Delta_3-\Delta_k+|\lambda_2|+|\lambda_3|+|\lambda_k|}{2}}\\
&\sim \sum_j \mathcal{O}_j(x_1,\mathbf{D}_{\mathbf{z_j}})\sum_k \frac{t(x_{12},\mathbf{z}_1,\mathbf{D}_{\mathbf{z_k}},\mathbf{z}_j)\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_k-\Delta_j+|\lambda_1|+|\lambda_k|+|\lambda_j|}{2}}\left(x_{23}^2\right)^{\frac{\Delta_2+\Delta_3-\Delta_k+|\lambda_2|+|\lambda_3|+|\lambda_k|}{2}}}\,.
\end{aligned} \tag{3.39}$$

When this is inserted into $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$, only $\mathcal{O}_j = \mathcal{O}_4$ contributes. When summed over all possible irreps k, the terms $t(x_{12},\mathbf{z}_1,\mathbf{D}_{\mathbf{z_k}},\mathbf{z}_4)\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)$ clearly contain all possible contractions of x_{12}, x_{23} and the four polarizations $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ and $\mathbf{z}_4$. To exclude contractions between x_{12} and x_{23}, which do not lead to new tensor structures, counting can be performed using the restricted tensor product defined in (2.66), that keeps only irreps that have the same number of indices as both irreps in the product. Again, this counting confirms that the tensor structures in a four-point function are counted by (2.82) with $w = n-2 = 2$. Now q_1 and q_2 count how often x_{12} and x_{23} are contracted with polarizations.

3.2 Conserved tensors

Let us now consider conserved tensors in arbitrary irreducible $SO(d)$ representations. Recall that the unitarity bound for mixed-symmetry tensors [39,40], that must be satisfied in unitary

CFTs, restricts the conformal dimension of primaries in the irrep λ to satisfy the condition

$$\Delta \geq l_1^\lambda - h_{l_1}^\lambda + d - 1\,, \tag{3.40}$$

where $h_{l_1}^\lambda$ is the height of the rightmost column (the number of upper rows with the same number of boxes). The dimension for which (3.40) is saturated is called the critical dimension.

Let us first recall that at the critical dimension the conservation condition on fully symmetric or fully antisymmetric tensors $f_{a_1\dots a_l}(x)$

$$\frac{\partial}{\partial x_{a_1}} f_{a_1\dots a_l}(x) = 0\,, \tag{3.41}$$

is conformally invariant. The question which equations are conformally invariant for more general representations of the conformal group was discussed in [41] and specifically for mixed-symmetry tensors of hook diagram type in [35]. We will show below that, for general mixed-symmetry tensors in irrep λ, the analogue of the conservation condition (3.41) can only be imposed with respect to indices that correspond to boxes in one of the lowest columns in the Young tableau, i.e. in the same basis as in (2.16) they can be written as

$$\frac{\partial}{\partial x_{g_1}} f_{[a_1\dots a_{h_1}][b_1\dots b_{h_2}]\dots[g_1\dots g_{h_{l_1}}]}(x) = 0\,. \tag{3.42}$$

We will see that this equation can be imposed directly in embedding space. At the same time this will allow us to see that it is conformally invariant only when the unitarity bound (3.40) is saturated, and that similar equations with the derivative contracted with a different index are not conformally invariant.

The computation was done in [15] for symmetric tensors and the only part that changes is when the index symmetries are used. Let us first write

$$\begin{aligned} \frac{\partial}{\partial x_{a_{|\lambda|}}} f_{a_1\dots a_{|\lambda|}}(x) &= \frac{\partial}{\partial x_{a_{|\lambda|}}}\left(\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|}}}{\partial x^{a_{|\lambda|}}} F_{A_1\dots A_{|\lambda|}}(P_x)\right) \\ &= \frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}} S_{A_1\dots A_{|\lambda|-1}}(P_x) + T_{a_1\dots a_{|\lambda|-1}}(x)\,, \end{aligned} \tag{3.43}$$

where the projection from $F_{A_1\dots A_{|\lambda|}}$ to $f_{a_1\dots a_{|\lambda|}}$ given in (2.123) with (2.125) was inserted and

$$S_{A_1\dots A_{|\lambda|-1}}(P) = \left[\frac{\partial}{\partial P_{A_{|\lambda|}}} - \frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}} - (d-1-\Delta)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right] F_{A_1\dots A_{|\lambda|}}(P)\,, \tag{3.44}$$

is obtained in the same way as in [15], with $\bar{P} = (0,2,0)$ in the light-cone coordinates introduced in (2.124). The part $T_{a_1\dots a_{|\lambda|-1}}(x)$ comprises terms where $\frac{\partial}{\partial x_a}$ acts on the $\frac{\partial P^A}{\partial x^b}$ and can be simplified using

$$\frac{\partial}{\partial x_a}\frac{\partial P^A}{\partial x^b} = \delta_{ab}\bar{P}^A. \tag{3.45}$$

This is the part where the index symmetries are important

$$
\begin{aligned}
T_{a_1\dots a_{|\lambda|-1}}(x) = &- \frac{1}{P\cdot\bar{P}}\frac{\partial P^{A_{|\lambda|}}}{\partial x^{a_{|\lambda|}}}\Bigg[\delta_{a_{|\lambda|}a_1}\bar{P}^{A_1}\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}+\dots \\
&\qquad + \frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-2}}}{\partial x^{a_{|\lambda|-2}}}\delta_{a_{|\lambda|}a_{|\lambda|-1}}\bar{P}^{A_{|\lambda|-1}}\Bigg]F_{A_1\dots A_{|\lambda|}}(P) \\
= &- \frac{1}{P\cdot\bar{P}}\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}\bar{P}^{A_{|\lambda|}} \\
&\Big[F_{A_{|\lambda|}A_2\dots A_{|\lambda|-1}A_1}+F_{A_1A_{|\lambda|}A_3\dots A_{|\lambda|-1}A_2}+\dots+F_{A_1\dots A_{|\lambda|-2}A_{|\lambda|}A_{|\lambda|-1}}\Big]\,.
\end{aligned}
\tag{3.46}
$$

The second identity here is just a relabelling of indices. The sum in the last brackets simplifies due to the index symmetries (2.17) and becomes

$$
\big((l_1-1)-(h_{l_1}-1)\big)F_{A_1\dots A_{|\lambda|}}\,. \tag{3.47}
$$

Note that this step is only possible since the derivative in (3.42) is contracted with an index in the rightmost column of the Young tableau. The lift of the conservation condition with respect to the last index is then

$$
0=\frac{\partial}{\partial x_{a_{|\lambda|}}}f_{a_1\dots a_{|\lambda|}}(x)=\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}R_{A_1\dots A_{|\lambda|-1}}(P_x)\,, \tag{3.48}
$$

where

$$
R_{A_1..A_{|\lambda|-1}}(P)=\left[\frac{\partial}{\partial P_{A_{|\lambda|}}}-\frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}}-\big(l_1-h_{l_1}+d-1-\Delta\big)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right]F_{A_1..A_{|\lambda|}}(P). \tag{3.49}
$$

This generalizes the result derived in [15] for symmetric tensors. As discussed in [15] the first two terms in (3.49) are $SO(d+1,1)$ invariant. The last term is not, but it vanishes for conserved tensors which saturate the unitarity bound (3.40). Because of the index symmetries (2.18) the derivative in the conservation condition (3.48) can be contracted with any index that belongs to a column in the Young diagram of the same height as the rightmost one. In particular it may be contracted with any index in the case of rectangular Young diagrams.

There is actually a second conformally invariant condition that can be imposed on mixed-symmetry tensors. This was found for hook diagrams in [35] and requires a value for Δ different from the critical dimension. It is now very easy to find the dimension where this condition can be imposed for general mixed-symmetry tensors simply by lifting the conservation condition to the embedding space. This is most easily seen in the symmetric basis, so now take f to be in the symmetric basis as in (2.19) and consider the conservation condition

$$
\frac{\partial}{\partial x_{g_1}}f_{(a_1\dots a_{l_1})(b_1\dots b_{l_2})\dots(g_1\dots g_{l_{h_1}})}(x)=0\,. \tag{3.50}
$$

The lift to embedding space (3.43–3.46) works exactly as before. Now (2.20) is used to bring the last bracket in (3.46) into a form analogous to (3.47)

$$
-\big((h_1-1)-(l_{h_1}-1)\big)F_{A_1\dots A_{|\lambda|}}\,. \tag{3.51}
$$

The conservation condition (3.50) becomes, analogously to (3.49)

$$0 = \left[\frac{\partial}{\partial P_{A_{|\lambda|}}} - \frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}} - \left(l_{h_1} - h_1 + d - 1 - \Delta\right)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right]F_{A_1\ldots A_{|\lambda|}}(P)\,. \quad (3.52)$$

This is conformally invariant for

$$\Delta = l_{h_1} - h_1 + d - 1\,. \quad (3.53)$$

For rectangular Young diagrams, where $h_{l_1} = h_1$ and $l_{h_1} = l_1$ this is again the critical dimension. However, in general, we have $h_{l_1} \leq h_1$ and $l_{h_1} \leq l_1$, hence the unitarity bound (3.40) is violated for non-rectangular diagrams and the operators for which (3.52) is conformally invariant are non-unitary.

3.3 Conformal blocks

In this section we shall show how the above methods can be used to compute the conformal blocks for arbitrary irreducible tensor representations of the conformal group. The basic idea is that a conformal block in the channel $\mathcal{O}_1\,\mathcal{O}_2 \to \mathcal{O}_3\,\mathcal{O}_4$ can be written as a conformal integral of the product of the 3-point function of the operators $\mathcal{O}_1$, $\mathcal{O}_2$ and the exchanged operator $\mathcal{O}$ of dimension Δ, times the 3-point function of the operators $\mathcal{O}_3$, $\mathcal{O}_4$ and the shadow of the exchanged operator $\tilde{\mathcal{O}}$ of dimension $d-\Delta$ [10,12,29]. This method makes use of the shadow formalism of [42–45]. In practice however, one needs to remove from the final expression the contribution of the shadow operator exchange to the conformal block, which has the wrong OPE limit. This can be done rather efficiently by doing a monodromy projection of the above conformal integral, as proposed in [14].[1]

Conformal blocks are known for many cases involving external scalar operators and the exchange of spin l symmetric tensors. These results can be reused for correlators of external spin l operators by acting with differential operators on the conformal blocks for external scalars [13], but new exchanged tensor representations can not be taken care of in this way. Here we will follow closely the approach detailed in [14] to compute the conformal blocks, and show with a non-trivial example that the embedding methods here presented can be used to compute conformal blocks with external and exchanged operators in arbitrary tensor representations of the conformal group.

The idea is to define a projector to the conformal multiplet of a given operator which, when inserted into a four-point function, produces the conformal partial wave for the exchange of that operator (and its descendants). For an operator $\mathcal{O}$ with conformal dimension Δ this projector has the form

$$|\mathcal{O}| = \frac{1}{\mathcal{N}_{\mathcal{O}}}\int D^dP_0 D^dP_5\,\big|\mathcal{O}\big(P_0;\mathbf{D}_{\mathbf{Z}_0}\big)\big\rangle\,\big\langle\mathcal{O}(P_0;\mathbf{Z}_0)\mathcal{O}\big(P_5;\mathbf{D}_{\mathbf{Z}_5}\big)\big\rangle\big|_{\Delta\to\tilde{\Delta}}\,\big\langle\mathcal{O}(P_5;\mathbf{Z}_5)\big|\,. \quad (3.54)$$

Note that we are schematically representing the index contraction of $\mathcal{O}$ with a differential operator acting on the polarization vectors, as explained in (2.116). The integrals appearing here are called conformal integrals and are defined as

$$\int D^dP = \frac{1}{\mathrm{Vol}\;GL(1,\mathbb{R})^+}\int_{P^+ + P^- \geq 0} \mathrm{d}^{d+2}P\;\delta(P^2)\,. \quad (3.55)$$

[1]Such split of the operator and its shadow exchanges can also be done using the Mellin space representation of the conformal partial wave [46].

Explicit expressions for these integrals are known for all functions that appear in the computation of conformal blocks (see for instance Appendix A.5 in [47]).

The projector (3.54) can be more compactly expressed in terms of the shadow operator $\tilde{\mathcal{O}}$, which is in the same $SO(d)$ irrep as $\mathcal{O}$ and has conformal dimension $\tilde{\Delta} = d - \Delta$

$$|\mathcal{O}| = \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d P_0 \, \big|\mathcal{O}(P_0; \mathbf{D}_{\mathbf{Z}_0})\big\rangle \big\langle \tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)\big| \,, \tag{3.56}$$

where

$$\big\langle \tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)\big| = \int D^d P_5 \, \langle \mathcal{O}(P_0; \mathbf{Z}_0) \mathcal{O}(P_5; \mathbf{D}_{\mathbf{Z}_5})\rangle\big|_{\Delta \to \tilde{\Delta}} \, \langle \mathcal{O}(P_5; \mathbf{Z}_5)\big| \,. \tag{3.57}$$

Consider for simplicity the case where the three-point functions have only one tensor structure. Inserting $|\mathcal{O}|$ into a four-point function one obtains the conformal partial wave

$$\begin{aligned} W_{\mathcal{O}} &= \big\langle \mathcal{O}_1(P_1; \mathbf{Z}_1) \mathcal{O}_2(P_2; \mathbf{Z}_2) \big| \mathcal{O} \big| \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\big\rangle \\ &= \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d P_0 \, \langle \mathcal{O}_1(P_1; \mathbf{Z}_1) \mathcal{O}_2(P_2; \mathbf{Z}_2) \mathcal{O}(P_0; \mathbf{D}_{\mathbf{Z}_0})\rangle \langle \tilde{\mathcal{O}}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \,. \end{aligned} \tag{3.58}$$

Since $\tilde{\mathcal{O}}$ is in the same $SO(d)$ irrep as $\mathcal{O}$, three-point functions containing either of them must be equal up to an overall constant and to the conformal dimensions of the operators, i.e.

$$\big\langle \tilde{\mathcal{O}}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\big\rangle = \mathcal{S}_\Delta \, \langle \mathcal{O}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle\big|_{\Delta \to \tilde{\Delta}} \,. \tag{3.59}$$

This constant $\mathcal{S}_\Delta$ is calculated by using the definition of the shadow operator (3.57) and by computing the corresponding conformal integral. The constant $\mathcal{N}_{\mathcal{O}}$ in (3.58) can then be calculated by demanding that $|\mathcal{O}|$ acts trivially when inserted into a three-point function, i.e. requiring

$$\big\langle \mathcal{O}(P_0; \mathbf{Z}_0) \big| \mathcal{O} \big| \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\big\rangle = \big\langle \mathcal{O}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\big\rangle \,. \tag{3.60}$$

Using (3.56) and (3.57) one sees that this insertion amounts to doing the shadow transformation twice, hence with (3.59) we have

$$\begin{aligned} \big\langle \mathcal{O}(P_0; \mathbf{Z}_0) |\mathcal{O}| \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\big\rangle &= \frac{1}{\mathcal{N}_{\mathcal{O}}} \, \langle \tilde{\tilde{\mathcal{O}}}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \\ &= \frac{\mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}}{\mathcal{N}_{\mathcal{O}}} \, \langle \mathcal{O}(P_0; \mathbf{Z}_0) \mathcal{O}_3(P_3; \mathbf{Z}_3) \mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \,, \end{aligned} \tag{3.61}$$

and thus $\mathcal{N}_{\mathcal{O}} = \mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}$.

3.3.1 Example: Hook diagram exchange

As an example we will compute the conformal block $g^{\Delta_i}_{\yng(2,1)}(u, v)$ for the exchange of the tensor with irreducible representation $[\Delta, \yng(2,1)]$ in the correlation function of two scalars and two vectors $\langle \mathcal{O}_1^{\bullet}\mathcal{O}_2^{\yng(1)}\mathcal{O}_3^{\bullet}\mathcal{O}_4^{\yng(1)}\rangle$. The conformal partial wave is

$$\begin{aligned}
W_{\yng(2,1)} &= \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}}\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}}\frac{g^{\Delta_i}_{\yng(2,1)}(u,v)}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}}P_{34}^{\frac{\Delta_3+\Delta_4}{2}}}\\
&= \left\langle \mathcal{O}_1^{\bullet}(P_1)\mathcal{O}_2^{\yng(1)}(P_2;Z_2)\middle|\mathcal{O}^{\yng(2,1)}\middle|\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\yng(1)}(P_4;Z_4)\right\rangle \qquad (3.62)\\
&= \frac{1}{\mathcal{S}^{\yng(2,1)}_{\tilde\Delta}}\int D^dP_0\langle \mathcal{O}_1^{\bullet}(P_1)\mathcal{O}_2^{\yng(1)}(P_2;Z_2)\mathcal{O}^{\yng(2,1)}(P_0;D_{Z_0},\partial_{\Theta_0})\rangle\\
&\qquad\qquad \langle \mathcal{O}^{\yng(2,1)}(P_0;Z_0,\Theta_0)\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\yng(1)}(P_4;Z_4)\rangle\Big|_{\Delta\to\tilde\Delta}\,,
\end{aligned}$$

where we recall that u,v are the cross ratios defined in (3.33) and that the function $g^{\Delta_i}_{\yng(2,1)}(u,v)$ also depends on the external polarization vectors Z_2 and Z_4. The ingredients for this calculation are the two- and three-point functions from (3.27) and (3.32) for which we choose the normalizations

$$\langle \mathcal{O}^{\yng(2,1)}(P_1;Z_1,\Theta_1)\,\mathcal{O}^{\yng(2,1)}(P_2;Z_2,\Theta_2)\rangle = \frac{2\left(H_{12}^{(\Theta,\Theta)}H_{12}^{(Z,Z)}-H_{12}^{(\Theta,Z)}H_{12}^{(Z,\Theta)}\right)H_{12}^{(Z,Z)}}{(P_{12})^{\Delta+3}}\,, \qquad (3.63)$$

$$\langle \mathcal{O}^{\yng(2,1)}(P_0;Z_0,\Theta_0)\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\yng(1)}(P_4;Z_4)\rangle = \frac{V_{0,34}^{(\Theta)}V_{0,34}^{(Z)}H_{04}^{(Z,Z)}-\left(V_{0,34}^{(Z)}\right)^2H_{04}^{(\Theta,Z)}}{(P_{03})^{\frac{\Delta+\Delta_3-\Delta_4+2}{2}}(P_{34})^{\frac{\Delta_3+\Delta_4-\Delta-2}{2}}(P_{40})^{\frac{\Delta_4+\Delta-\Delta_3+4}{2}}}\,,$$

the differential operator D_Z from (2.115) which encodes the projection to the irrep $\yng(2,1)$, the constant $\mathcal{S}^{\yng(2,1)}_{\Delta}$ and the solution of the conformal integrals.

The constant $\mathcal{S}^{\yng(2,1)}_{\Delta}$ is computed using (3.59) and evaluating the conformal integral

$$\begin{aligned}
&\langle \tilde{\mathcal{O}}^{\yng(2,1)}(P_0;Z_0,\Theta_0)\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\yng(1)}(P_4;Z_4)\rangle \qquad (3.64)\\
&= \int D^dP_5\,\langle \mathcal{O}^{\yng(2,1)}(P_0;Z_0,\Theta_0)\,\mathcal{O}^{\yng(2,1)}(P_5;D_{Z_5},\partial_{\Theta_5})\rangle\Big|_{\Delta\to\tilde\Delta}\langle \mathcal{O}^{\yng(2,1)}(P_5;Z_5,\Theta_5)\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\yng(1)}(P_4;Z_4)\rangle\,.
\end{aligned}$$

All the integrals here are of the type

$$\int D^dP_5\,\frac{P_5^{A_1}\dots P_5^{A_n}}{(P_{50})^a(P_{53})^b(P_{54})^c}\,, \qquad (3.65)$$

and their explicit solution can be found for instance in [47, 48].[2] Comparing the integral in (3.64) with the three-point function, the resulting constant is

$$\mathcal{S}^{\yng(2,1)}_{\Delta} = \frac{\pi^h(\Delta-2)\Delta\,\Gamma(\Delta-h)}{\Gamma(\tilde\Delta+2)}\frac{\Gamma\left(\frac{\tilde\Delta+\Delta_{34}+2}{2}\right)\Gamma\left(\frac{\tilde\Delta-\Delta_{34}+2}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_{34}+2}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}+2}{2}\right)}\,. \qquad (3.67)$$

[2] To give an impression of how these integrals look like, here is the case with $n=1$

$$\begin{aligned}
\int D^dP_5\frac{P_5^A}{(P_{50})^a(P_{53})^b(P_{54})^c} &= \frac{\Gamma\left(\frac{b+c-a+1}{2}\right)\Gamma\left(\frac{c+a-b+1}{2}\right)\Gamma\left(\frac{a+b-c+1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}\frac{\pi^h}{(P_{34})^{\frac{b+c-a+1}{2}}(P_{40})^{\frac{c+a-b+1}{2}}(P_{03})^{\frac{a+b-c+1}{2}}}\\
&\times\left(\frac{P_{34}P_0^A}{\frac{1}{2}(b+c-a+1)}+\frac{P_{40}P_3^A}{\frac{1}{2}(c+a-b+1)}+\frac{P_{03}P_4^A}{\frac{1}{2}(a+b-c+1)}\right). \qquad (3.66)
\end{aligned}$$

Note that this is very similar to the corresponding constant for the exchange of the antisymmetric two-tensor $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, given below in (3.74), which was calculated first in [14]. As a small consistency check observe that the constant $\mathcal{N}_{\mathcal{O}} = \mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}$ appearing in (3.54) is independent of Δ_{34}.

To calculate the conformal partial wave (3.62) it is enough to know the conformal integrals

$$\begin{aligned}
&\int D^d P_0 \, \frac{(P_0 \cdot Z_2)(P_0 \cdot Z_4)}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f} \,, \qquad && \int D^d P_0 \, \frac{P_0 \cdot Z_2}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f} \,, \\
&\int D^d P_0 \, \frac{P_0 \cdot Z_4}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f} \,, \qquad && \int D^d P_0 \, \frac{1}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f} \,,
\end{aligned} \tag{3.68}$$

which much like the example (3.66) can be brought into a form where the polarizations are contracted with P_1, P_2, P_3 and P_4, or with each other. Just as in [14], after doing the monodromy projection to eliminate the shadow block, the final expression depends on functions of the cross ratios u, v given by

$$J^{(i)}_{j,k,l} = \frac{\Gamma(h+i-f)\,\Gamma(f)\sin(\pi f)}{\sin\big(\pi(e+f-h-i)\big)} \int_0^\infty \frac{\mathrm{d}x}{x} \int_{x+1}^\infty \frac{\mathrm{d}y}{y} \, \frac{x^b y^e}{(y+vxy-ux)^{h+i-f}(y-x-1)^f} \,, \tag{3.69}$$

with

$$\begin{aligned}
b &= \alpha + i + j - 1 \,, \\
e &= \beta - \Delta + h + i + k - l \,, \\
f &= 1 - \beta + h - k \,,
\end{aligned} \tag{3.70}$$

and

$$\alpha = \frac{\Delta - \Delta_{12} - 2}{2} \,, \qquad \beta = \frac{\Delta + \Delta_{34} - 2}{2} \,, \tag{3.71}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$ and $h = d/2$. In even dimensions, $h \in \mathbb{N}$, the functions $J^{(i)}_{j,k,l}$ can be expressed in terms of ${}_2F_1$ hypergeometric functions, see [14].

Doing the computation we arrived at the following expression for the conformal block defined in (3.62)

$$\begin{aligned}
g^{\Delta_i}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(u,v) &= \frac{u^{\Delta/2-1}\Gamma(\Delta+2)}{4P_{24}(\tilde{\Delta}-2)\tilde{\Delta}(2h-1)\,\Gamma(\alpha+2)\,\Gamma(\beta+2)\,\Gamma(\Delta-\alpha)\,\Gamma(\Delta-\beta)\,\Gamma(h-\Delta)} \\
&\times \left[V^{(Z)}_{2,14} V^{(Z)}_{4,12}\, u F_1 + V^{(Z)}_{2,14} V^{(Z)}_{4,23}\, v F_2 + V^{(Z)}_{2,34} V^{(Z)}_{4,12}\, u F_3 + V^{(Z)}_{2,34} V^{(Z)}_{4,23}\, v F_4 + \frac{1}{2} H^{(Z,Z)}_{24} F_H \right] .
\end{aligned} \tag{3.72}$$

As expected, this conformal block is organized into tensor structures that are analogous to the ones discussed for this four-point correlator in Section 2.7.2. The functions $F_i = F_i(u,v)$ depend on h, Δ, α and β, and are expressed in terms of a finite number of the integrals $J^{(i)}_{j,k,l}$ given in (3.69) above. For clarity of exposition we decided to present these functions in the Appendix A.2.

The example at hand shows that we have a well defined algorithm to compute any conformal block. However, before going on to compute even more complicated conformal blocks it would be helpful to study the functions $J^{(i)}_{j,k,l}$ in detail. This is done to some extent in Appendix A.3 where a number of relations between these functions are derived. That these relations can lead to shorter expressions is demonstrated with the following example.

3.3.2 Example: Two-form exchange

The conformal block for exchange of a two-form tensor in the irrep $\tiny\yng(1,1)$ was computed analogously in [14, 24], however the result is an expression half a page long. The normalizations for the two- and three-point functions of [14] are in our notation

$$
\begin{aligned}
\left\langle \mathcal{O}^{\tiny\yng(1,1)}(P_1,\Theta_1)\mathcal{O}^{\tiny\yng(1,1)}(P_2,\Theta_2)\right\rangle &= \frac{1}{4}\frac{\left(H_{12}^{(\Theta,\Theta)}\right)^2}{(P_{12})^{\Delta+2}}\,, \\
\left\langle \mathcal{O}^{\tiny\yng(1,1)}(P_0,\Theta_0)\mathcal{O}_3^{\bullet}(P_3)\mathcal{O}_4^{\tiny\yng(1)}(P_4,Z_4)\right\rangle &= \frac{V_{0,34}^{(\Theta)}H_{04}^{(\Theta,Z)}}{(P_{03})^{\frac{\Delta+\Delta_3-\Delta_4+1}{2}}(P_{34})^{\frac{\Delta_3+\Delta_4-\Delta-1}{2}}(P_{40})^{\frac{\Delta_4+\Delta-\Delta_3+3}{2}}}\,,
\end{aligned}
\tag{3.73}
$$

and the contraction of two-forms is now done using the normalized derivative $\partial_\Theta/\sqrt{2}$. The constant $\mathcal{S}_\Delta$ is given by

$$
\mathcal{S}_\Delta^{\tiny\yng(1,1)} = \frac{\pi^h(\Delta-2)\,\Gamma(\Delta-h)}{4\,\Gamma(\tilde{\Delta}+1)}\frac{\Gamma\left(\frac{\tilde{\Delta}+\Delta_{34}+1}{2}\right)\Gamma\left(\frac{\tilde{\Delta}-\Delta_{34}+1}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_{34}+1}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}+1}{2}\right)}\,. \tag{3.74}
$$

After doing carefully the conformal integrals and using the relations derived in Appendix A.3 one finds an expression considerably shorter than the previously published results

$$
\begin{aligned}
g_F^{\Delta_i}(u,v) &= \frac{2u^{\Delta/2-1/2}\,\Gamma(\Delta+1)}{P_{24}(2-\tilde{\Delta})\,\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\Delta-\alpha)\,\Gamma(\Delta-\beta)\,\Gamma(h-\Delta)} \\
&\times\Bigg[V_{2,14}^{(Z)}V_{4,12}^{(Z)}\,u\left(J_{0,0,1}^{(2)}-vJ_{0,1,1}^{(2)}\right)-V_{2,14}^{(Z)}V_{4,23}^{(Z)}\,v(v-1)J_{0,1,1}^{(2)}+V_{2,34}^{(Z)}V_{4,23}^{(Z)}\,vu\left(J_{0,1,1}^{(2)}-J_{1,1,1}^{(2)}\right) \\
&\quad+V_{2,34}^{(Z)}V_{4,12}^{(Z)}\,u\left((h-2)J_{1,1,1}^{(1)}+J_{0,0,0}^{(2)}+uJ_{0,1,1}^{(2)}+uJ_{1,1,2}^{(2)}\right) \\
&\quad-\frac{H_{24}^{(Z,Z)}}{2}\left((v-1)\left((h-2)J_{1,1,1}^{(1)}+J_{0,0,0}^{(2)}+uJ_{1,1,2}^{(2)}\right)+u\left(J_{0,0,1}^{(2)}-J_{0,1,1}^{(2)}-J_{1,0,1}^{(2)}+vJ_{1,1,1}^{(2)}\right)\right)\Bigg]\,,
\end{aligned}
\tag{3.75}
$$

where $J_{j,k,l}^{(i)}$ is defined in (3.69), but now with

$$
\alpha = \frac{\Delta-\Delta_{12}-1}{2}\,,\qquad \beta = \frac{\Delta+\Delta_{34}-1}{2}\,. \tag{3.76}
$$

Chapter 4

Perturbative unitarity in string theory

In this chapter on-shell recursion relations for string amplitudes based on tree-level unitarity cuts are studied. Due to the arbitrary tensor states in the string spectrum this requires a general understanding of tensor structures in scattering amplitudes, which are constructed using the methods of Chapter 2. Apart from this unitarity of the string S-matrix is studied by inspecting the reality of coupling constants.

4.1 Review

4.1.1 Spectrum and conventions

There is an elegant and efficient method to compute the spectrum of a string theory as generating functions in terms of group characters, described in [49]. We quote here the open bosonic string spectrum in $D = 26$ dimensions up to mass level $A = 5$

$$\begin{aligned} Z_{Bosonic} = {} & [0]_{25}q^{-1} + [1]_{24} + [2]_{25}q + ([3]_{25} + [0,1]_{25})q^2 \\ & + ([4]_{25} + [2]_{25} + [1,1]_{25} + [0]_{25})q^3 \\ & + ([5]_{25} + [3]_{25} + [2,1]_{25} + [1,1]_{25} + [1]_{25} + [0,1]_{25})q^4 + O(q^5). \end{aligned} \tag{4.1}$$

The exponent of q indicates the mass $\alpha' m^2 = A-1$ of a state and $SO(n)$ irreps are labelled by Dynkin labels as introduced in Section 2.1. The integer A is called the mass level. Except from the massless vector, the spectrum contains only massive particles, so mainly the formalism of Section 2.8.1 is used to embed irreps of $SO(d)$ into $\mathbb{R}^{d,1}$ where $d = D - 1$.

A state is given by a mass level A and a $SO(d)$ irrep λ, which will be collectively labelled by $\chi = [A, \lambda]$. Tachyon states will be labeled by $T \equiv [0, \bullet]$. Later on superstrings will be studied as well, here mass levels are defined by $\alpha' m^2 = A$. Hence the lowest mass particle is always the one at level $A = 0$.

Instead of the capital P^A of Section 2.8.1, momenta in $\mathbb{R}^{d,1}$ will now be denoted k^A. Furthermore, define for the product of two momenta

$$k_{ij} = 2\alpha' k_i \cdot k_j, \tag{4.2}$$

and the Mandelstam invariants

$$s_{ij} = -\alpha'(k_i + k_j)^2 = \alpha'(m_i^2 + m_j^2) - k_{ij}, \tag{4.3}$$

$$s_{1\ldots a} = -\alpha'(k_1 + \ldots + k_a)^2 = \sum_{i=1}^{a} \alpha' m_i^2 - \sum_{i=1}^{a-1} \sum_{j=i+1}^{a} k_{ij}. \tag{4.4}$$

The full string amplitude is given as a sum over all non-cyclic permutations of so-called color-ordered amplitudes times single trace factors,

$$\mathcal{A}_n = \sum_{\sigma \in \mathbb{P}_n/\mathbb{Z}_n} \mathcal{A}_{\text{colour-ordered}}(\sigma_1, \ldots, \sigma_n) \mathrm{Tr}\left(T^{\sigma_1} \ldots T^{\sigma_n}\right). \tag{4.5}$$

Throughout open string amplitudes will be assumed to be color-ordered amplitudes. This decomposition is natural in string theory: the traces of matrices T in the fundamental representation of $U(N)$ are simply the Chan-Patton factors. The string theory picture played a large [50] but not exclusive [51] role in introducing the concept of color-ordering in field theory. See [52] for a derivation of color-ordering from the more modern D-brane picture of string theory. General properties of color-ordered amplitudes are well-known [50] and will not be reviewed here.

4.1.2 On-shell recursion in string theory

The main idea of on-shell recursion as introduced in [19] is to introduce a single auxiliary complex parameter into scattering amplitudes, while keeping the amplitudes physical. To this end, one picks two legs and deforms their momenta

$$k_i \to \hat{k}_i \equiv k_i + q\,z, \qquad k_j \to \hat{k}_j \equiv k_j - q\,z. \tag{4.6}$$

This automatically satisfies momentum conservation. If one then also imposes

$$q^2 = q \cdot k_i = q \cdot k_j = 0, \tag{4.7}$$

the two singled-out legs remain on their original mass-shell. These equations can always be solved in four or more dimensions. In four dimensions two solutions exist (this is easily verified in the center-of-mass frame [53]). The deformations in equation (4.6) are collectively known as a BCFW-shift. Note that this shift makes momenta automatically complex.

The point of single complex variables in physics is invariably the possibility to use Cauchy's theorem. In the present context, one would like to compute the original amplitude, $\mathcal{A}(0)$, which may be computed as

$$\mathcal{A}(0) = \oint_{z=0} \frac{\mathcal{A}(z)}{z} = -\sum_{z_I \text{ finite}} \mathrm{Res}_{z=z_I} \frac{\mathcal{A}(z)}{z} - \mathrm{Res}_{z=\infty} \frac{\mathcal{A}(z)}{z}. \tag{4.8}$$

Here and in the following, all residue-type integrals contain $\frac{1}{2\pi \mathrm{i}}$ factors.

The residues at finite values of z have an interpretation through perturbative unitarity in terms of products of lower point scattering amplitudes, as explained in the introduction (1.7). If therefore the residue at infinity is absent, equation (4.8) constitutes an explicit on-shell recursion relation. The behavior at infinity was studied generically for string theories in [20, 21] and will be reviewed below in Section 5.1.2.

4.1.3 The pole structure of $\mathcal{A}(z)$

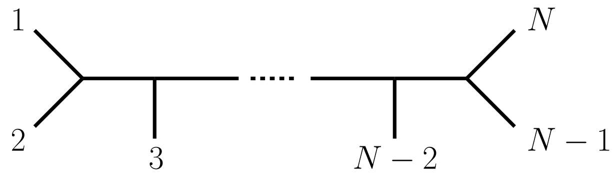
As a function of z, the amplitude $\mathcal{A}(z)$ can have physical poles[1]. In fact, for generic external momenta it will only have single poles. Say one takes a certain channel defined by a set σ of adjacent particles which includes the shifted particle i but not the other one j. The pole in z in this channel occurs when the associated internal propagator goes on-mass-shell, i.e.

$$-(\hat{k}_i + \sum_{l\in\sigma\setminus\{i\}} k_l)^2 = m^2, \tag{4.9}$$

for some mass of a particle in the particular theory under study. Note that the location of this pole is at a finite value of z. The residue of the amplitude at this pole is given by perturbative unitarity (1.7), which is repeated here for convenience as a residue in the Mandelstam invariant $s_{\hat{i},\{l\in\sigma\setminus\{i\}\}} = -\alpha'\left(\hat{k}_i + \sum_{l\in\sigma\setminus\{i\}} k_l\right)^2$

$$-\operatorname{Res}_{s_{\hat{i},\{l\in\sigma\setminus\{i\}\}}\to\alpha' m^2}\mathcal{A}_{\chi_1\cdots\chi_n}(z) = \sum_{\lambda}\mathcal{A}_{\{\chi_l,l\in\sigma\}[m,\lambda]}\mathcal{A}_{[m,\lambda]\{\chi_r,r\in\{1,\ldots,n\}\setminus\sigma\}}\,, \tag{4.10}$$

where the right-hand side must be evaluated at the value of z for which (4.9) holds. By iteration this formula relates the $(N-3)$-fold residue of any N-point amplitude to 3-point amplitudes only. String theory amplitudes have the property that they are dual, meaning that any tree diagram with the same external states in the same order describes the same amplitude. This means that much less diagrams have to be considered than in in field theory, where all possible tree diagrams have to be summed. We will mostly work with residues belonging to multiperipheral configurations



. (4.11)

The necessary input for using the resulting on-shell recursion relations is the knowledge of all 3-point amplitudes. In the same way as conformal 3-point functions these are given by tensor structures multiplied by constant coefficients. Indeed, it will be shown below that the tensor structures are essentially the same as for conformal correlators. The coefficients multiplying the tensor structures in these manifestly $SO(D-1)$ covariant 3-point amplitudes are not known in general, with some exception for the state with maximal spin at each mass level [54]. Better known are $SO(D-2)$ covariant expressions which can be generated by DDF operators [55].

However, the maximal residues of Koba-Nielsen amplitudes (N-tachyon amplitudes) can be easily obtained. They are derived from monodromy relations in the following chapter and from the worldsheet in Appendix A.4. It is in principle possible to derive the coefficients in the 3-point amplitudes using (4.10) and the residues of Koba-Nielsen amplitudes as input for the left-hand side. Then one could use the 3-point amplitudes to compute the residues of any amplitude using the same equation. Due to the infinite sums this is not really feasible

[1] In this thesis it will be implicitly assumed that amplitudes do not have unphysical poles. Moreover, it is assumed that the poles originate in nothing more exotic than Feynman-type propagators going on mass-shell.

in general, however it will be explored in this chapter to some extent, for the following two reasons.

Firstly, it is an example for using the formalism for general mixed-symmetry tensors of Chapter 2 in the context of scattering amplitudes, illustrating the matching of tensor structures in CFT correlators and scattering amplitudes that was found for symmetric tensors in [15] and generalizing it to general irreps.

Secondly, the coefficients of 3-point amplitudes are directly related to unitarity. They are equal - up to possible combinatorial factors - to the coefficients of the 3-point coupling terms in the interacting Hamiltonian. Using standard conventions (real fields), the Hamiltonian is hermitian if and only if all these coefficients are real, and a hermitian Hamiltonian implies unitarity of the S-matrix. To study unitarity in string theory, one therefore has to inspect all three point amplitudes. For the examples studied it is found that the obtained 3-point couplings are real if the no-ghost theorem conditions hold.

To this end, the formalism for dealing with general tensor structures is first adapted to scattering amplitudes of mixed-symmetry tensor fields. Then the relation between 3-point amplitudes of arbitrary states and the well-known N-tachyon amplitudes is explored in various ways in Section 4.3.

4.2 Tensor structures in scattering amplitudes

This section introduces building blocks similar to the ones introduced in Section 2.7 implementing the additional constraints for on-shell scattering amplitudes. The statements regarding tensor structures in scattering amplitudes are completely general and related to string theory only in that string theory is one of the few theories with general $SO(d)$ irreps in its spectrum.

4.2.1 Massive scattering amplitudes

First amplitudes of massive particles on $\mathbb{R}^{d,1}$ are considered. Since the massive little group of this space is $SO(d)$ such states are given by $\chi = [A, \lambda]$ with λ being $SO(d)$ irreps. The transverse embedding of such states into $\mathbb{R}^{d,1}$ was explained in Section 2.8.1. Amplitudes are encoded as polynomials analogously to CFT correlators (3.4)

$$\mathcal{A}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{Z}_i\}\big) = \prod_{j=1}^{n}\prod_{p=1}^{n_Z^j}\prod_{q=1}^{\min(l_p^j,n_\Theta^j)}\left(Z_j^{(p)}\cdot\partial_{\Theta_j^{(q)}}\right)\bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{\Theta}_i\}\big) \tag{4.12}$$

As in the discussion of CFT correlators in Section 3.1.1 we will again concentrate on the construction of $\bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{\Theta}_i\}\big)$ in terms of tensor structures

$$\bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{\Theta}_i\}\big) = \sum_k f_{\chi_1\dots\chi_n,k}(s_a)\,\bar{F}^k_{\lambda_1\dots\lambda_n}\big(\{k_i;\mathbf{\Theta}_i\}\big)\,, \tag{4.13}$$

where $f_{\chi_1\dots\chi_n,k}(s_a)$ are functions of the $\frac{n(n-3)}{2}$ independent Mandelstam invariants[2] (for $n \geq 4$) or constants (for $n = 3$). $\bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{\Theta}_i\}\big)$ should satisfy an equation analogous to (3.6)

$$\bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\boldsymbol{\beta}_i\mathbf{\Theta}_i)\}\big) = \bar{\mathcal{A}}_{\chi_1\dots\chi_n}\big(\{k_i;\mathbf{\Theta}_i\}\big)\prod_{i=1}^{n}\left(\beta_i^{(1)}\right)^{h_1^i}\dots\left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}\,, \tag{4.14}$$

[2] See Appendix A.5 for the set of Mandelstams we choose to use later.

and be transverse (analogous to (3.8))

$$\bar{\mathcal{A}}_{\chi_1\cdots\chi_n}\big(\{k_i;\boldsymbol{\Theta}_i+\boldsymbol{\gamma}_i k_i\}\big)=\bar{\mathcal{A}}_{\chi_1\cdots\chi_n}\big(\{k_i;\boldsymbol{\Theta}_i\}\big)\,, \tag{4.15}$$

however there is no condition demanding homogeneity in the momenta as in (3.5).

The counting of possible building blocks $\mathcal{V}_{ij}^{(p)}$ to replace (2.80) works slightly differently than in the case of CFT correlators. While for CFT correlators (or massless amplitudes) the polarizations must be transverse to two of the available embedding space coordinates, in the massive amplitude case they are transverse only to the corresponding momentum $\Theta_i^{(p)}\cdot k_i=0$. However, for scattering amplitudes momentum conservation (4.16)

$$k_1+k_2+\ldots+k_n=0\,, \tag{4.16}$$

restricts the number of building blocks further to a set of $n-2$ independent terms, due to

$$\Theta_1^{(p)}\cdot k_n=-\Theta_1^{(p)}\cdot k_2-\ldots-\Theta_1^{(p)}\cdot k_{n-1}. \tag{4.17}$$

As before, manifestly transverse building blocks can be constructed by inserting the projector (2.122) to the transverse space, this time for massive states. As an example consider a 3-point amplitude with momenta k_i,k_j,k_k and momentum conservation $k_i+k_j+k_k=0$

$$\begin{aligned}
V_{i,jk}^{(p)}&\propto\Theta_{i\,A}^{(p)}\left(\eta^{AB}-\frac{k_i^Ak_i^B}{k_i^2}\right)k_{j\,B}\\
&=\Theta_{i\,A}^{(p)}\left(\eta^{AB}-\frac{(k_j+k_k)^A(k_j+k_k)^B}{(k_j+k_k)^2}\right)k_{j\,B}\\
&=\left(\Theta_i^{(p)}\cdot k_j\right)\frac{k_k^2+k_j\cdot k_k}{(k_j+k_k)^2}-\left(\Theta_i^{(p)}\cdot k_k\right)\frac{k_j^2+k_j\cdot k_k}{(k_j+k_k)^2}\\
&=\frac{-\big(\Theta_i^{(p)}\cdot k_j\big)\big(k_i\cdot k_k\big)+\big(\Theta_i^{(p)}\cdot k_k\big)\big(k_i\cdot k_j\big)}{k_i^2}\,.
\end{aligned} \tag{4.18}$$

This produces the same building blocks as constructed above for CFT correlators (3.19) up to a factor. For $n>3$ repeating the same computation produces linear combinations of these building blocks so the same $n-2$ element basis as for CFT correlators (3.20) can be used.

For three-point amplitudes all products of different momenta can be expressed in terms of masses $k_i^2=-m_i^2$

$$V_{i,jk}^{(p)}=\sqrt{\frac{\alpha'}{2}}\;\frac{\big(\Theta_i^{(p)}\cdot k_j\big)\big(m_i^2+m_k^2-m_j^2\big)-\big(\Theta_i^{(p)}\cdot k_k\big)\big(m_i^2+m_j^2-m_k^2\big)}{m_i^2}\,. \tag{4.19}$$

Here we also chose a coefficient that will be convenient for the string theory computations below.

For the building blocks $H_{ij}^{(p,q)}$ we simply use the same expression as for CFT correlators (3.17), but with an extra factor α' to cancel the mass dimension introduced by the momenta

$$H_{ij}^{(p,q)}=2\alpha'\bigg(\Big(k_j\cdot\Theta_i^{(p)}\Big)\Big(k_i\cdot\Theta_j^{(q)}\Big)-\Big(\Theta_i^{(p)}\cdot\Theta_j^{(q)}\Big)\Big(k_i\cdot k_j\Big)\bigg). \tag{4.20}$$

Since there are $n-2$ independent $V_{i,jk}^{(p)}$ building blocks, the number of tensor structures is again given by $n_{\text{structures}}^{\lambda_1\ldots\lambda_n}(n-2)$ as defined in (2.82).

In this thesis the formalism will mostly be used for 3-point amplitudes of fully symmetric tensors, so let us scale down our machinery a bit and write the simpler formulae for these. In this case no anticommuting polarizations are needed and the amplitude is directly given by

$$\mathcal{A}_{\chi_1\chi_2\chi_3}\big(\{k_i;Z_i\}\big)=\sum_k c_{\chi_1\chi_2\chi_3,k}\,F^k_{\lambda_1\lambda_2\lambda_3}\big(\{k_i;Z_i\}\big)\,, \tag{4.21}$$

and each symmetric tensor is encoded by a single vector Z_i, leading to the building blocks

$$\begin{aligned} H_{ij}&=2\alpha'\Big(\big(k_j\cdot Z_i\big)\big(k_i\cdot Z_j\big)-\big(Z_i\cdot Z_j\big)\big(k_i\cdot k_j\big)\Big)\\ V_{i,jk}&=\sqrt{\frac{\alpha'}{2}}\;\frac{\big(Z_i\cdot k_j\big)\big(m_i^2+m_k^2-m_j^2\big)-\big(Z_i\cdot k_k\big)\big(m_i^2+m_j^2-m_k^2\big)}{m_i^2}\,. \end{aligned} \tag{4.22}$$

4.2.2 Massless scattering amplitudes

For amplitudes of massless particles the embedding of irreducible $SO(d)$ irreps into $\mathbb{R}^{d+1,1}$ can be performed in the same way as for conformal field theories in embedding space. As a result, the construction of transverse building blocks for tensor structures works in the same way as for CFT correlators, described in Section 3.1.1. In analogy to the previous section, momentum conservation further reduces the number of independent building blocks $\mathcal{V}^{(p)}_{ij}$ for a given i by one, so the number of massless scattering amplitudes is given by $n^{\lambda_1\ldots\lambda_n}_{\text{structures}}(n-3)$ as given in (2.82). To make this more clear, note that the essential part of the building blocks

$$V^{(p)}_{i,jk}\propto-\big(\Theta^{(p)}_i\cdot k_j\big)\big(k_i\cdot k_k\big)+\big(\Theta^{(p)}_i\cdot k_k\big)\big(k_i\cdot k_j\big)\,, \tag{4.23}$$

is linear in the momenta k_j and k_k, so using momentum conservation one of them can be easily expressed in terms of the others.

Note that a correspondence between the number of massless scattering amplitudes and conformal correlators of conserved tensors was conjectured in [15] at least for fully symmetric tensors. It would be interesting to study whether this correspondence also holds true for more general representations, leading to a general result for the number of tensor structures in correlators of conserved tensors.

4.3 Unitarity cuts of tree-level string amplitudes

With the tensor structures in place, the coefficients appearing in 3-point amplitudes can now be computed using (4.10) and the known residues of Koba-Nielsen amplitudes. It will be checked for a few examples that the ansatz matches the known residues and that the missing coefficients can be computed unambiguously. Various consistency checks will be performed: The same coefficients must appear when the same 3-point amplitude is part of a different (higher point) tachyon amplitude and the coefficients must be real, which is required for unitarity.

4.3.1 One tensor, two tachyons

First the unitarity cut (4.10) of the Veneziano amplitude is studied

$$-\operatorname{Res}_{s_{12}\to A-1}\mathcal{A}_{TTTT}=\sum_\lambda\mathcal{A}_{TT[A,\lambda]}\big(k_1,k_2;\mathbf{D_Z}\big)\mathcal{A}_{[A,\lambda]TT}\big(k_3,k_4;\mathbf{Z}\big)\,. \tag{4.24}$$

Here the sum runs over all states $[A,\lambda]$ in the spectrum at mass level A and the differential operator $\mathbf{D_Z}$ restores tracelessness of the tensors. The sum over an orthonormal basis of polarization tensors has been turned into a contraction between the two amplitudes, as explained under (2.116).[3]

The three-point amplitudes in (4.24) involve two scalars each, hence the number of tensor structures is given by (2.51)

$$n^{\bullet\bullet\alpha}_{\text{structures}}(1) = \sum_{q\geq 0} b_{\bullet\bullet\alpha[q]} = \sum_{q\geq 0} \delta^{\alpha}_{[q]} \, . \tag{4.25}$$

This implies that the third state in such amplitudes generally has to be a fully symmetric tensor. Furthermore we know about the actual spectrum (4.1) that the rank of the tensors is limited by the mass level. Using this (4.24) becomes

$$-\operatorname{Res}_{s_{12}\to A-1}\mathcal{A}_{TTTT} = \sum_{a=0}^{A} \mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;D_Z\big)\mathcal{A}_{[A,[a]]TT}\big(k_3,k_4;Z\big) \tag{4.26}$$

with the form of the three-point amplitudes as defined in (4.21) and the operator D_Z generating the projector to traceless symmetric tensors is defined in (2.40). The 3-point amplitudes appearing here are given by (4.21)

$$\mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;Z\big) = c_{TT[A,[a]]}F_{\bullet\bullet[a]}\big(k_1,k_2;Z\big)\, , \tag{4.27}$$

with the single tensor structure

$$F_{\bullet\bullet[a]}\big(k_1,k_2;Z\big) = (V_{3,12})^a \, . \tag{4.28}$$

Inserting $V_{3,12}$ from (4.22) one finds

$$\mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;Z\big) = c_{TT[A,[a]]}\left(\sqrt{\frac{\alpha'}{2}}(k_1-k_2)\cdot Z\right)^a \, , \tag{4.29}$$

in agreement with [56]. Note that although the building block $V_{3,12}$ for massive states is used, this formula can be used for the massless state at $A=1$ as well, because the mass m_3^2 in the denominator of (4.22) drops out in the case $m_1^2 = m_2^2$.

As recalled above, one way to prove unitarity of the Veneziano amplitude is to show that all the $c_{TT[A,[a]]}$ are real and more generally, real coefficients in all 3-point amplitudes imply unitarity of the complete S-matrix. It is not obvious from (4.26) that the $c_{TT[A,[a]]}$ are real since the coefficients always appear in this formula as squares $c^2_{TT[A,[a]]}$ which in principle could be negative. By matching the right-hand side of (4.26) to the known residue of the 4-point tachyon amplitude, the coefficients $c_{TT[A,[a]]}$ can be calculated. While straightforward for low levels, this computation gets harder for the general case. This computation can be found in Appendix A.7 with the result

$$c^2_{TT[A,[a]]} = \begin{cases} \displaystyle\sum_{l=0}^{\frac{A-a}{2}} V_{\frac{A-a}{2}-l,A}\left(\frac{A+3}{4}\right)^{2l}\frac{(a+1)^{(2l)}}{l!(\frac{d}{2}+a)^{(l)}} & A-a \text{ even,} \\ 0 & A-a \text{ odd,} \end{cases} \tag{4.30}$$

[3] Whenever such contractions are performed, momentum conservation and the on-shell conditions have to be used to write the result in terms of the remaining $\frac{(N-2)(N-3)}{2}$ independent kinematic variables k_{ij} with $1<i<j<N$. This procedure is briefly described in Appendix A.5.

where $V_{k,A\,\mathrm{even}}$ and $V_{k,A\,\mathrm{odd}}$ are essentially the central factorial numbers $t(A,k)$ and $t_2(A,k)$, (sequences A008955 and A008956 in the Online Encyclopedia of Integer Sequences [57] and also defined in (A.56) and (A.57))

$$V_{k,A\,\mathrm{even}} = \frac{(-1)^k}{A!4^k} t_2\left(\frac{A}{2}, k\right), \qquad 0 \le k \le \left\lfloor \frac{A}{2} \right\rfloor, \tag{4.31}$$

$$V_{k,A\,\mathrm{odd}} = \frac{(-1)^k}{A!} t\left(\frac{A-1}{2}, k\right), \qquad 0 \le k \le \left\lfloor \frac{A}{2} \right\rfloor. \tag{4.32}$$

The zero in equation (4.30) is in agreement with the spectrum given in (4.1). In order to prove tree-level unitarity of the Veneziano amplitude, one has to show

$$c^2_{TT[A,[a]]} \overset{?}{\ge} 0, \qquad \forall A, a \qquad \text{(to be shown)}\,. \tag{4.33}$$

Despite having formula (4.30), this is not straightforward since the $V_{k,A}$ contains an alternating sign and the central factorial numbers complicate the issue. Explicit checks for $D = 26$ and all states up to $A = 400$ show that the squared coefficients are positive.

The no-ghost theorem conditions

Before continuing to 3-point amplitudes with two massive legs, it will be shown that the techniques which led to (4.26) can be used to (re)derive the no-ghost theorem conditions. For this, consider the Veneziano amplitude for arbitrary 'intercept' α_0

$$\mathcal{A}_4 = \frac{\Gamma(-s_{12} - \alpha_0)\Gamma(-s_{23} - \alpha_0)}{\Gamma(-s_{12} - s_{23} - 2\alpha_0)}, \tag{4.34}$$

and arbitrary dimension D. The intercept α_0 appears in the residues

$$\lim_{s_{12} \to A - \alpha_0} \mathcal{A}_4 = \frac{1}{A!} \frac{-1}{s_{12} - A + \alpha_0} \prod_{i=1}^{A} (s_{23} + \alpha_0 + i), \quad A \in \mathbb{N}_0. \tag{4.35}$$

In the case $A = 0$ a scalar particle with minimal mass $\alpha' m^2 = -\alpha_0$ is exchanged. It will be assumed this is the same as the external particle. Therefore the external tachyons also have this mass.

The left-hand side of equation (4.26) becomes with $A = 1$

$$-\operatorname{Res}_{s_{12} \to 1 - \alpha_0} \mathcal{A}_4 = s_{23} + \alpha_0 + 1. \tag{4.36}$$

For the right-hand side of equation (4.26) a vector and a scalar particle have to be considered in this case

$$\begin{aligned}
&\lim_{s_{12} \to 1 - \alpha_0} \mathcal{A}_{TT[1,\square]}\big(k_1, k_2; \partial_Z\big) \mathcal{A}_{[1,\square]TT}\big(k_3, k_4; Z\big) + \mathcal{A}_{TT[1,\bullet]}\big(k_1, k_2\big) \mathcal{A}_{[1,\bullet]TT}\big(k_3, k_4\big) \\
&= \lim_{s_{12} \to 1 - \alpha_0} c^2_{TT[1,\square]} \frac{\alpha'}{2} (k_1 - k_2) \cdot (k_3 - k_4) + c^2_{TT[1,\bullet]} \\
&= c^2_{TT[1,\square]} \left(s_{23} + \frac{3}{2}\alpha_0 + \frac{1}{2} \right) + c^2_{TT[1,\bullet]}.
\end{aligned} \tag{4.37}$$

Matching up (4.36) and (4.37), it is seen that the overall coefficient is $c^2_{TT[1,\square]} = 1$ and that the intercept is fixed at

$$\alpha_0 = 1 - 2c^2_{TT[1,\bullet]}. \tag{4.38}$$

Unitarity requires all couplings c to be real which implies one of the conditions of the no-ghost theorem

$$\alpha_0 \leq 1. \tag{4.39}$$

For the remainder of this thesis we set $\alpha_0 = 1$.

For the next mass level start by reading off the residue of the Veneziano amplitude (4.34)

$$-\operatorname{Res}_{s_{12}\to 1}\mathcal{A}_4 = \frac{1}{2}\left(s_{23}^2 + 5s_{23} + 6\right). \tag{4.40}$$

The right-hand side of (4.26) yields

$$\begin{aligned}
&\lim_{s_{12}\to 1}\Big\{\mathcal{A}_{TT[2,\square\square]}(k_1,k_2;D_Z)\mathcal{A}_{[2,\square\square]TT}(k_3,k_4;Z) \\
&\quad + \mathcal{A}_{TT[2,\square]}(k_1,k_2;\partial_Z)\mathcal{A}_{[2,\square]TT}(k_3,k_4;Z) + \mathcal{A}_{TT[2,\bullet]}(k_1,k_2)\mathcal{A}_{[2,\bullet]TT}(k_3,k_4)\Big\} \\
&= \lim_{s_{12}\to 1}\left\{c^2_{TT[2,\square\square]}\left(\frac{\alpha'}{2}\right)^2\left(\{(k_1-k_2)\cdot(k_3-k_4)\}^2 - \frac{1}{d}(k_1-k_2)^2(k_3-k_4)^2\right)\right. \\
&\qquad\qquad \left. + c^2_{TT[1,\square]}\frac{\alpha'}{2}(k_1-k_2)\cdot(k_3-k_4) + c^2_{TT[1,\bullet]}\right\} \\
&= c^2_{TT[2,\square\square]}\left(s_{23}^2 + 5s_{23} + \frac{25}{4} - \frac{25}{4d}\right) + c^2_{TT[2,\square]}\left(s_{23} + \frac{5}{2}\right) + c^2_{TT[2,\bullet]}.
\end{aligned} \tag{4.41}$$

This time (4.40) and (4.41) agree for

$$c^2_{TT[2,\square\square]} = \frac{1}{2}, \qquad c^2_{TT[2,\square]} = 0, \qquad c^2_{TT[2,\bullet]} = \frac{1}{2}\left(6 - \frac{25}{4} + \frac{25}{4d}\right) = \frac{26-(d+1)}{8d}. \tag{4.42}$$

Only for $D = d+1 = 26$ the symmetric traceless 2-tensor is the only particle appearing, as stated in (4.1). For $D < 26$ unitarity requires a scalar particle at this mass level[4], while for $D > 26$ the required coupling $c_{TT[2,\bullet]}$ becomes imaginary, which conflicts with unitarity of the S-matrix. So by unitarity there is an upper bound for D that agrees with the result known from the no-ghost theorem

$$D \leq 26. \tag{4.43}$$

This subsection contains a direct derivation of the dimension and intercept bounds of the no-ghost theorem from the Veneziano amplitude using nothing but locality, unitarity and Poincaré invariance. Closest to this in the literature as far as we are aware comes a derivation in [59] which does still use some worldsheet input about the spectrum.

4.3.2 Two tensors, one tachyon

Now consider the amplitude $\mathcal{A}_{[A,[a]]T[B,[b]]}$ of one tachyon and two massive particles on mass levels A, B with in the irreducible representations $[a], [b]$. Since the calculation of the coefficients was already cumbersome in the case of 3-point amplitudes with one massive leg, a general computation of the coefficients appearing in the amplitude will not be attempted here.

[4]known as a Brower state [58] in non-critical string theory

This amplitude appears first in the double residue of the 5-point tachyon amplitude

$$\begin{aligned}&\mathrm{Res}_{s_{12}\to A-1}\mathrm{Res}_{s_{45}\to B-1}\mathcal{A}_{T^5} \\ &= \sum_{a=0}^{A}\sum_{b=0}^{B}\mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;D_{Z_A}\big)\mathcal{A}_{[A,[a]]T[B,[b]]}\big(k_A,k_3,k_B;Z_A,D_{Z_B}\big)\mathcal{A}_{[B,[b]]TT}\big(k_4,k_5;Z_B\big)\,.\end{aligned} \tag{4.44}$$

The labels A and B are also used to label momenta $k_A = k_1 + k_2$ and $k_B = k_4 + k_5$ and the corresponding polarizations. The new 3-point amplitudes are given by

$$\mathcal{A}_{[A,[a]]T[B,[b]]}\big(k_A,k_3,k_B;Z_A,Z_B\big) = \sum_k c_{[A,[a]]T[B,[b]],k}F^k_{[a]\bullet[b]}\big(k_A,k_3,k_B;Z_A,Z_B\big)\,, \tag{4.45}$$

and depend on the tensor structures

$$F^k_{[a]\bullet[b]}\big(k_A,k_3,k_B;Z_A,Z_B\big) = (H_{AB})^k\,(V_{A,3B})^{a-k}\,(V_{B,A3})^{b-k}\ , \quad k\in\{0,1,\ldots,\min(a,b)\}\,. \tag{4.46}$$

The reason why only fully symmetric irreps appear in (4.44) is that both other 3-point amplitudes that appear involve two scalar states. This suggests that in terms of unitarity the 5-point amplitude does not add anything to the story of the 4-point amplitude. And indeed, assuming $c_{TT[A,[a]]}\in\mathbb{R}$, everything in (4.44) (including the know left hand side) is real and the right-hand side is linear in the $c_{[A,[a]]T[B,[b]],k}$, which are each multiplied by linearly independent polynomials of Mandelstams. This implies $c_{[A,[a]]T[B,[b]],k}\in\mathbb{R}$ and thus unitarity of the 5-point amplitude follows trivially from unitarity of the 4-point amplitude.

A nice consistency check is the case $A = B = 2$. It is known from (4.1) that at this mass level only one irep appears if $D = 26$, the symmetric traceless matrices $\square\square$.

$$\begin{aligned}&\mathrm{Res}_{s_{12}\to 1}\mathrm{Res}_{s_{45}\to 1}\mathcal{A}_5 \\ &= \mathcal{A}_{TT[2,\square\square]}\big(k_1,k_2;D_{Z_A}\big)\mathcal{A}_{[2,\square\square]T[2,\square\square]}\big(k_A,k_3,k_B;Z_A,D_{Z_B}\big)\mathcal{A}_{[2,\square\square]TT}\big(k_4,k_5;Z_B\big)\ .\end{aligned} \tag{4.47}$$

This has to be compared to the known result (A.27)

$$\sum_{a=0}^{2}(-1)^a\binom{k_{24}}{a}\binom{k_{23}}{2-a}\binom{k_{34}}{2-a} = \frac{1}{2}k_{24}^2 - k_{24}k_{23}k_{34} + \frac{1}{4}k_{23}^2k_{34}^2 + \ldots\ , \tag{4.48}$$

where only the highest order terms were written on the right-hand side. Since $c_{TT[2,\square\square]}$ was computed above (4.30) from the residue of the 4-point amplitude, the factorized expression (4.47) is determined up to the three coefficients $c_{[2,\square\square]T[2,\square\square],k}$, $k = 0,1,2$ in the amplitude (4.45). These coefficients are uniquely fixed by matching the coefficients of the monomials k_{24}^2, $k_{24}k_{23}k_{34}$ and $k_{23}^2k_{34}^2$ in the expressions (4.47) and (4.48), with the result

$$c_{[2,\square\square]T[2,\square\square],0} = -\frac{1}{144},\qquad c_{[2,\square\square]T[2,\square\square],1} = -\frac{1}{9},\qquad c_{[2,\square\square]T[2,\square\square],2} = \frac{1}{18}. \tag{4.49}$$

With these coefficients the expressions (4.47) and (4.48) match exactly, which is a non-trivial consistency check.

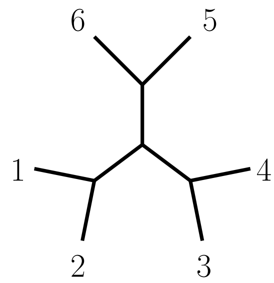
4.3.3 Going further

We checked for the mass levels up to $A = B = C = 3$ that the coefficients $c_{[A,[a]]T[B,[b]],k}$ obtained from the 5 point amplitude give the correct contribution to the factorized six point amplitude in a multiperipheral configuration, which is related to 3-point amplitudes by

$$
\begin{aligned}
&-\mathrm{Res}_{s_{12}\to A-1}\mathrm{Res}_{s_{123}\to B-1}\mathrm{Res}_{s_{56}\to C-1}\mathcal{A}_{T^6} \\
&= \sum_{a=0}^{A}\sum_{\beta}\sum_{c=0}^{C}\mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;D_{Z_A}\big)\mathcal{A}_{[A,[a]]T[B,\beta]}\big(k_A,k_3,k_B;Z_A,D_{\mathbf{Z}_B}\big) \\
&\qquad\qquad \mathcal{A}_{[B,\beta]T[C,[c]]}\big(k_B,k_4,k_C;\mathbf{Z}_B,D_{Z_C}\big)\mathcal{A}_{[C,[c]]TT}\big(k_5,k_6;Z_C\big)\,.
\end{aligned}
\tag{4.50}
$$

Here antisymmetric or mixed-symmetry tensors appear in the central propagator starting at $B=3$ (see the spectrum (4.1)). Since these states do not appear in 5-point amplitudes, these residues of the six point amplitude cannot be fully reconstructed from data obtained from the five point amplitude. Furthermore, the six point amplitude does not provide all the data to construct higher point amplitudes, because it does not involve enough momenta to contain all highly antisymmetric representations itself.

One could also study amplitudes of three symmetric tensors by considering the star configuration of a six-point amplitude



$$. \tag{4.51}$$

The maximal residue is related to 3-point amplitudes by

$$
\begin{aligned}
&-\mathrm{Res}_{s_{12}\to A-1}\mathrm{Res}_{s_{34}\to B-1}\mathrm{Res}_{s_{56}\to C-1}\mathcal{A}_{T^6} \\
&= \sum_{a=0}^{A}\sum_{b=0}^{B}\sum_{c=0}^{C}\mathcal{A}_{TT[A,[a]]}\big(k_1,k_2;D_{Z_A}\big)\mathcal{A}_{TT[B,[b]]}\big(k_3,k_4;D_{Z_B}\big)\mathcal{A}_{TT[C,[c]]}\big(k_5,k_6;D_{Z_C}\big) \\
&\qquad\qquad \mathcal{A}_{[A,[a]][B,[b]][C,[c]]}\big(k_A,k_B,k_C;Z_A,Z_B,Z_C\big)\,.
\end{aligned}
\tag{4.52}
$$

The most general 3-point amplitudes in 26 dimensions involve three $SO(25)$ irreps labeled by Young diagrams with up to 12 rows (2.2) and appear first in a configuration where 13 tachyons are attached to each leg of a central 3-point amplitude. Given that even the numerical coupling constants of one massive particle and two tachyons (4.30) are not that simple, computing general $SO(D-1)$ covariant 3-point amplitudes in this way is obviously very complex and would require considerable motivation. In the next chapter we study a shortcut to the residues which avoids this complexity by exploiting a property that is unique to string amplitudes, namely that they obey monodromy relations.

Chapter 5

String amplitudes from monodromy relations

The analysis in the previous chapter was based on general properties of scattering amplitudes, while string theory merely played the role of an example for a theory with a rich particle spectrum. In this chapter features that are typical to string theory play a major role, namely monodromy and Kawai-Lewellen-Tye (KLT) [60] relations and the particular scaling behavior of string amplitudes under BCFW shifts which is closely related to Regge behavior. We begin by reviewing these properties and continue by showing how they can be used to compute the residues of tree-level string theory amplitudes. Finally it is argued that the constraints used to fix the amplitudes constitute a definition of the tree-level string S-matrix in a flat background without any reliance on the worldsheet.

5.1 Properties of string theory amplitudes

5.1.1 Monodromy relations

Central to the discussion will be the monodromy relations first discussed in [23]. The two basic monodromy relations for color-ordered open string tree amplitudes in a flat background read

$$\boxed{\mathcal{A}(\beta, 1, 2, \ldots, N) = -\sum_{i=1}^{N-2} \exp\left[\pm i\pi(\sum_{j=1}^{i} k_{\beta,j})\right] \mathcal{A}(1, \ldots, i, \beta, i+1, \ldots, N),} \tag{5.1}$$

for an amplitude involving N bosonic particles. Basically the particle labelled β is moved through the other color-ordered particles, picking up a 'sign' for every interchange. In string theory this follows from the braid relation for flat background vertex operators [21]. Note there are two relations: one for each choice of sign in the exponent. For complex momenta these two relations are not complex conjugate.

From the basic relations others may be derived [61]. In modern language [62, 63] the relations needed below can be written as

$$\mathcal{A}(\beta^T, 1, \alpha, N) = (-1)^s \sum_{\sigma \in OP(\{\beta\},\{\alpha\})} \mathcal{P}_{\{\beta^T,1,\alpha,N\},\{1,\sigma,N\}} \mathcal{A}(1, \sigma, N), \tag{5.2}$$

where $\beta = \{\beta_1, ..., \beta_s\}$ is now an ordered set of particle labels and β^T indicates the inversion of the ordered set β. In the formula $\alpha = \{\alpha_1, ..., \alpha_{N-s-2}\}$ is an ordered set of particle labels and $OP(\{\beta\},\{\alpha\})$ are the ordered permutations of β and α i.e. the permutations of the union $\beta \cup \alpha$ that preserve the order of both subsets. The sum over $OP(\{\beta\},\{\alpha\})$ is known as the shuffle product $\beta \sqcup\!\sqcup \alpha$.

The phase factor $\mathcal{P}$ can be neatly expressed in terms closely related to the so-called momentum kernel [64]. In the notation of [61], it is given as a function of two permutations σ, τ as

$$\mathcal{P}_{\{\sigma\},\{\tau\}} = \exp\left[i\pi \sum_{i,j} k_{ij}\theta(\sigma^{-1}(i) - \sigma^{-1}(j))\theta(\tau^{-1}(j) - \tau^{-1}(i))\right], \tag{5.3}$$

where

$$\theta(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases}. \tag{5.4}$$

The θ's are there to let any k_{ij} appear in the exponent if and only if i and j appear in a different order in σ and τ. Some examples are

$$\mathcal{P}_{\{\sigma\},\{\sigma\}} = 1, \qquad \mathcal{P}_{\{1,2,3\},\{2,1,3\}} = \exp\left[i\pi k_{12}\right], \qquad \mathcal{P}_{\{\sigma\},\{\sigma^T\}} = \exp\left[i\pi \sum_{i<j} k_{ij}\right]. \tag{5.5}$$

For fermionic particles an additional minus sign appears every time a pair of fermions is interchanged, see [65] for more details.

The relations are universal in that they do not depend on the particle content of the open string amplitude. Moreover, also the 'conjugate' relations hold:

$$\mathcal{A}(\beta^T, 1, \alpha, N) = (-1)^s \sum_{\sigma \in OP(\{\beta\},\{\alpha\})} \mathcal{P}^*_{\{\beta^T,1,\alpha,N\},\{1,\sigma,N\}} \mathcal{A}(1,\sigma,N), \tag{5.6}$$

with only the sign of the exponent changed

$$\mathcal{P}^*_{\{\sigma\},\{\tau\}} = \exp\left[-i\pi \sum_{i,j} k_{ij}\theta(\sigma^{-1}(i) - \sigma^{-1}(j))\theta(\tau^{-1}(j) - \tau^{-1}(i))\right]. \tag{5.7}$$

These relations hold also for complex momenta: in the worldsheet derivation the exact phase simply corresponds to a choice of branch cut, while the amplitudes should be independent of this choice. Relations (5.2) and (5.6) can be subtracted to give

$$\sum_{\sigma \in OP(\{\beta\},\{\alpha\})} \mathcal{S}_{\{\beta^T,1,\alpha,N\},\{1,\sigma,N\}} \mathcal{A}(1,\sigma,N) = 0, \tag{5.8}$$

where

$$\mathcal{S}_{\{\sigma\},\{\tau\}} = \operatorname{Im} \mathcal{P}_{\{\sigma\},\{\tau\}} = \sin\left[\pi \sum_{i,j} k_{ij}\theta(\sigma^{-1}(i) - \sigma^{-1}(j))\theta(\tau^{-1}(j) - \tau^{-1}(i))\right]. \tag{5.9}$$

In this chapter, equation (5.8) will be used to study the residues of amplitudes $\mathcal{A}(123\ldots N)$ in the variables $s_{12}, s_{123}, \ldots, s_{1\ldots N-2}$. To this end it is useful to rewrite the expression in a way which exposes the pole in the $s_{1,\beta}$ channel by splitting off the first element of α (this will be labeled $s+2$) and separating the sum over its positions. Although not all particle labels will be specified in the following formulae, set $\beta = \{2, \ldots, s+1\}, \alpha = \{s+3, \ldots, N-1\}$ for the remainder of the chapter.

The relations in (5.8) are graded by the size of the set β. For instance, the relation that makes the pole in $s_{1\beta_1}$ manifest is

$$\begin{aligned} &\mathcal{A}(1, \beta_1, 3, \alpha, N) \\ &= -\frac{1}{\mathcal{S}_{\{\beta_1,1,3,\alpha,N\},\{1,\beta_1,3,\alpha,N\}}} \sum_{\sigma \in OP(\{\beta_1\},\{\alpha\})} \mathcal{S}_{\{\beta_1,1,3,\alpha,N\},\{1,3,\sigma,N\}} \mathcal{A}(1, 3, \sigma, N) \\ &= \frac{(-1)^{\alpha' m_{\beta_1}^2 + \alpha' m_1^2}}{\sin(\pi s_{1,\beta_1})} \sum_{\sigma \in OP(\{\beta_1\},\{\alpha\})} \mathcal{S}_{\{\beta_1,1,3,\alpha,N\},\{1,3,\sigma,N\}} \mathcal{A}(1, 3, \sigma, N). \end{aligned} \tag{5.10}$$

where the definition of the Mandelstam variables in equation (4.4) was used. Note that none of the amplitudes on the right-hand side has a pole in the $s_{1\beta_1}$ channel. Since the sine functions in the numerator cannot cause poles, all poles must be captured by the sine in the denominator. Similarly, the pole in $s_{1\beta_1\beta_2}$ is manifest in

$$\begin{aligned} &\mathcal{A}(1, \beta_1, \beta_2, 4, \alpha, N) \\ &= \frac{(-1)^{\alpha'(m_{\beta_1}^2 + m_{\beta_2}^2 + m_1^2)}}{\sin(\pi s_{1,\beta_1,\beta_2})} \Bigg[\sum_{\sigma \in OP(\{\beta_1,\beta_2\},\{\alpha\})} \mathcal{S}_{\{\beta_2,\beta_1,1,4,\alpha,N\},\{1,4,\sigma,N\}} \mathcal{A}(1, 4, \sigma, N) \\ &\quad + \sum_{\sigma \in OP(\{\beta_2\},\{\alpha\})} \mathcal{S}_{\{\beta_2,\beta_1,1,4,\alpha,N\},\{1,\beta_1,4,\sigma,N\}} \mathcal{A}(1, \beta_1, 4, \sigma, N) \Bigg]. \end{aligned} \tag{5.11}$$

The general form of this relation is

$$\begin{aligned} &\mathcal{A}(1, \beta_1, ..., \beta_s, s+2, \alpha_1, ..., \alpha_{N-s-3}, N) \\ &= \frac{(-1)^{\alpha'\left(m_1^2 + \sum_{i=1}^{s} m_{\beta_i}^2\right)}}{\sin(\pi s_{1\beta_1\ldots\beta_s})} \Bigg[\sum_{\sigma \in OP(\{\beta_1,\ldots,\beta_s\},\{\alpha\})} \mathcal{S}_{\{\beta^T,1,s+2,\alpha,N\},\{1,s+2,\sigma,N\}} \mathcal{A}(1, s+2, \sigma, N) \\ &\quad + \sum_{l=1}^{s-1} \sum_{\sigma \in OP(\{\beta_{l+1},\ldots,\beta_s\},\{\alpha\})} \mathcal{S}_{\{\beta^T,1,s+2,\alpha,N\},\{1,\beta_1,\ldots,\beta_l,s+2,\sigma,N\}} \mathcal{A}(1, \beta_1, ..., \beta_l, s+2, \sigma, N) \Bigg]. \end{aligned} \tag{5.12}$$

The sine in the denominator captures the complete pole in the $(1, \beta)$-channel. It should be clear these relations may be nested to uniquely express a given open string amplitude in terms of a particular set of basis amplitudes with the positions of three particles fixed, e.g. $\mathcal{A}(1, 2, \sigma, N)$. This particular form of the monodromy relations has first appeared in [61], as far as we are aware.

Roots of amplitudes

The monodromy relations can be used to find the roots of amplitudes as studied in [22]. Their argument to find the roots has to be slightly extended here to allow for complex momenta.

In (5.2) each factor $\mathcal{P}_{\{\beta^T,1,\alpha,N\},\{1,\sigma,N\}}$ depends on the Mandelstam $s_{1,\beta}$ and additional momentum invariants

$$\begin{aligned} \{k\}_\sigma = \{k_{ij}\theta(\sigma^{-1}(i) - \sigma^{-1}(j))\theta(\tau^{-1}(j) - \tau^{-1}(i)) \mid i, j \in \{\sigma\}\}, \\ \text{where} \qquad \tau = \{\beta \cup \alpha\}, \qquad \sigma \in OP(\{\beta\}, \{\alpha\}). \end{aligned} \tag{5.13}$$

If all elements of all $\{k\}_\sigma$ are taken to non-negative integer values

$$\{k\}_\sigma \subset \mathbb{N}_0 \quad \forall \sigma \in OP(\{\beta\}, \{\alpha\}), \tag{5.14}$$

while $s_{1,\beta}$ is kept arbitrary the equations (5.2) and (5.6) become

$$\mathcal{A}(\beta, 1, \alpha, N) = \exp(-i\pi s_{1,\beta})F = \exp(i\pi s_{1,\beta})F, \tag{5.15}$$

for some function F. This can only be satisfied for generic $s_{1,\beta}$ if both $\mathcal{A}(\beta, 1, \alpha, N)$ and F vanish. The restriction to non-negative integers was to avoid hitting poles in the amplitudes which appear in the monodromy relations.

A second remark is that using a more general form of the monodromy relations should allow us to obtain additional sets of roots more straightforwardly. In [22] only monodromy relations were used where β has only one element which means there is one set of roots per amplitude that can trivially be read off as in (5.14). Further sets of roots are obtained by combining monodromy relations and can contain conditions on multi-particle Mandelstams. For instance, a table in [22] lists five sets of roots of the 6-point amplitude. Two of them are given by (5.14) when β has one or two elements. The remaining sets of roots in the table involve conditions on multi-particle Mandelstams and it still seems to be necessary to combine multiple monodromy relations to derive these. A general and simple way to derive all sets of roots is a worthwhile direction to explore, but will not be needed here. Below the form of the monodromy relations reviewed above will be used to study the roots.

The field theory limit[1] of the monodromy relations results in the BCJ-relations [66], which can alternatively be derived using a non-adjacent BCFW shift [67]. It would be interesting to see if the string monodromy relations could also be derived from a non-adjacent BCFW shift.

5.1.2 Scaling at infinite BCFW shifts

Let us briefly review the scaling of BCFW-shifted amplitudes around infinity. A sufficient condition for the residue at infinity in equation (4.8) to vanish is that $\mathcal{A}(z) \to 0$ for $z \to \infty$. In principle if one can compute the residue at infinity explicitly there is also an effective recursion relation, but examples of this type tend to be quite involved. Hence a vanishing residue will be aimed at henceforth. One needs to study the expansion of $\mathcal{A}(z)$ around $z = \infty$. In field theory a very direct analysis [53] in 4 or more dimensions yields

$$\mathcal{A}_{\text{ym}}(z) \sim \hat{\xi}_{1,\mu}\hat{\xi}_{2,\nu}\mathcal{A}^{\mu\nu}(z), \tag{5.16}$$

for the BCFW shift of two color-adjacent gluons labelled one and two in a Yang-Mills amplitude (possibly minimally coupled to matter). Here the ξ vectors are the polarization vectors of the shifted gluons, whose large z behavior is easily analyzed. The tensor $\mathcal{A}^{\mu\nu}$ is given as:

$$\mathcal{A}^{\mu\nu}(z) = z\left(\eta^{\mu\nu} f_0\left(\frac{1}{z}\right) + \frac{1}{z}B^{\mu\nu}\left(\frac{1}{z}\right) + \mathcal{O}\left(\frac{1}{z}\right)^2\right), \tag{5.17}$$

where $f(w)$ and $B^{\mu\nu}(w)$ are polynomials in w with generically non-zero constant term and the tensor $B^{\mu\nu}$ is anti-symmetric in its indices. Combining the $\mathcal{A}^{\mu\nu}$ tensor with the behavior

[1] Loosely speaking, this is the "$\alpha' \to 0$" limit. More correctly, this is the limit where $\alpha' s_{ij} \to 0$ for any i, j

of the polarization vectors then gives the result that for any choice of helicities of the singled-out two gluons a shift exists such that the amplitude may be computed through on-shell recursion. For this shift one obtains

$$\frac{\mathcal{A}(z)}{z} \sim \frac{1}{z^2} \text{ for } z \to \infty \ . \tag{5.18}$$

In string theory the result for the large z shift is very similar to the field theory result. As shown in [20] and [21], in the superstring

$$\mathcal{A}_{\text{open},gg}(z) \sim \hat{\xi}_{1,\mu}\hat{\xi}_{2,\nu} z^{-2\alpha' k_1 \cdot k_2} \mathcal{A}^{\mu\nu}(z), \tag{5.19}$$

holds for the shift of two color-adjacent gluons, with arbitrary field content on the other legs. The difference to the field theory is in the Regge-like prefactor. In the bosonic string, this result for the BCFW shift of two color-adjacent gluons is structurally the same, but the tensor $\mathcal{A}^{\mu\nu}$ is modified to $\tilde{\mathcal{A}}^{\mu\nu}$ as

$$\tilde{\mathcal{A}}^{\mu\nu}(z) \equiv \mathcal{A}^{\mu\nu}(z) + \, z\, \alpha' k^\mu k^\nu f_1\left(\frac{1}{z}\right), \tag{5.20}$$

with $k^\mu = k_1^\mu + k_2^\mu$ and $f_1(w)$ a polynomial of w with non-zero constant term. This particular term is forbidden in any supersymmetric field theory as it generates amplitudes with all helicities equal which is perturbatively impossible in a supersymmetric field theory [68]. For shifts of two tachyons, the result reads

$$\mathcal{A}_{\text{open},TT}(z) \sim z^{s_{12}+1}\left(f_1\left(\frac{1}{z}\right)\right), \tag{5.21}$$

again with arbitrary field content on the other legs. It is easy to show that the general structure of a BCFW shift for arbitrary choice of matter content on the two legs will always be a Regge-type factor times a polynomial in $1/z$. This can be computed directly from the OPE, see [20] and [21] for details.

The shifts of color-non-adjacent particles on an open string amplitude follow from the use of monodromy relations, see [21]. The BCFW shift of two particles on a closed string amplitude follows basically by either the same worldsheet based argument or from the use of the KLT relations [60].

5.2 String amplitudes from monodromy relations

In this section it will be shown that the residues at kinematic poles can be derived from the monodromy relations. These are then used in the on-shell recursion relations to construct the complete amplitude.

Instrumental are the location of the roots of the residues of amplitudes. Below it is shown that the form of the monodromy relations discovered more recently and reviewed above allow for a more natural approach to studying roots than was possible in the original [22] paper. In their new form the monodromy relations allow the systematic study of the roots of the *residues* of the amplitude, a possibility that was not obvious from the original monodromy relations. As a further input this section uses the behavior under BCFW-shifts reviewed above.

To provide some orientation the four point amplitudes will be discussed extensively. At four points the monodromy relation (5.10) can be written as

$$\mathcal{A}(1234) = (-1)^{\alpha'(m_1^2+m_4^2)} \frac{\sin(\pi s_{13})}{\sin(\pi s_{12})} \mathcal{A}(1324). \tag{5.22}$$

This relation is easily checked for the Veneziano amplitude in equation (1.8). A simple consistency check is to consider the pole structure: the amplitude on the left-hand side has poles in the s_{12} and s_{23} channels, but not in the s_{13} channel. Similarly, the amplitude on the right-hand side has poles in the s_{23} and s_{13} channels, but not in the s_{12} channel. This discrepancy is solved by the roots of the sine functions.

In equation (5.22) it is obvious that all poles in the s_{12}-channel of the left-hand side amplitude are contained in the sine-function in the denominator on the right-hand side. As a bonus, the equation also displays possible roots of the amplitudes. These are contained in the sine-function in the numerator. A restriction here is that for sufficiently large integer values of s_{13} the amplitude on the right-hand side develops a pole, leading to a finite, non-vanishing result. In the bosonic string case for instance the amplitude $\mathcal{A}(1234)$ generically has a series of roots at

$$s_{13} \in \{-2, -3, -4, \ldots\}. \tag{5.23}$$

Comparing to the Veneziano amplitude in equation (1.8) it is seen that *all* roots of this particular amplitude arise this way. Note that the starting location of the row of roots of the amplitude on the left-hand side is determined by the location of the lowest mass pole of the amplitude on the right-hand side. The argument just given applies to all possible choices of external states within the string spectrum and to the superstring. The precise starting location of the roots depends on the external masses and the spectrum, as some states will for instance not couple to two tachyons (see Section 4.3.1).

5.2.1 Four tachyons in the open bosonic string

The previous reasoning can be extended to compute the residues at poles. For definiteness the focus will first be on the Veneziano amplitude with four external tachyons. From equation (5.22) it follows that

$$\mathrm{Res}_{s_{12}\to A-1} \mathcal{A}(1234) = \frac{(-1)^{A-1}}{\pi} \left[\sin(\pi s_{13}) \mathcal{A}(1324)\right]_{s_{12}=A-1}, \tag{5.24}$$

for some non-negative integer A. By perturbative unitarity, Poincaré invariance and locality the left-hand side of this equation must be a polynomial in s_{13}. It is not manifest the right-hand side is. Note however that as a function of s_{13} it no longer has an *infinite* series of roots since by momentum conservation

$$(A-1) + s_{23} + s_{13} = \sum m_i^2 = -4. \tag{5.25}$$

Hence, if s_{13} is $\in \{-1, 0, 1, \ldots\}$ it will hit the pole in the amplitude $\mathcal{A}(1324)$ in the $(1,3)$ channel while if s_{13} is $\in \{-2-A, -3-A, \ldots\}$ it will hit a pole in the $(2,3)$ channel. For four tachyons, this implies the residue is a polynomial of at least degree A, with roots at $\{-2, -3, \ldots, -1-A\}$. For $A = 0$, the polynomial is a constant. The maximal degree of the polynomial in s_{13} appearing in the residue at this pole is set by the maximal spin of the spectrum at mass level A which is known to be A itself. Actually, this can be demonstrated by studying a $(1,2)$ channel BCFW shift of the residue. By equation (5.21) one obtains for the residue under this shift in a cross-channel

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1234) \sim z^A \left(f_1\left(\frac{1}{z}\right)\right). \tag{5.26}$$

Note that technically, one should study a non-adjacent BCFW shift for the amplitude on the right-hand side of equation (5.24). How to do this was explained in [21], which in this particular case simply reduces to reading of the large z shift from the left-hand side of equation (5.24). It will be assumed the BCFW large z-limit in the $(1,2)$ channel and taking the residue in this channel commute[2]. Since the residue must be a function of s_{13} only, the BCFW shift fixes the maximal spin of the spectrum at level A to be A.

By the main theorem of algebra, these observations fix the residue up to an overall constant

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1234) = c(s_{13}+2)\dots(s_{13}+A+1). \tag{5.27}$$

This constant can be fixed by tuning s_{13} to the value -1 in equation (5.24). The right-hand side in this case does not vanish but factorizes by unitarity into two 3-tachyon amplitudes,

$$\lim_{s_{13}\to -1}\left[\frac{(-1)^{A-1}}{\pi}\sin(\pi s_{13})\mathcal{A}(1324)\right] = (-1)^{A-1}\mathcal{A}_3(T,T,T)\mathcal{A}_3(T,T,T) = (-1)^{A-1}g_o^2, \tag{5.28}$$

these 3 point amplitudes are just the open string coupling constant g_o. Combining this expression for the right-hand side of equation (5.24) with equation (5.27) for the left-hand side at $s_{13}=-1$ now fixes the constant c to be

$$c = g_o^2\frac{(-1)^{A-1}}{\Gamma[A+1]}. \tag{5.29}$$

Note this computation has fixed the numerical coefficient of all the tachyon-tachyon-massive-state couplings in terms of the three tachyon coupling. As a result the complete residue is fixed by equation (5.24), a combination of unitarity, locality, Poincaré invariance as well as Regge behavior. The string coupling constants will mostly be suppressed in the following.

The complete four point function through on-shell recursion

The stage is now set for the derivation of the Veneziano amplitude through on-shell recursion by assembling the above building blocks. Since the s_{12} channel poles have been worked out it is natural to study a shift on particles 2 and 3. This will keep s_{23} invariant. Hence it is advantageous to express the residues in equation (5.27) in terms of s_{23} instead of s_{13},

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1234) = g_o^2\frac{(-1)^{A-1}}{\Gamma[A+1]}(-s_{23}-A-1)\dots(-s_{23}-2). \tag{5.30}$$

The on-shell recursive expression in this case simply gives (suppressing g_o)

$$\begin{aligned}\mathcal{A}(1,2,3,4) &= -\sum_{A=0}^{\infty}\frac{1}{s_{12}-A+1}(-1)^A\frac{\Gamma[-s_{23}-1]}{\Gamma[A+1]\Gamma[-s_{23}-1-A]}\\ &= -\sum_{A=0}^{\infty}\frac{1}{s_{12}-A+1}(-1)^A\binom{k_{23}}{A}\\ &= \frac{\Gamma[-s_{12}-1]\Gamma[-s_{23}-1]}{\Gamma[-s_{12}-s_{23}-2]},\end{aligned} \tag{5.31}$$

[2] This can be proven from the worldsheet point of view using the full result for the large z-shift in [21].

as the result of the in string theory very well-known summation formulae for the β function. In the second line the binomial coefficient was used.

In the rest of this chapter the Veneziano amplitude calculation will widely be extended. To motivate more general remarks further example computations will be presented first.

5.2.2 Three tachyons, one gluon in the open bosonic string

In general string scattering amplitudes will involve particles with polarization vectors. To show how this fits into the calculation first study the example of an amplitude with three tachyons and a gluon. Residues of the amplitude $\mathcal{A}(1,2,3,4_g)$ with three tachyons labelled 1, 2, 3 and a gluon 4_g in the $(1,2)$ channel can depend on one momentum invariant, say s_{23}, and terms containing the polarization $\xi_4 \cdot k_1$, $\xi_4 \cdot k_2$, $\xi_4 \cdot k_3$. Due to momentum conservation and orthogonality of the polarization vector w.r.t. it's own momentum, one of these can be expressed in terms of the other two, e.g.

$$\xi_4 \cdot k_2 = -\xi_4 \cdot (k_1 + k_3) \; . \tag{5.32}$$

Momentum conservation gives in this case

$$s_{12} + s_{23} + s_{13} = \sum \alpha' m_i^2 = -3. \tag{5.33}$$

By the same monodromy relation as before (5.22), repeated here for convenience,

$$\mathcal{A}(1,2,3,4_g) = (-1)^{\alpha'(m_1^2+m_4^2)} \frac{\sin(\pi s_{13})}{\sin(\pi s_{12})} \mathcal{A}(1,3,2,4_g), \tag{5.34}$$

the amplitude $\mathcal{A}(1,3,2,4_g)$ has no poles (and thus $\mathcal{A}(1,2,3,4_g)$ has roots) for

$$s_{13} \in \mathbb{Z} \quad \wedge \quad s_{23} \leq -2 \quad \wedge \quad s_{13} \leq -2. \tag{5.35}$$

This becomes at the residue $s_{12} = A - 1$, using (5.33)

$$s_{23} \in \mathbb{Z} \quad \wedge \quad s_{23} \leq -2 \quad \wedge \quad s_{23} \geq -A. \tag{5.36}$$

Hence there is, again, only a finite number of roots. This fixes a polynomial of degree $A-1$. Similar to the Veneziano example the residues have to be proportional to the following polynomials which exhibit all the required roots

$$\frac{\Gamma[-s_{23}-1]}{\Gamma[A]\Gamma[-s_{23}-A]} = \begin{pmatrix} k_{23} \\ A-1 \end{pmatrix} \qquad A > 0. \tag{5.37}$$

The poles at $A = 0$ and $A = 1$ deserve special attention. For $A = 0$ the exchanged particle in the $(1,2)$ channel is a tachyon. Hence the polarization of the gluon can only be contracted to the momentum which belongs to the tachyon on the same 3-point amplitude (up to momentum conservation). This gives

$$\mathrm{Res}_{s_{12}\to -1} \mathcal{A}(1,2,3,4_g) = c_0 \, \xi_4 \cdot k_3, \tag{5.38}$$

up to a numerical constant c_0 by dimensional analysis. The constant can be fixed from the $T^2 g$ and T^3 three point amplitude found in the Veneziano amplitude computation, so that $c_0 \propto g_0^2$. At $A = 1$ the residue at the pole is parametrized by

$$\mathrm{Res}_{s_{12}\to 0} \mathcal{A}(1,2,3,4_g) = c_1 \, \xi_4 \cdot k_1 + c_1'(s_{23} + c_1'') \, \xi_4 \cdot k_3, \tag{5.39}$$

with numerical constants c_1 and c_1'. Here the fact that the maximal spin of the exchanged particle is 1 at this level was used. This either gives a contraction of the polarization vector into a momentum 'at the other side of the pole', i.e. the $\xi_4 \cdot k_1$ term, or an additional power of momentum.

Tuning to $s_{23} = -1, s_{13} = -2$ gives by equation (5.34) the pole in the $(2,3)$ channel of the right-hand side amplitude which leads to

$$c_1 \xi_4 \cdot k_1 + c_1'(-1 + c_1'')\xi_4 \cdot k_3 = -c_0\, \xi_4 \cdot k_1 \tag{5.40}$$

so that immediately $c_1'' = 1$ follows. Tuning $s_{13} = -1, s_{23} = -2$ gives similarly

$$c_1 \xi_4 \cdot k_1 + c_1'(-2 + 1)\xi_4 \cdot k_3 = c_0\, \xi_4 \cdot k_2. \tag{5.41}$$

Hence there are two equations in two unknowns which can be solved

$$c_1' = -c_1 = c_0, \tag{5.42}$$

so that (5.39) becomes

$$\mathrm{Res}_{s_{12}\to 0}\mathcal{A}(1,2,3,4_g) = -c_0\left(\xi_4 \cdot k_1 - (s_{23}+1)\, \xi_4 \cdot k_3\right). \tag{5.43}$$

Note that this computation has in effect fixed the numerical coefficient of the tachyon-gluon-gluon coupling in terms of the tachyon-tachyon-tachyon coupling. Generalizing to higher values of A is straightforward since the ansatz in equation (5.39) captures all possible polarization structures. At a generic level then the roots appearing in (5.37) can be included as multiplicative factors. To fix the coefficients at level A, one tunes to the two data-points $s_{23} = -1, s_{13} = -1 - A$ as well as $s_{13} = -1, s_{23} = -1 - A$. The result is

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1,2,3,4_g) = c_A\,(-1)^A\left(-\,\xi_4 \cdot k_1 + \frac{1}{A}(s_{23}+1)\,\xi_4 \cdot k_3\right)\binom{k_{23}}{A-1}, \tag{5.44}$$

where c_A is a constant that can be different for each A. This completes the calculation of all residues in the $(1,2)$ channel.

The complete four point function through on-shell recursion

At this stage on-shell recursion can be used to obtain the complete four point amplitude from its residues. As above, a shift in the (2,3) channel will be implemented. Following the same steps this yields

$$\begin{aligned}\mathcal{A}(1,2,3,4_g) &= c_A A\left(\xi_4 \cdot k_1 \sum_{A=1}^{\infty} \frac{(-1)^{A-1}}{s_{12}-A+1}\binom{k_{23}}{A-1} + \xi_4 \cdot k_3 \sum_{A=0}^{\infty} \frac{(-1)^{A}}{s_{12}-A+1}\binom{k_{23}+1}{A}\right)\\ &= (g_o')^2\left(\xi_4 \cdot k_1 \frac{\Gamma[-s_{12}]\Gamma[-s_{23}-1]}{\Gamma[-s_{12}-s_{23}-1]} + \xi_4 \cdot k_3 \frac{\Gamma[-s_{12}-1]\Gamma[-s_{23}]}{\Gamma[-s_{12}-s_{23}-1]}\right).\end{aligned} \tag{5.45}$$

As a cross-check it can be verified straightforwardly that this color-ordered amplitude is invariant under interchange of particles $1 \leftrightarrow 3$ as it must be since

$$\mathcal{A}(1234) = \mathcal{A}(4321) = \mathcal{A}(3214). \tag{5.46}$$

Since the particles 1 and 3 are tachyons, this amounts simply to an exchange of their momenta. In particular $s_{12} \leftrightarrow s_{23}$. The string coupling constant squared $(g')_o^2$ can be traced to a tachyon factorization channel where two amplitudes appear which was already computed above: tachyon-tachyon-gluon and $(\text{tachyon})^3$.

5.2.3 Four gluons in the open superstring

Since the monodromy relations hold for all string amplitudes, they are relations between superamplitudes which contain all amplitudes that are related by supersymmetry as components. It is useful for computational purposes to use an on-shell superspace formalism. Here the formalism of [69] will be used for massless fields which necessarily involves complex chiral spinors. The minimal on-shell superspace in 10 dimensions constructed through this method therefore has $(2, 0)$ supersymmetry. For open strings one has to restrict all momenta to a $D = 8$ subspace to be able to employ unrestricted massless on-shell superfields. Note that this is only a (kinematic) restriction above 9 points. It will mostly be important below that the massless superfields used here are scalar.

The superamplitudes are given by a kinematic function $\tilde{\mathcal{A}}$ times a momentum conserving delta function $\delta^8(K)$ which depends on the kinematic variables K and a fermionic supermomentum conserving delta function $\delta^8(Q)$ which assures that the Ward identities of on-shell supersymmetry are satisfied

$$\mathcal{A}^{D=8} = \delta^8(K)\delta^8(Q)\tilde{\mathcal{A}}(Q, K). \tag{5.47}$$

For four points, the function $\tilde{\mathcal{A}}(Q, K)$ has no fermionic weight,

$$\tilde{\mathcal{A}}(Q, K) = \tilde{\mathcal{A}}(K), \qquad \text{four points.} \tag{5.48}$$

As a function of the momenta $\tilde{\mathcal{A}}(K)$ has roots and poles. The sums over parts of the states at the residues of the poles can be performed using a fermionic integral. As here the interest is in the result of this integral, it actually mostly does not have to be considered. See [56] for an explanation of the massive spinor helicity formalism in higher dimensions. The only thing important for the discussion here is that this makes the computation manifestly on-shell supersymmetric. In field theory, the four point function reads:

$$\mathcal{A}^{D=8,\mathrm{YM}} = \delta^8(K)\delta^8(Q)\frac{g_{\mathrm{ym}}}{k_{12}k_{23}}, \qquad \text{field theory.} \tag{5.49}$$

As the delta functions are completely symmetric the functions $\tilde{\mathcal{A}}(K)$ satisfy the same monodromy relations as before. Hence the roots can be derived analogously, with the poles starting at 0 instead of -1[3].

$$\tilde{\mathcal{A}}(1, 2, 3, 4) = \frac{\sin(\pi k_{13})}{\sin(\pi k_{12})}\tilde{\mathcal{A}}(1, 3, 2, 4), \tag{5.50}$$

leads to $\tilde{\mathcal{A}}(1, 2, 3, 4) = 0$ at $k_{12} = -A$ for

$$\begin{gathered} k_{23} \in \mathbb{Z}, \\ 0 < k_{13} \quad \wedge \quad 0 < k_{23} \qquad \Leftrightarrow \qquad 0 < k_{23} < A. \end{gathered} \tag{5.51}$$

This gives us the following $A - 1$ roots for $A \geq 1$

$$\mathrm{Res}_{k_{12}\to -A}\tilde{\mathcal{A}} \propto \binom{k_{23} - 1}{A - 1}. \tag{5.52}$$

[3]This is actually not an essential assumption. There is a more complicated version of this derivation which takes an arbitrary starting point for the series of poles, basically introducing an 'intercept'. Then, as illustrated by the discussion in Section 4.3.1, unitarity restricts the starting point to be 0.

The maximum power of k_{23} can be determined from a BCFW supershift in the $(1,2)$ channel. Compared to the residue of the tree level Yang-Mills amplitude at the s_{12}-channel pole, $(\frac{g_{\rm ym}}{k_{23}})$, this power is A.

This can also be argued on the basis of the known spectrum. The spectrum for the open superstring in 10 dimensions was worked out in [49]. Structurally, the highest spin field in the spectrum at mass level A transforms as the symmetric traceless $A+1$-tensor of the massive little group $SO(9)$. In the massive superfield formalism this translates into a $A-1$-tensor massive on-shell superfield. The fermionic integral in this case contributes an overall constant [56]. This shows that the obtained polynomials at the residues contain the complete dependence on kinematic invariants and that the overall numerical constants are all that is left to be determined.

These overall constants can, as before, be fixed by unitarity in the cross-channel. That is, first take the residue of (5.50),

$$\mathrm{Res}_{s_{12}\to A}\tilde{\mathcal{A}}(1234) = \frac{(-1)^{A-1}}{\pi}\left[\sin(\pi s_{13})\tilde{\mathcal{A}}(1324)\right]_{s_{12}=A}. \tag{5.53}$$

Then one inserts the ansatz for the left-hand side,

$$c\binom{k_{23}-1}{A-1} = \frac{(-1)^{A-1}}{\pi}\left[\sin(\pi s_{13})\tilde{\mathcal{A}}(1324)\right]_{s_{12}=A}. \tag{5.54}$$

and tunes $s_{13} = -k_{13} \to 0$ to obtain

$$c = \frac{(-1)^A}{A}, \tag{5.55}$$

where instead of writing the unitarity expression for the s_{13} pole on the right-hand side the known expression of equation (5.49) was used. Note this last step fixes the residues of the four point superstring amplitude in terms of the field theory limit.

Assembling the full amplitude through on-shell recursion now follows by repeating basically the same computation as in the Veneziano amplitude case and simply yields

$$\mathcal{A}^{D=8} = \delta^8(K)\delta^8(Q)\frac{\Gamma[-s_{12}]\Gamma[-s_{23}]}{\Gamma[-s_{12}-s_{23}+1]}. \tag{5.56}$$

5.2.4 Four tachyons in the closed bosonic string

Closed string amplitudes are defined by the KLT relations. For four points these can be written as

$$M(1234) = \sin(\pi k_{23})\mathcal{A}(1234)\mathcal{A}(1324), \tag{5.57}$$

with all coupling constants stripped off. In this subsection the direct application of a similar reasoning as above to determine the residues at poles is briefly explored for closed strings.

The closed string amplitude has poles in all channels and is completely symmetric. Consider without loss of generality the residue at the s_{12} channel pole,

$$\mathrm{Res}_{s_{12}\to A-1}M(1234) = \sin(\pi k_{23})\left(\mathcal{A}(1324)\right)_{s_{12}\to A-1}\left(\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1234)\right). \tag{5.58}$$

Now by the following analog of equation (5.24),

$$\left[\mathcal{A}(1324)\right]_{s_{12}=A-1} = (-1)^{A-1}\frac{\pi}{\sin(\pi k_{13})}\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(1234) \tag{5.59}$$

the residues of the closed string amplitudes simply reduce to a double copy of the residues of the open string amplitude by momentum conservation at the residue,

$$\mathrm{Res}_{s_{12}\to A-1} M(1234) = -\pi \left(\mathrm{Res}_{s_{12}\to A-1} \mathcal{A}(1234)\right)^2 . \tag{5.60}$$

By the holomorphic factorization property of the closed string worldsheet vertex operators this is expected.

The residues of the open string amplitudes were determined above. This fixes the residue at the pole of the closed string amplitudes. The overall numerical factor is now the product of the two open string coupling constants squared. This can now be defined as the closed string coupling constant. It should be clear a similar reasoning will go through for tree level closed string amplitudes with arbitrary field content.

Of course, one can also use monodromy relations to write the KLT relation here as

$$M(1234) = \frac{\sin(\pi k_{12})\sin(\pi k_{23})}{\sin(\pi k_{13})} \mathcal{A}(1234)\mathcal{A}(1234). \tag{5.61}$$

Now all poles in the s_{13} channel are explicitly factored into the sin denominator. This generalizes to multiple points: there is always an expression of the closed string amplitudes in terms of a $(N-3)!$ basis of open string amplitudes with three particles fixed in consecutive positions. If these particles are labelled $1, 2, 3$, then all the poles of the closed string amplitude which involve momentum k_2 and multiple momenta not equal to k_1 or k_3 will be explicit in the denominator. This simply follows since the open string amplitudes in the chosen basis do not have poles in these channels.

Further and more direct exploration of the closed string sector is left to future work, save for one comment. By Bose symmetry, the complete closed string tachyon amplitude must be completely symmetric. Note that in equation (5.61) there are roots of the closed string amplitude manifest in the s_{12} and s_{23} channel while those in the s_{13} channel are contained in the open string amplitude squared, moderated by corresponding poles from the sine function in the denominator. In the first way of writing in equation (5.57) only one series of roots is manifest.

5.2.5 Other four-point amplitudes

The main technical complication in extending the argument given above to four point amplitudes with other external states is the appearance of more and more polarization tensors. These may be treated by parametrizing the residues in terms of all possible tensor structures built out of metrics and external momenta on the three point amplitudes which appear at the residue. Since these tensor structures are independent, their coefficient polynomials can be fixed as in the example above from the roots at least to some extent. If the monodromy relations are strong enough[4], this leaves fixing the overall constants at each mass level. We strongly suspect that one needs all three point amplitudes up to the level of the highest level external particle involved in the scattering to fix all coefficients: this ensures all possible tensor structures appear on the residue.

In the superstring case the same complications start to appear in the massive sector as long as one considers superfields. Massless vector fields are components of scalar on-shell superfields, which are treated analogously to tachyons in the bosonic string, at least in the 8 dimensional formalism.

[4]This will be shown below for Koba-Nielsen amplitudes.

From the structure of the argument it should be clear that in the four point case one always ends up with sums over β function type functions times possibly complicated coefficients. This is of course well known from the worldsheet formalism.

5.2.6 Five tachyons in the open bosonic string

At five points the monodromy relations can be solved to give

$$\mathcal{A}(12345) = \frac{1}{\sin(\pi s_{12})} \left[\sin(\pi(-s_{12} + k_{23}))\mathcal{A}(13245) + \sin(\pi(-s_{12} + k_{23} + k_{24}))\mathcal{A}(13425)\right], \tag{5.62}$$

so the residues of the amplitude in the s_{12} channel are

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(12345) = \frac{1}{\pi} \left[\sin(\pi k_{23})\mathcal{A}(13245) + \sin(\pi(k_{23} + k_{24}))\mathcal{A}(13425)\right]_{s_{12}=A-1}. \tag{5.63}$$

This has roots for

$$k_{23}, k_{24} \in \mathbb{Z}, \tag{5.64}$$

but only if the two amplitudes on the right-hand side do not have a pole at these values, which leads to the conditions

$$\begin{aligned} &k_{23} \geq 0\ , \\ &k_{24} \geq 0\ , \\ &k_{25} \geq 0 \quad \Leftrightarrow \quad k_{23} + k_{24} \leq A - 1\ . \end{aligned} \tag{5.65}$$

The condition for k_{25} is required because k_{25} becomes an integer due to momentum conservation when k_{12}, k_{23}, k_{24} are integers.

The conditions are solved by the polynomials

$$\binom{k_{23}}{A-a}\binom{k_{24}}{a}, \qquad 0 \leq a \leq A. \tag{5.66}$$

Each of this terms contains A powers of k_2, the maximally allowed number. So multiplying them by further polynomials containing k_{23} or k_{24} is not allowed.

The polynomials just written down are a basis of the space of polynomials of total order $\leq A$ which vanish under the conditions (5.64) and (5.65). Since the main theorem of algebra does not hold for functions of more than one variable proving this requires some work. For this, note that

$$\binom{k_{23}}{B-a}\binom{k_{24}}{a}, \qquad 0 \leq a \leq B, \qquad B \leq A, \tag{5.67}$$

is a basis for all polynomials of maximal total degree A labelled by indices B and a. This follows as they are linear combinations of the natural basis monomials $(k_{23})^i(k_{24})^j$ for $i+j \leq A$. The most generic polynomial of maximal total degree A is therefore a linear combination of this basis. Now consider the set of roots in equation (5.65). By first setting k_{23} and k_{24} to zero it is easy to see there can be no constant term. Then, considering the two points $(k_{23}, k_{24}) = (0, 1)$ and $(1, 0)$ one can rule out all linear polynomials. Continuing along these lines one sees that none of the polynomials in (5.67) with $B < A$ has the required roots s.t. equation (5.66) is the basis of all polynomials which satisfy the conditions of equation (5.65).

For the channel $s_{123} = B - 1$ the monodromy relation (5.11) can be used,

$$\mathcal{A}(12345) = \frac{1}{\sin(\pi s_{123})}\left[\sin(\pi(-s_{123} + k_{34}))\mathcal{A}(12435) + \sin(\pi(-s_{123} + k_{34} + k_{24}))\mathcal{A}(14235)\right], \tag{5.68}$$

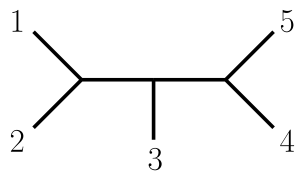
which implies the following conditions for a vanishing residue $\mathrm{Res}_{s_{123}\to B-1}\mathcal{A}(12345)$

$$\begin{aligned} k_{24}, k_{34} &\in \mathbb{Z}\ , \\ k_{24} &\geq 0\ , \\ k_{34} &\geq 0\ , \\ k_{14} &\geq 0 \quad \Leftrightarrow \quad k_{24} + k_{34} \leq B - 1\ . \end{aligned} \tag{5.69}$$

The polynomials solving them are

$$\binom{k_{24}}{a}\binom{k_{34}}{B-a}, \qquad 0 \leq a \leq B. \tag{5.70}$$

If both internal particles are send on-shell, that is the channel



, (5.71)

is considered, the residues have to vanish when either conditions (5.64, 5.65) or (5.69) are satisfied. At the same time, k_2 is only allowed to appear to the Ath power and k_4 to the Bth power. These conditions follow from considering BCFW shifts of the residue in the $(1,2)$ channel as well as the $(4,5)$ channel. Just as in the four-point case one finds

$$\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}(12345) \sim z^A\left(f_1\left(\frac{1}{z}\right)\right), \tag{5.72}$$

$$\mathrm{Res}_{s_{45}\to B-1}\mathcal{A}(12345) \sim z^B\left(f_1\left(\frac{1}{z}\right)\right). \tag{5.73}$$

The only polynomials fulfilling all roots as well as the power counting constraints just derived are

$$\binom{k_{23}}{A-a}\binom{k_{24}}{a}\binom{k_{34}}{B-a}, \qquad 0 \leq a \leq \min(A,B). \tag{5.74}$$

Fixing the coefficients

The coefficient for each of these polynomials can be fixed by using the monodromy relations again or, alternatively, by assuming cyclicity of the amplitude which is shown in Appendix A.6. As a warm-up for the the N-point case discussed below, the exact linear combination of polynomials (5.74) that is the double residue of $\mathcal{A}(12345)$ will be determined. Just as in the four point case, the overall factors will follow by considering the right-hand side of equation (5.62) at an integer-valued kinematic point where it does *not* vanish. It is convenient to take this point to be

$$k_{23} + k_{24} = A, \tag{5.75}$$

with k_{23} and k_{24} non-negative integers. The polynomials (5.74) are special at this point. To see this, assume w.l.o.g. that $A \leq B$ and consider the expression

$$\binom{A-k_{24}}{A-a}\binom{k_{24}}{a}\binom{k_{34}}{B-a}, \qquad 0 \leq a \leq A. \tag{5.76}$$

Now, the second binomial coefficient vanishes at these integer values when $k_{24} < a$, while the first vanishes when $A - k_{24} < A - a$. Hence at this particular kinematic point the only one of these polynomials that is non-zero is the one with $a = k_{24}$. By choosing different integers for k_{24} the coefficients of all the polynomials can now be calculated. This can be done by calculating the right-hand side of equation (5.62) at this particular kinematic point. The term containing $\mathcal{A}(13245)$ vanishes while the amplitude $A(13425)$ develops a tachyonic pole in the (25) channel which cancels against the root from the sine function that multiplies it

$$\begin{aligned}\lim_{k_{23}+k_{24}\to A}\lim_{k_{24}\to a}\left[\frac{1}{\pi}\sin(\pi(k_{23}+k_{24}))\mathcal{A}(1,3,4,2,5)\right]_{s_{12}=A-1} &= \\ -(-1)^A\left[\mathcal{A}(1,3,4,P)\,\mathcal{A}(-P,2,5)\right]_{\left\{\begin{array}{c} s_{12}=A-1 \\ k_{23}+k_{24}=A \\ k_{24}=a\end{array}\right\}}&.\end{aligned} \tag{5.77}$$

Note the amplitudes in this equation all involve tachyons only and the open string coupling constant has been suppressed. The four point amplitude is easy to evaluate on a further special kinematic point. Now use monodromy relation (5.10) again to expose the residue in $s_{13} = s_{123} - a$

$$\mathrm{Res}_{s_{123}\to B-1}\mathcal{A}(1,3,4,P) = \frac{1}{\pi}\left[\sin(\pi k_{34})\mathcal{A}(1,4,3,P)\right]_{s_{123}=B-1}. \tag{5.78}$$

Setting $k_{34} = B - a \in \mathbb{Z}$ will hit a root of the sine and a tachyon pole in the amplitude $\mathcal{A}(1,4,3,P)$ because at this value of k_{34} the equation $s_{3P} = s_{235} = -1$ holds. This lead to

$$\begin{aligned}&\lim_{k_{34}\to B-a}\left[\frac{1}{\pi}\sin(\pi k_{34})\mathcal{A}(1,4,3,P)\right]_{\left\{\begin{array}{c} s_{12}=A-1 \\ s_{123}=B-1 \\ k_{23}+k_{24}=A \\ k_{24}=a\end{array}\right\}} \\ &= -(-1)^{B-a}\left[\mathcal{A}(1,4,Q)\,\mathcal{A}(-Q,3,P)\right]_{\left\{\begin{array}{c} s_{12}=A-1 \\ s_{123}=B-1 \\ k_{23}+k_{24}=A \\ k_{24}=a \\ k_{34}+k_{24}=B\end{array}\right\}}.\end{aligned} \tag{5.79}$$

Plugging everything back into (5.62) the final result reads

$$\mathrm{Res}_{s_{12}\to A-1}\mathrm{Res}_{s_{123}\to B-1}\mathcal{A}(12345) = \sum_{a=0}^{\infty}\binom{k_{23}}{A-a}\binom{k_{24}}{a}\binom{k_{34}}{B-a}(-1)^{A+B-a}. \tag{5.80}$$

The complete five point function through on-shell recursion

In this example it will now be shown explicitly how the double residues can be combined with BCFW on-shell recursion to obtain the full amplitude. First perform a BCFW shift on particles 1 and 5 by a vector q_{15} scaled by a complex parameter z_{15}

$$\hat{k}_1 = k_1 + z_{15}q_{15}, \qquad \hat{k}_5 = k_5 - z_{15}q_{15}, \tag{5.81}$$

where

$$k_1 \cdot q_{15} = k_5 \cdot q_{15} = q_{15}^2 = 0. \tag{5.82}$$

Using BCFW on-shell recursion,

$$\begin{aligned}\mathcal{A}(12345) = &-\sum_{A=0}^{\infty}\sum_{\alpha}\sum_{\text{polarizations}} \frac{\mathcal{A}(\hat{1},2,\hat{M}^{A,\alpha})\mathcal{A}(\hat{M}^{A,\alpha},3,4,\hat{5})}{s_{12}-A+1} \\ &-\sum_{B=0}^{\infty}\sum_{\beta}\sum_{\text{polarizations}} \frac{\mathcal{A}(\hat{1},2,3,\hat{M}^{B,\beta})\mathcal{A}(\hat{M}^{B,\beta},4,\hat{5})}{s_{123}-B+1},\end{aligned} \tag{5.83}$$

is obtained. For details about the sums over irreps α, β and polarizations of the intermediate particles see Section 4.3. Now implement another shift for each of the four point amplitudes, namely for the first term

$$\tilde{k}_3 = k_3 + z_{34}q_{34}, \qquad \tilde{k}_4 = k_4 - z_{34}q_{34}, \tag{5.84}$$

and for the second term

$$\bar{k}_2 = k_2 + z_{23}q_{23}, \qquad \bar{k}_3 = k_3 - z_{23}q_{23}. \tag{5.85}$$

Using this

$$\begin{aligned}\mathcal{A}(12345) = &\sum_{A,B=0}^{\infty}\sum_{\alpha,\beta}\sum_{\text{polarizations}} \frac{\mathcal{A}(\hat{1},2,\hat{M}^{A,\alpha})\mathcal{A}(\hat{M}^{A,\alpha},\tilde{3},\tilde{M}^{B,\beta})\mathcal{A}(\tilde{M}^{B,\beta},\tilde{4},\hat{5})}{(s_{12}-A+1)(s_{\hat{1}2\tilde{3}}-B+1)} \\ &+\sum_{A,B=0}^{\infty}\sum_{\alpha,\beta}\sum_{\text{polarizations}} \frac{\mathcal{A}(\hat{1},\bar{2},\bar{M}^{A,\alpha})\mathcal{A}(\bar{M}^{A,\alpha},\bar{3},\hat{M}^{B,\beta})\mathcal{A}(\hat{M}^{B,\beta},4,\hat{5})}{(s_{\hat{1}\bar{2}}-A+1)(s_{123}-B+1)},\end{aligned} \tag{5.86}$$

is obtained. In each term the BCFW shifts are tuned in such a way that in the first line

$$s_{\hat{1}2} = A-1, \qquad s_{\hat{1}2\tilde{3}} = B-1, \tag{5.87}$$

and in the second line

$$s_{\hat{1}\bar{2}} = A-1, \qquad s_{\hat{1}\bar{2}\bar{3}} = B-1. \tag{5.88}$$

The first and second line in (5.86) are very similar up to a difference in the BCFW shifts and the rather subtle difference in denominators. Practically this means that taking first a $s_{12} \to A'-1$ and then a $s_{123} \to B'-1$ limit of the full result selects the first term, while doing this in the opposite order selects the second. This follows as the second expression for instance generically does not have a pole at $s_{12} = a$ for *any* integer a unless $s_{123} = B-1$ holds.

The residues appearing in both terms were derived from the monodromy relations above in equation (5.80). These can be plugged in

$$\begin{aligned}\mathcal{A}(12345) = &\sum_{A,B=0}^{\infty}\sum_{a=0}^{\infty}\binom{k_{2\tilde{3}}}{A-a}\binom{k_{2\tilde{4}}}{a}\binom{k_{34}}{B-a}\frac{(-1)^{A+B-a}}{(s_{12}-A+1)(s_{\hat{1}23}-B+1)} \\ &+\sum_{A,B=0}^{\infty}\sum_{a=0}^{\infty}\binom{k_{23}}{A-a}\binom{k_{\bar{2}4}}{a}\binom{k_{\bar{3}4}}{B-a}\frac{(-1)^{A+B-a}}{(s_{\hat{1}2}-A+1)(s_{123}-B+1)}.\end{aligned} \tag{5.89}$$

Note that the secondary BCFW shifts can be chosen[5] such that $q_{34} \cdot k_2 = q_{23} \cdot k_4 = 0$. In this case the dependence on these shifts trivially drops out of the numerator. This is significant as a form of internal recursion relations for open string tachyon amplitudes were already proposed more than 40 years ago by Hopkinson and Plahte [70]. Here the full amplitude is just the maximal residue summed over the mass levels. These results seem to suggest much simpler formulae are possible. We leave this for future work.

5.2.7 N tachyons in the open bosonic string

Equation (5.12) can be used to derive the residue of the N tachyon amplitude in $s_{1\ldots l}$. This amplitude will be referred to as the Koba-Nielsen amplitude. First note that none of the amplitudes on the right-hand side has a pole in $s_{1\ldots l}$, where $2 \leq l \leq N-2$. Furthermore, all sines vanish at a pole at $s_{1\ldots l} = A_l - 1$ under the condition

$$k_{ij} \in \mathbb{Z} \qquad \forall\, i \in \{2, \ldots, l\}, j \in \{l+1, \ldots, N-1\}. \tag{5.90}$$

as in the previous examples, these momenta must be in the range where the amplitudes on the right-hand side do not have poles

$$k_{ij} \geq 0 \qquad \forall\, i \in \{2, \ldots, l\}, j \in \{l+1, \ldots, N-1\}. \tag{5.91}$$

There is one further pole in one of the amplitudes that has to be taken into account, namely the one in $s_{2\ldots l,N}$, because this Mandelstam variable becomes integer at the considered configuration

$$s_{2\ldots l,N} = \sum_{\substack{1<i\leq l\\ l<j<N}} k_{ij} - s_{1\ldots l} + \alpha'(m_1^2 + m_N^2). \tag{5.92}$$

Avoiding the pole leads to the condition

$$s_{2\ldots l,N} \leq -2 \quad \Leftrightarrow \quad \sum_{\substack{1<i\leq l\\ l<j<N}} k_{ij} \leq A_l - 1. \tag{5.93}$$

The combined conditions are naturally solved by the polynomials

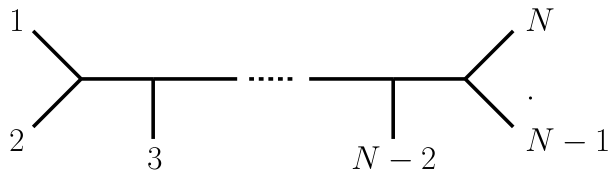
$$\prod_{\substack{1<i\leq l\\ l<j<N}} \binom{k_{ij}}{a_{ij}}, \qquad \text{where } a_{ij} \in \mathbb{N}_0 \wedge \sum_{\substack{1<i\leq l\\ l<j<N}} a_{ij} = A_l. \tag{5.94}$$

To obtain the multiple residue where all the internal particles in the multiperipheral channel are on-shell $s_{1\ldots l} = A_l - 1\ \forall l \in \{2, \ldots, N-2\}$ take the polynomials that solve the above conditions for all those l

$$\prod_{\substack{i,j\\ 1<i<j<N}} \binom{k_{ij}}{a_{ij}}, \qquad \text{where } a_{ij} \in \mathbb{N}_0 \wedge \sum_{\substack{1<i\leq l\\ l<j<N}} a_{ij} = A_l \quad \forall l. \tag{5.95}$$

[5] Choosing BCFW shift vectors like this should always be done with care, the obtained poles must always be at finite values of the shift parameters. For these particular shifts this is the case.

The multiperipheral channel is visualized by the diagram



(5.96)

Although it is possible to consider other channels, for our purposes the multiperipheral channel is enough: this channel has enough information to determine the full amplitude through on-shell recursion.

Completeness of the basis

The bases of polynomials (5.94) and (5.95) are complete. This follows as the spin is limited by the mass level and the spin determines how many indices can be contracted across an internal line that is put on-shell (this was discussed in much more detail in Chapter 4).

For amplitudes in the multiperipheral channel as discussed above the limit on spin by level implies that the residue at $s_{1\ldots l} = A_l - 1$ is proportional to a polynomial of degree A_l in Lorentz invariants which involve a contraction across the pole under consideration. For tachyon amplitudes this statement can be written as

$$\mathrm{Res}_{s_{1\ldots l}\to A_l-1}\mathcal{A}_N \propto \mathrm{Pol}[k_{ij}|1 \le i \le l, l < j \le N] \text{ of degree } A_l. \tag{5.97}$$

For other external particles the same statement holds true, but now the Lorentz invariants can also be constructed from polarizations as for example in (5.44). The polynomials in (5.94) saturate condition (5.97) for one choice of residue l while the polynomials in (5.95) saturate the condition for each l individually. Hence they constitute bases of polynomials fulfilling the requirements for roots and degree.

The same result can also be derived by utilizing a modified BCFW-type shift. Note that two-particle shifts were enough to fix the polynomials up to five external particles. This is because any pole in the multiperipheral channel for a 5 particle amplitude splits the external particles into at least one set with two particles. Above 5 points however one also has a multiperipheral pole which splits the external lines into two sets, both of which contain more than two particles. Consider such a pole with particles 1 through l in the left-hand side set. Consider the shift

$$k_1 \to k_1 - (l-1)qz, \quad k_2 \to k_2 + qz, \quad \ldots, \quad k_l \to k_l + qz, \tag{5.98}$$

for a non-trivial vector q for which $q^2 = q\cdot k_1 = \ldots = q \cdot k_l = 0$, but for which also $q \cdot k_{l+1} \neq 0$. This shift always exists for up to 27 particle kinematics in the bosonic string, above it requires an analytic continuation in the dimension[6]. The large z behavior of a string scattering amplitude can be argued for using a saddle-point-type argument just as in [20] and [21] which leads to

$$\lim_{z\to\infty} \mathrm{Res}_{s_{1\ldots l}\to A_l-1}\mathcal{A}_N \sim z^{A_l}\left(f_1\left(\frac{1}{z}\right)\right), \tag{5.99}$$

under this shift which is equivalent to the statement above.

[6] This argument will only be used to estimate the maximal degree of a polynomial, so this continuation will not have drastic consequences at string tree level. Moreover, in the analysis of Chapter 4 it became manifest that the target space dimension only affects unitarity in sub-leading coefficients.

Fixing the coefficients

The residues can contain only polynomials from the basis (5.95) which are labeled by the mass levels $\{A_l\}$ and further parameters $\{a_{ij}\}$. All that is left to do is to fix the coefficient $h_{\{A_l\},\{a_{ij}\}}$ for each basis element. For this, start with the ansatz

$$\left(\prod_{l=2}^{N-2} \mathrm{Res}_{s_{1\dots l}\to A_l-1}\right)\mathcal{A}_N = \sum_{a_{23},\dots,a_{N-2,N-1}=0}^{\infty} h_{\{A_l\},\{a_{ij}\}} \prod_{\substack{i,j\\1<i<j<N}} \binom{k_{ij}}{a_{ij}} \prod_{l=2}^{N-2} \delta_{A_l,\sum\limits_{\substack{1<u\le l\\ l<v<N}} a_{uv}}. \tag{5.100}$$

To fix the coefficients consider certain kinematic limits where the ansatz reduces to a single coefficient $h_{\{A_l\},\{a_{ij}\}}$. These limits are reached when the Mandelstams $s_{2\dots l,N}$ that were considered in (5.93) are set to -1 which implies $\sum\limits_{\substack{1<i\le l\\ l<j<N}} k_{ij} = A_l$. With this constraint the ansatz becomes

$$\left(\prod_{l=2}^{N-2} \mathrm{Res}_{s_{1\dots l}\to A_l-1}\right)\mathcal{A}_N = \sum_{a_{24},\dots,a_{N-3,N-1}=0}^{\infty} h_{\{A_l\},\{a_{ij}\}} \prod_{\substack{i,j\\1<i<j-1<N-1}} \binom{k_{ij}}{a_{ij}} \prod_{l=2}^{N-2} \begin{pmatrix} A_l - \sum\limits_{\substack{1<u\le l\\ l<v<N\\ v-u\ge 2}} k_{uv} \\ A_l - \sum\limits_{\substack{1<u\le l\\ l<v<N\\ v-u\ge 2}} a_{uv} \end{pmatrix}. \tag{5.101}$$

The next step is to set all the remaining k_{ij} to non-negative integer values. For every term in the sum the first product of binomial coefficients vanishes if $a_{ij} > k_{ij}$ for any values of i, j. A binomial coefficients in the second product vanishes if $\sum a_{uv} < \sum k_{uv}$ for some summation range as given in (5.101). Every relevant pair u, v appears in the sum of such a condition at least once. Together these two observations imply that the only term that does not vanish is the one for $a_{ij} = k_{ij}\ \forall i, j$. The coefficient $h_{\{A_l\},\{a_{ij}\}}$ is extracted from the relation by tuning the $\{k_{ij}\}$ to the desired $\{a_{ij}\}$

$$\left(\prod_{l=2}^{N-2} \mathrm{Res}_{s_{1\dots l}\to A_l-1}\right)\mathcal{A}_N \Bigg|_{\substack{\{s_{2\dots l,N}=-1\}_{1<l<N-1}\\ \{k_{ij}\}_{1<i<j-1<N-1}\subset\mathbb{N}_0}} = h_{\{A_l\},\{k_{ij}\}}. \tag{5.102}$$

To determine the number on the left-hand side (which must be a number because all momentum invariants are fixed) the monodromy relations can be employed again. Only the first relation, equation (5.10), is needed. At the s_{12} residue only the s_{2N} pole is hit in the last amplitude and all other terms vanish due to the sines, so

$$\mathrm{Res}_{s_{12}\to A_2-1}\mathcal{A}_N(1,2,\dots,N) = \frac{1}{\pi}\sin\left(\pi\sum_{i=3}^{N-1} k_{2i}\right)\mathcal{A}_N(1,3,4,\dots,N-1,2,N), \tag{5.103}$$

follows. The remaining amplitude on the right-hand side factorizes in the tachyon channel since $s_{2N} = -1$. This leaves a $N-1$ tachyon amplitude where one external leg has the momentum $k_2 + k_N$. The argument of the sine function equals πA_2 and determines the sign

$$\mathrm{Res}_{s_{12}\to A_2-1}\mathcal{A}_N(1,2,\dots,N) = -(-1)^{A_2}\mathcal{A}_{N-1}(1,3,4,\dots,N-1,(2+N)). \tag{5.104}$$

The same monodromy relation can be used again to move leg 3 to the right. This time the sine in the denominator has the argument πs_{13} but this can also be related to the s_{123} channel since k_{12} and k_{23} are integer

$$\mathrm{Res}_{s_{123}\to A_3-1}\mathcal{A}_{N-1}(1,3\ldots N-1,(2+N)) = \frac{1}{\pi}\sin\left(\pi\sum_{i=4}^{N-1}k_{3i}\right)\mathcal{A}_{N-1}(1,4\ldots N-1,3,(2+N)). \tag{5.105}$$

Using $\sum_{i=4}^{N-1} k_{3i} = A_3 - \sum_{i=4}^{N-1} k_{2i}$ and the factorization in the s_{23N} tachyon channel

$$\mathrm{Res}_{s_{123}\to A_3-1}\mathcal{A}_{N-1}(1,3\ldots N-1,(2+N)) = -(-1)^{A_3-\sum_{i=4}^{N-1}k_{2i}}\mathcal{A}_{N-2}(1,4\ldots N-1,(3+2+N)) \tag{5.106}$$

is obtained. This procedure can be repeated until one arrives at the 3-tachyon amplitude which is 1. In general each step contributes a factor

$$-(-1)^{A_l-\sum_{\substack{1<i<l\\ l<j<N}}k_{ij}} \tag{5.107}$$

so that

$$h_{\{A_l\},\{a_{ij}\}} = (-1)^{N-3}(-1)^{\sum_{l=2}^{N-2}\left(A_l-\sum_{\substack{1<i<l\\ l<j<N}}a_{ij}\right)} \tag{5.108}$$

follows. This agrees with the known result from the worldsheet computation, equation (A.26). Note that this result is again basically a simple sign, an indication that the polynomial basis chosen is very natural.

5.2.8 Five gluons in the open superstring

As a further illustrative example it will be explored below how the conditions posed above can be solved in the superstring case. As the four point case was discussed above, let us focus on five particles. Although the approach used previously will also work, for variety here the explicit formula

$$\begin{aligned}\mathrm{Res}_{s_{12}\to A}\mathrm{Res}_{s_{123}\to B}\left[\mathcal{A}(12345)\right] = (-1)^{A+B}\pi^{-2}\sin(\pi k_{34})& \\ \left[\sin(\pi k_{24})\mathcal{A}(14235)+\sin(\pi(k_{23}+k_{24}))\mathcal{A}(14325)\right],&\end{aligned} \tag{5.109}$$

will be used. This is obtained by solving the monodromy relations for $\mathcal{A}(14235)$ and $\mathcal{A}(14325)$. The amplitudes on the right-hand side have poles as a function of the following variables:

$$\begin{array}{cc} A(14235): & A(14325): \\ k_{14} = B - k_{24} - k_{34}, & k_{14} = B - k_{24} - k_{34}, \\ k_{24}, & k_{34}, \\ k_{23}, & k_{23}, \\ k_{35} = B - A - k_{34}, & k_{25} = A - k_{23} - k_{24}, \\ k_{15} = k_{23} + k_{24} + k_{34}, & k_{15} = k_{23} + k_{24} + k_{34}. \end{array} \tag{5.110}$$

Isolating the roots and fixing an ansatz

Analyzing the right-hand side of equation (5.109) gives roots for instance for

$$k_{24} > 0, \qquad k_{23} > 0, \qquad k_{24}+k_{23} < A, \qquad \{k_{24}\in\mathbb{N}, k_{23}\in\mathbb{N}\}, \tag{5.111}$$

by avoiding the poles of the amplitudes on the right-hand side. These conditions are less strong compared to the bosonic string case. However, there is much more information left unused in the above equation. First, on any massless pole the superstring amplitudes factorize into massless amplitudes with less legs. Since it was already shown the four point amplitude is proportional to the field theory amplitude $\mathcal{A}^F$, the massless residues of the five point string amplitude are proportional to the massless residues of the field theory amplitude. Hence it is natural to write as an ansatz for the residue,

$$\mathrm{Res}_{s_{12}\to A, s_{123}\to B}\left[\mathcal{A}(12345)\right] = \mathrm{Res}_{s_{12}\to A, s_{123}\to B}\left[F_1\mathcal{A}^F(12345) + F_2\mathcal{A}^F(13245)\right]. \quad (5.112)$$

Although this ansatz is natural, it helps to know by the results in [71] that it will be enough. Let us furthermore introduce the notation

$$G_1 = \mathrm{Res}_{s_{12}\to A, s_{123}\to B}F_1, \qquad G_2 = \mathrm{Res}_{s_{12}\to A, s_{123}\to B}F_2. \quad (5.113)$$

The right-hand side of equation (5.109) now gives the functions G_i an important property: they have roots for

$$k_{24} \geq 0, \qquad k_{23} \geq 0, \qquad k_{24} + k_{23} < A, \qquad \{k_{24} \in \mathbb{N}, k_{23} \in \mathbb{N}\}, \quad (5.114)$$

and

$$k_{24} \geq 0, \qquad k_{34} \geq 0, \qquad k_{24} + k_{34} < B, \qquad \{k_{24} \in \mathbb{N}, k_{34} \in \mathbb{N}\}. \quad (5.115)$$

Note the appearance of the equality signs. This property can be argued as follows: first consider $k_{24} \to 0$, while $A > k_{23} + k_{24} \in \mathbb{N}$. The right-hand side of equation (5.109) gives a contribution proportional to the massless residue of the $\mathcal{A}(14235)$ amplitude in the $(4,2)$ channel. This is proportional to the residue of the corresponding field theory amplitude, $\mathcal{A}^F(14235)$ in the $(4,2)$ channel, which can be expressed in terms of $\mathcal{A}^F(12345)$ and $\mathcal{A}^F(13245)$ by the field theory BCJ relations. This picks up the residue of the $\mathcal{A}^F(13245)$ amplitude, while $\mathcal{A}^F(12345)$ does not diverge in the limit so that it does not contribute. This is to be compared to (5.112). $G_2 \to 0$ is required to extract the residue of $\mathcal{A}^F(13245)$ and $G_1 \to 0$ follows because $\mathcal{A}^F(12345)$ does not vanish by itself in this limit.

For $k_{34} \to 0$ a similar reasoning gives that with k_{24} a positive integer for which $k_{34}+k_{24} < B$ one has to demand $G_1 \to 0$ and $G_2 \to 0$. This isolates the k_{34} residue in the second amplitude on the right-hand side of equation (5.109). The remaining zero for $k_{23} = 0$ follows from the solution of the monodromy relations in terms of $\mathcal{A}(13425)$ and $\mathcal{A}(14325)$,

$$\begin{aligned}\mathrm{Res}_{s_{123}\to B}\mathrm{Res}_{s_{12}\to A}\left[\mathcal{A}(12345)\right] = (-1)^{A+B}\pi^{-2}\sin(\pi k_{23}) \\ \left[\sin(\pi k_{24})\mathcal{A}(13425) + \sin(\pi(k_{34}+k_{24}))\mathcal{A}(14325)\right],\end{aligned} \quad (5.116)$$

where the relevant variables are

$$\begin{array}{cc} A(13425): & A(14325): \\ k_{13} = A - B - k_{23}, & k_{14} = B - k_{24} - k_{34}, \\ k_{34}, & k_{34}, \\ k_{24}, & k_{23}, \\ k_{25} = A - k_{23} - k_{24}, & k_{25} = A - k_{23} - k_{24}, \\ k_{15} = k_{23} + k_{24} + k_{34}, & k_{15} = k_{23} + k_{24} + k_{34}. \end{array} \quad (5.117)$$

Just as before, there is a maximal spin at each mass level. In the above notation, this is $A+1$ and $B+1$ in the respective channels. Therefore, one would expect that for instance an

$\mathcal{A}^F(12345)$ amplitude would be multiplied by a polynomial of maximal degree $A+B$, with a similar spin-induced fine-structure of powers of k_{34}, k_{24} and k_{23} as elucidated above for the bosonic string. To be more precise, under a $(1,2)$ channel BCFW shift the residue scales as

$$\lim_{z\to\infty} \mathrm{Res}_{s_{12}\to A}\mathcal{A}(12345) \sim z^A \mathcal{A}^F(12345)(z)\left(f_1\left(\frac{1}{z}\right)\right), \tag{5.118}$$

while for a $(4,5)$ channel shift

$$\lim_{z\to\infty} \mathrm{Res}_{s_{45}\to B}\mathcal{A}(12345) \sim z^B \mathcal{A}^F(12345)(z)\left(f_1\left(\frac{1}{z}\right)\right), \tag{5.119}$$

holds. Note the analogy to (5.72) in the bosonic string case. Just as in that bosonic string case it is natural to use the following basis of polynomials,

$$f_a(k_{23},k_{24},k_{34}) = \binom{k_{23}}{A-a}\binom{k_{24}}{a}\binom{k_{34}}{B-a}, \quad 0\le a\le \min(A,B), \tag{5.120}$$

which scales as z^A under a $(1,2)$ BCFW shift and as z^B under a $(4,5)$ BCFW shift.

The analysis of the maximal spin gives for the ansatz of equation (5.112) the existence of vectors of numbers G_1^a, G_2^a and $\tilde{G}_2^a$ such that

$$G_1 = G_1^a f_a(k_{23},k_{24},k_{34}), \qquad G_2 = \left(-k_{23}G_2^a + \tilde{G}_2^a\right) f_a(k_{23},k_{24},k_{34}). \tag{5.121}$$

Here the possibility of a first order polynomial in k_{23} in the G_2 polynomial follows as the $(1,2)$ shift of the field theory amplitude $A(13245)$ is suppressed by one power of z as it is a non-adjacent BCFW shift, while the $(4,5)$ shift of this amplitude is color-adjacent. Having fixed the complete functional form of the amplitude, it remains to compute the above three vectors of numbers.

Solving consistency constraints to obtain full result

Some constraints follow by the requirement that the combination on the right-hand side of equation (5.112) have no poles, while the field theory amplitudes have residual kinematic poles at the residue. This immediately forces

$$G_1^B = 0, \qquad G_2^0 = 0, \qquad \tilde{G}_2^0 = 0, \tag{5.122}$$

by absence of poles in the $(3,4)$ and $(2,4)$ channels respectively. To see this, take for instance $k_{34}=0$ in (5.112). The pole in $\mathcal{A}^F(12345)$ is not cancelled by a factor of k_{34} in $f_B(k_{23},k_{24},k_{34})$ hence $G_1^B=0$. For these channels only one of the field theory amplitudes in the basis has a potential pole. In the $(2,3)$ channel both field theory amplitudes develop a pole, leading to the constraint

$$G_1^A - G_2^A = 0. \tag{5.123}$$

Here it was used that in this channel both field theory amplitudes in (5.112) factorize into the same lower point amplitudes $\mathcal{A}^F(23P)\mathcal{A}^F(P451)$. If $A<B$, then avoiding the pole in the k_{13} channel forces

$$\tilde{G}_2^c = k_{23}G_2^c = (A-B)\,G_2^c \qquad \text{for } A<B. \tag{5.124}$$

The $(1,5)$ channel yields further information as the vanishing of the residue of the pole in the $(1,5)$ channel implies

$$G_1 + \frac{k_{34}}{k_{24}} G_2 = 0 \qquad \text{for } k_{15} = k_{23} + k_{34} + k_{24} = 0. \tag{5.125}$$

Here the BCJ relation $k_{34}\mathcal{A}^F(P234) + (k_{34} + k_{23})\mathcal{A}^F(P324) = 0$ has been used to pull out an overall $\mathcal{A}^F(P234)\mathcal{A}^F(P51)$. Evaluating this constraint on the kinematic point $k_{23} = A - k_{24}, k_{24} = c$ and $k_{34} = -A$ with c an integer $0 < c < A$ gives

$$c\, G_1^c - A\left((c-A)\, G_2^c + \tilde{G}_2^c\right) = 0, \tag{5.126}$$

while evaluating it on the compatible kinematic point $k_{34} = B - k_{24}, k_{24} = c$ and $k_{23} = -B$ with c an integer $0 < c < B$ gives

$$c\, G_1^c + (B-c)\left(B\, G_2^c + \tilde{G}_2^c\right) = 0. \tag{5.127}$$

These equations can be solved for $\tilde{G}_2^c$ and G_1^c to give

$$G_1^c = \frac{1}{c} A\,(c-B)\, G_2^c, \qquad \tilde{G}_2^c = (A-B)\, G_2^c. \tag{5.128}$$

The cases $c = 0$ and $c = A$ or $c = B$ are special as they would hit poles of the residue in the $(1,5)$ channel. The correct approach is to first take the kinematic limits, obtaining for instance

$$\frac{\partial G_1}{\partial k_{34}} + \frac{1}{k_{24}} G_2 = 0 \qquad \text{for } k_{34} = 0,\ k_{23} + k_{24} = 0, \tag{5.129}$$

as well as

$$\frac{1}{k_{34}} G_1 + \frac{\partial G_2}{\partial k_{24}} = 0 \qquad \text{for } k_{24} = 0,\ k_{23} + k_{34} = 0, \tag{5.130}$$

by the requirement that the residues at these poles have to vanish. Here the derivatives single out the terms linear in the corresponding variable since that variable is taken to be zero in both cases. Again a BCJ relation was used to pull out an overall factor containing three 3-point amplitudes. From the first

$$\sum_{a=0}^{\min(A,B-1)} \frac{G_1^a (-1)^{B-a-1}}{B-a} \binom{k_{23}}{A-a}\binom{-k_{23}}{a} - \frac{1}{k_{23}}\left(-k_{23} G_2^B + \tilde{G}_2^B\right)\binom{k_{23}}{A-B}\binom{-k_{23}}{B} = 0, \tag{5.131}$$

while from the second

$$-\frac{G_1^0}{k_{23}}\binom{k_{23}}{A}\binom{-k_{23}}{B} + \sum_{a=1}^{\min(A,B)}\left(-k_{23} G_2^a + \tilde{G}_2^a\right)\frac{(-1)^{a-1}}{a}\binom{k_{23}}{A-a}\binom{-k_{23}}{B-a} = 0, \tag{5.132}$$

is obtained. Both equations can be solved *uniquely* for G_1^a resp. G_2^a since the polynomials these coefficients multiply differ by two powers of k_{23}. Starting from the maximal power term of degree $A + B$ one can solve for G_2^a and $\tilde{G}_2^a$. To read off the solution we use

$$\begin{aligned} \sum_{a=0}^{\min(A,B-1)} \binom{k_{23}}{A-a}\binom{-k_{23}}{a} &= \frac{B(A-B-k_{23})}{Ak_{23}}\binom{k_{23}}{A-B}\binom{-k_{23}}{B}, \\ \sum_{a=1}^{\min(A,B)} \binom{k_{23}}{A-a}\binom{-k_{23}}{B-a} &= \frac{AB}{k_{23}(A-B-k_{23})}\binom{k_{23}}{A}\binom{-k_{23}}{B}, \end{aligned} \tag{5.133}$$

and obtain the solutions

$$\tilde{G}_2^a = (A-B)\, G_2^a, \qquad G_1^a = (-1)^{B-a-1} \frac{(B-a)A}{B}\, G_2^B, \tag{5.134}$$

and

$$\tilde{G}_2^a = (A-B)\, G_2^a, \qquad G_2^a = (-1)^{a-1} \frac{a}{A\,B}\, G_1^0, \tag{5.135}$$

for the two equations. Note that together with (5.128) already more than enough constraints were found to fix the residue up to a constant and all redundant constraints that were obtained in different ways are compatible. The result for G_1 and G_2 is

$$\begin{aligned} G_1 &= \sum_{a=0}^{\infty} \frac{(B-a)}{B} (-1)^a G_1^0 f_a(k_{23}, k_{24}, k_{34}), \\ G_2 &= \sum_{a=0}^{\infty} \frac{(A-B-k_{23})a}{A\,B} (-1)^{a-1} G_1^0 f_a(k_{23}, k_{24}, k_{34}). \end{aligned} \tag{5.136}$$

Hence the ansatz for the A, B residue is fixed up to an overall constant by consistency requirements. Note that with this solution a factor of k_{13} factors out of the G_2 function. From this result it is also manifest that there is no pole in the $(1,3)$ channel within the ansatz remaining. The remaining constant G_1^0 is the string coupling constant times a numerical factor which can only depend on A and B. This is easily determined from equation (5.109).

The result just obtained corresponds indeed to the residue of the known open superstring theory five point amplitude. The simplest form in the literature can be found in [71], for which

$$\begin{aligned} F_1 &= k_{12}k_{34} \int_0^1 \int_0^1 dx dy x^{k_{45}} y^{k_{12}-1} (1-x)^{k_{34}-1} (1-y)^{k_{23}} (1-xy)^{k_{24}} \\ &= \sum_{a=0}^{\infty} \frac{(-1)^{A+B-a} A k_{34}}{(k_{12}+A)(k_{45}+B)} \binom{k_{23}}{A-a} \binom{k_{24}}{a} \binom{k_{34}-1}{B-a-1} \\ &= \sum_{a=0}^{\infty} \frac{(-1)^{A+B-a} A(B-a)}{(k_{12}+A)(k_{45}+B)} \binom{k_{23}}{A-a} \binom{k_{24}}{a} \binom{k_{34}}{B-a}, \end{aligned} \tag{5.137}$$

and

$$\begin{aligned} F_2 &= k_{13}k_{24} \int_0^1 \int_0^1 dx dy x^{k_{45}} y^{k_{12}} (1-x)^{k_{34}} (1-y)^{k_{23}} (1-xy)^{k_{24}-1} \\ &= \sum_{a=0}^{\infty} \frac{(-1)^{A+B-a-1} (k_{12}+k_{23}-k_{45}) k_{24}}{(k_{12}+A)(k_{45}+B)} \binom{k_{23}}{A-a-1} \binom{k_{24}-1}{a} \binom{k_{34}}{B-a-1} \\ &= \sum_{a=0}^{\infty} \frac{(-1)^{A+B-a} (k_{12}+k_{23}-k_{45}) a}{(k_{12}+A)(k_{45}+B)} \binom{k_{23}}{A-a} \binom{k_{24}}{a} \binom{k_{34}}{B-a}, \end{aligned} \tag{5.138}$$

hold. The residues studied above are easily read off from these equations. As these are correctly obtained, the full result follows by BCFW on-shell recursion.

Note that the derivation above almost exclusively uses physical input such as locality, unitarity and Regge-behavior. It should be stressed there is a non-trivial step in the above at the point where an ansatz for the five point string theory amplitude is written in terms of field theory amplitudes. Note that by the consistency conditions that the residue is local it is easy to see that an ansatz with a single field theory amplitude would not work. Although the ansatz is very natural, in principle it could be in the above approach that it would ultimately turn out to be insufficient. While it is known [71] that this does not occur, it would be interesting to have a target space understanding of this.

5.2.9 Higher points & other matters in the superstring

For more than five points as well as for fermionic matter the above analysis can be repeated. For higher points the results are at least easy to sketch: one expects to be able to write the residue in terms of a $(N-3)!$ element basis of field theory amplitudes, multiplied by a polynomial in the remaining kinematic variables. At a chosen pole in the multiperipheral channel the maximum powers of these variables are set by the scaling of the string theory amplitude under the relevant BCFW-like shift of equation (5.99). For more than 11 particles this will require a formal analytic continuation to higher dimensional amplitudes. The monodromy relations can then be used to obtain roots of the chosen residue. This, taken together with the cases where the residues do not vanish but instead involve lower point string amplitudes with massless matter only is fully expected to completely fix the residue. Using on-shell recursion, this then fixes the amplitude.

5.3 Target space definition of the string S-matrix at tree level in a flat background

The results obtained above can be gathered into a working definition of the S-matrix of string theory at tree level in a flat background. Although an explicit and self-contained proof that the set of conditions given below leads to a fully consistent S-matrix is still lacking, from the results obtained so far such as the Koba-Nielsen amplitude obtained above it is plausible that the produced S-matrix will be identical to the worldsheet-derived one. The definition can be summarized as follows:

The tree level S-matrix for open strings in a flat background is determined by:

- unitarity
- locality
- D dimensional (super-)Poincaré invariance
- standard tree level color-ordering (equation (4.5))
- universal monodromy relations (equation (5.1))
- under (generalized) BCFW shifts of color-adjacent particles, the amplitude behaves as it does in the corresponding string theory
- a strict ordering between the location of poles and of roots. In particular there is a unique smallest mass (super)particle.

The last three requirements are those that are special to string theory. Color-ordering forces scattering amplitudes to have poles only in adjacent channels. Locality enters by the requirement that three point amplitudes are polynomial functions of the external momenta. This translates by unitarity to 'polynomity' of residues of higher point amplitudes in forbidden channel momentum invariants. This was a crucial ingredient in extending the Veneziano amplitude to higher multiplicity [72, 73]. Note that for four particles the requirement of BCFW shift behavior is almost literally the same as Regge behavior. For two tachyons for instance, equation (5.21) holds.

The ordering requirement for roots and poles amounts to the following: there is a number x such for any momentum invariant $s_{i\ldots j}$ the roots of the amplitudes as a function of this variable are located at $s_{i\ldots j} < x$ and the poles at $s_{i\ldots j} \geq x$. This number is the mass of the smallest mass (super)particle, which will be taken to be unique.

Closed strings can simply be defined by the KLT relations. The KLT relations are closely related to the monodromy relations. This is already clear from a close reading of the KLT paper: the monodromy relations are used implicitly. A more modern and precise connection is through the 'momentum kernel' of [64]. A particularly neat geometrical observation about the relation between monodromy and KLT for the four point amplitude which can also be used to find the roots is made in [74]. A higher point generalization is unknown.

A further comment concerns Poincaré invariance, which for superstring theory should naturally be enlarged to super-Poincaré invariance. As noted above, unitarity forces in both cases the $D \leq D_{\text{crit}}$ constraint. The value of the critical dimension is dependent on the theory (26 and 10 for vanilla bosonic or superstrings respectively).

A final comment is that the possibility of using the monodromy relations as an extension of duality for foundational purposes was already conjectured for five point amplitudes in [22]. What is added here is a calculational path to make their argument precise for, in principle, any external matter content and any number of particles.

On minimality

The definition given above is certainly sufficient for the bosonic string: the output S-matrix is the same as in the worldsheet approach. As already indicated above, it is highly desirable to obtain known properties of the string theory S-matrix such as unitarity (reality of couplings) without invoking conformal symmetry. On the other side, there is a pressing question if the above definition is minimal. That is, is there perhaps a smaller set of criteria possible?

There are physical reasons to suspect such a smaller set is indeed possible. The main motivation for this is the 'folk-theorem' that there are no interacting quantum field theories with a finite number of particles with spins bigger than two. From an on-shell perspective, this 'theorem' has been discussed in a series of papers [75], [76] and [77] (see also [78]). Suggestively, the path to a consistent theory proposed in the last two references involves roots of amplitudes. It is suspected that combining the above analysis with this line of reasoning might lead to the elimination of the requirement of imposing monodromy relations. Similarly, in the close string sector there might be an argument which does away with the assumption of the KLT relations, perhaps in favor of some form of what would be called holomorphic factorization on the worldsheet. The requirement that there is a unique lowest mass (super)particle might also be unnecessary.

The above set of conditions is a working definition. As is usual in high energy physics there are a number of hidden assumptions. One of these is for instance that only standard, causal, Feynman type propagators are allowed (this feeds into the "residues at poles from unitarity" argument). It would be interesting to reach a definition up to more rigorous mathematical standards. It will be interesting to see where this differs from the much more axiomatic approach of [79]: as shown above, any physical theory which contains the Veneziano amplitude has a critical dimension by unitarity.

On extendability

The above definition is tailored to flat backgrounds. Analogs of at least some of the assumptions can be worked out however in quite generic backgrounds. Unitarity for instance should have an analog in any background. Furthermore, monodromy relations for open strings can in principle be derived in any background, see [21]. Crucial here is that it is known that vertex operators in the open string generically obey a braid relation,

$$: V_1 :: V_2 := R_{12} : V_2 :: V_1 : \ . \tag{5.139}$$

As pointed out first in [80], the R factors obey generically the Yang-Baxter equation. This simply follows from consistency of the three point scattering amplitude in string theory. Hence analogs of the monodromy relations should exist for open strings in any background by following the same steps as in [21]. Actually, for closed strings one can also repeat the step in deriving a KLT-like relation between open and closed string amplitudes in any background. This follows as the KLT paper is basically only concerned with relating the measure of the integration over the moduli space of the N-punctured sphere to that of two N-punctured discs. The braid relation in equation (5.139) can then be inserted for the proper (but very formal) form of the curved background KLT relations.

It would be extremely interesting if these short observations could be turned into a tool to study scattering amplitudes in non-trivial string backgrounds. This is however far beyond the scope of the present thesis.

Chapter 6

Discussion

As discussed in the introduction a major goal of modern theoretical physics is to find a theory of quantum gravity by theoretical means. To this end it is common practice to study highly symmetric theories which do not necessarily provide a realistic description of nature but promise to solve the specific problem at hand. Here this would be the inability of standard perturbation theory applied to the Einstein-Hilbert action of general relativity to provide a complete description of quantum gravity. The "highly symmetric theories" studied in this thesis are CFTs and string theories. While CFTs are quantum field theories with additional external symmetries, we took the standpoint (and proved to some extent) that string theories can be viewed as field theories constrained by a specific set of internal symmetries including monodromy and KLT relations. All these symmetries played a major role in this thesis. The two concrete problems studied were to find a convenient description of general CFT correlators and conformal blocks and a purely target space based approach to string theory scattering amplitudes.

Towards the first goal an efficient formalism for describing irreducible tensor representations of $SO(d)$ in terms of polynomials was developed. With this formalism and the help of representation theory, tensor structures in CFT correlators and scattering amplitudes become tangible. We gave an algorithm for counting the number of independent tensor structures in any CFT correlator (or massive scattering amplitude) of bosonic operators (or particles), allowing for a systematic construction of the tensor structures for any given correlator. A first conformal block describing the exchange of a rank 3 mixed-symmetry tensor was computed as an example.

It was shown that the same formalism can be used to implement on-shell recursion relations for string theory based on tree-level unitarity. However due to their considerable complexity it seemed favorable to avoid using these recursion relations. This was achieved by exploiting an internal symmetry that is unique to string theory amplitudes, notably that they obey monodromy relations. This allowed us to give a target space based definition for the string S-matrix on a flat background, based on a list of constraints on the amplitudes.

As addressed in the introduction a major motivation to study correlators of mixed-symmetry tensors was the conformal bootstrap for the correlator of four stress-tensors. The conformal blocks can be computed by means of the shadow formalism as integrals over two three-point functions, each involving the exchanged operator and two of the external operators. With the insights about three-point correlators from this work it is easy to outline what needs to be done to compute all the necessary conformal blocks. An overview of all the irreps appearing in the correlator of four stress-tensors is given in Table 6.1, where each listed correlator contains conformal blocks for exchange of the irreps that are in the same

correlator	new exchanged $SO(d)$ irreps
$\langle \mathcal{O}_1^{\bullet}\mathcal{O}_2^{\bullet}\mathcal{O}_3^{\bullet}\mathcal{O}_4^{\bullet}\rangle$	$\begin{ytableau} \ & \ & \ & \cdots & \ \end{ytableau}$
$\langle \mathcal{O}_1^{\bullet}\mathcal{O}_2^{\square}\mathcal{O}_3^{\bullet}\mathcal{O}_4^{\square}\rangle$	$\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ \end{ytableau}$
$\langle \mathcal{O}_1^{\square}\mathcal{O}_2^{\square}\mathcal{O}_3^{\square}\mathcal{O}_4^{\square}\rangle$	$\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ \\ \ \end{ytableau}$, $\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ & \ \end{ytableau}$
$\langle \mathcal{O}_1^{\square}\mathcal{O}_2^{\square\square}\mathcal{O}_3^{\square}\mathcal{O}_4^{\square\square}\rangle$	$\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ & \ \\ \ \end{ytableau}$, $\begin{ytableau} \ & \ & \ & \ & \cdots & \ \\ \ & \ & \ \end{ytableau}$
$\langle \mathcal{O}_1^{\square\square}\mathcal{O}_2^{\square\square}\mathcal{O}_3^{\square\square}\mathcal{O}_4^{\square\square}\rangle$	$\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ & \ \\ \ & \ \end{ytableau}$, $\begin{ytableau} \ & \ & \ & \ & \cdots & \ \\ \ & \ & \ \\ \ \end{ytableau}$, $\begin{ytableau} \ & \ & \ & \ & \ & \cdots & \ \\ \ & \ & \ & \ \end{ytableau}$

Table 6.1: Exchanged irreps in correlators of currents and stress-tensors, following the discussion of possible tensor structures for three-point functions in Section 3.1.3 and the construction of conformal blocks in Section 3.3.

or in previous rows of the table. The computation of conformal blocks corresponding to a correlator in the table and exchange of an irrep from the same line is comparatively easy, since in those cases the three-point function between external operators and the exchanged operator has only one tensor structure. It is likely that the remaining conformal blocks can be expressed in terms of derivatives of those "easy" blocks with the same exchanged irrep, since it is possible to increase the spin of external operators by derivatives. This was shown in [15] at least for the cases where the exchanged irrep is a symmetric tensor. For example, the conformal blocks for exchange of $\begin{ytableau} \ & \ & \ & \cdots & \ \end{ytableau}$ in $\langle \mathcal{O}_1^{\square\square}\mathcal{O}_2^{\square\square}\mathcal{O}_3^{\square\square}\mathcal{O}_4^{\square\square}\rangle$ are given by derivatives of the conformal blocks of $\langle \mathcal{O}_1^{\bullet}\mathcal{O}_2^{\bullet}\mathcal{O}_3^{\bullet}\mathcal{O}_4^{\bullet}\rangle$.

The only ingredient in the computation of these conformal blocks through the shadow formalism that is not given explicitly in this thesis or in the literature are the projectors to traceless mixed-symmetry tensors. An efficient way to derive recursion relations for these projectors will appear in a subsequent publication. So far, we computed the relation for the family of mixed-symmetry tensors with one box in the second row $\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ \end{ytableau}$ and also managed to turn it into a recursion relation that expresses the conformal blocks for exchange of these irreps in terms of known conformal blocks. The same method can be applied to other families of mixed-symmetry tensors. For example conformal bootstrap of four stress-tensors in four or five dimensions requires an additional recursion relation for each family in Table 6.1 with up to two rows: $\begin{ytableau} \ & \ & \ & \cdots & \ \\ \ & \ \end{ytableau}$, $\begin{ytableau} \ & \ & \ & \ & \cdots & \ \\ \ & \ & \ \end{ytableau}$ and $\begin{ytableau} \ & \ & \ & \ & \ & \cdots & \ \\ \ & \ & \ & \ \end{ytableau}$.

Another interesting direction would be the extension of the techniques of Chapter 2 to general irreducible representations of superconformal groups. Superconformal blocks for some of the simpler cases were derived by solving the superconformal Casimir eigenvalue equation [81, 82]. A first application of the shadow formalism to superconformal blocks in four dimensions appeared in [83], however the supertwistor superspace used there can only describe $\mathcal{N} = 1$ and some $\mathcal{N} > 1$ supermultiplets. It would be interesting to apply the method to superspaces that can describe interesting $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supermultiplets, for example the $\mathcal{N} = 4$ stress-tensor multiplet.

An extension of our formalism to general spinor representations of $SO(d)$ should also be possible. This would complete the construction of all tensor structures in CFT correlators and make it possible to study conformal bootstrap for operators with half-integer spin.

Regarding string theory in target space an interesting direction would be the extension to loop level. A starting point would be the Feynman tree theorem [84, 85], which was used in the first calculation of the open string one-loop amplitude, before the worldsheet formulation was known [86]. As in the tree-level case, the crucial step that is required to make this method feasible would be to find a way to avoid doing the infinite sum over the spectrum. Since closed strings appear naturally and inevitably in open string loop amplitudes and KLT relations relate open and closed string amplitudes, it would be highly interesting to study whether KLT relations play part of the role that monodromy relations have at tree level.

Finally, there might be a deeper connection between the two parts of this thesis than just the appearance of the same tensor structures in CFT correlators and string scattering amplitudes. We already mentioned above that a generalization of on-shell recursion relations to curved backgrounds seems promising. In the case of AdS backgrounds a form of BCFW recursion was studied in [87, 88] for field theories. A similar approach for string theory on AdS would require in general mixed-symmetry tensor fields, which could be described by extending the techniques of this thesis to AdS, in the spirit of [89]. It seems reasonable to expect that the contributions from the different representations in a future version of on-shell recursion on AdS will be related to conformal blocks on the boundary.

Chapter 7

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Appendix A

Appendices

A.1 Code for tensor product computations

This appendix provides an introduction to tensor product computations using the program lrcalc [90] for computing Littlewood-Richardson coefficients and SageMath [91] for a Python based interface to lrcalc. The $SO(d)$ tensor product coefficients are computed using the Newell-Littlewood formula (2.63)[1]

$$b_{efg} = \sum_{h,i,j} c_{ih}{}^{e} c_{hj}{}^{f} c_{ji}{}^{g} \,. \tag{A.1}$$

The following SageMath code uses this formula to compute at once all b_{efg} which appear in the tensor product

$$\lambda_e \otimes \lambda_f = \bigoplus_{\lambda_g} b_{efg} \lambda_g \,. \tag{A.2}$$

```
import sage.libs.lrcalc.lrcalc as lr

def b_mult(rep_e, rep_f):
    result = dict()
    for reps_hi, c_ihe in lr.coprod(rep_e).iteritems():
        rep_h, rep_i = reps_hi
        for rep_j, c_hjf in lr.skew(rep_f,rep_h).iteritems():
            for rep_g, c_jig in lr.mult(rep_i,rep_j).iteritems():
                n = result[rep_g] if (rep_g in result) else 0
                result[rep_g] = n + c_ihe*c_hjf*c_jig
        if rep_h != rep_i:   # lr.coprod() provides only one ordering
            rep_i, rep_h = reps_hi
            for rep_j, c_hjf in lr.skew(rep_f,rep_h).iteritems():
                for rep_g, c_jig in lr.mult(rep_i,rep_j).iteritems():
                    n = result[rep_g] if (rep_g in result) else 0
                    result[rep_g] = n + c_ihe*c_hjf*c_jig
    return result
```

This computes for instance the example (2.90)

```
sage: b_mult([2,1],[1])
{[1, 1]: 1, [2]: 1, [2, 1, 1]: 1, [2, 2]: 1, [3, 1]: 1}
```

[1] Note the assumption that d is sufficiently large (2.64). While finishing this thesis I noticed that SageMath is also capable of calculating the tensor products for general d via the WeylCharacterRing class.

Using the function `b_mult()` and the corresponding function for Littlewood-Richardson coefficients `lr.mult()` all the formulae for counting tensor structures that appear in this thesis can be implemented.

For further illustration consider the coefficients appearing in the tensor product of three irreps

$$\lambda_e \otimes \lambda_f \otimes \lambda_g = \bigoplus_{\lambda_h} b_{efgh} \lambda_h \,, \tag{A.3}$$

given by (2.65)

$$b_{efgh} = \sum_i b_{efi} b_{igh} . \tag{A.4}$$

The implementation is straightforward

```
def b_mult_3(rep_e, rep_f, rep_g):
    result = dict()
    for rep_i, b_efi in b_mult(rep_e, rep_f).iteritems():
        for rep_h, b_igh in b_mult(rep_i, rep_g).iteritems():
            n = result[rep_h] if (rep_h in result) else 0
            result[rep_h] = n + b_efi*b_igh
    return result
```

An example that can be computed by this function is the tensor product in (2.89)

```
sage: b_mult_3([1,1],[1],[1])
{[]: 1, [1, 1]: 3, [1, 1, 1, 1]: 1, [2]: 2, [2, 1, 1]: 2,
 [2, 2]: 1, [3, 1]: 1}
```

A.2 Functions in the conformal block for hook diagram exchange

The following are the functions appearing in the conformal block (3.72) for exchange of the primary in the irreducible representation $[\Delta, \yng(2,1)]$ in the correlator of two scalars and two vectors $\langle \mathcal{O}_1^{\bullet} \mathcal{O}_2^{\square} \mathcal{O}_3^{\bullet} \mathcal{O}_4^{\square} \rangle$.

$$\begin{aligned}
F_1 = (\alpha - \Delta + 1) \Big[& (\beta - \Delta + h + 1)\Big(- (2h-1) J^{(1)}_{2,1,2} (\beta - \Delta + h) - (\alpha + 1) J^{(1)}_{1,1,1} \\
& - J^{(2)}_{1,1,2} \big((2h-1)(v-1) + u\big) + 2(2h-1) v J^{(1)}_{2,2,2} (-\beta + h - 1)\Big) \\
& + (\beta - h + 1)\, v \Big((2h-1)\Big(v J^{(1)}_{2,3,2} (-\beta + h - 2) - (v-1) J^{(2)}_{1,2,2}\Big) + u J^{(2)}_{1,2,2}\Big)\Big] \\
& + (\alpha + 1)\Big[(\beta - \Delta + h + 1)\Big(2\alpha J^{(0)}_{1,2,1}(-\beta + h - 1) + (1 - 2h) J^{(1)}_{1,1,2} (\beta - \Delta + h) \\
& - J^{(1)}_{1,2,2} (-\beta + h - 1)\big((1-2h)(v+1) + 2u\big) + J^{(2)}_{0,1,2}\big((1-2h)(v-1) + u\big) - \alpha J^{(1)}_{0,1,1}\Big) \\
& + (\beta - h + 1)\Big(J^{(2)}_{0,2,2} \big((1-2h)(v-1) - u\big) \\
& + (2h-1)\, v J^{(1)}_{1,3,2} (-\beta + h - 2) + \alpha J^{(1)}_{0,2,1} + v(\alpha - \Delta + 1) J^{(1)}_{1,2,1}\Big)\Big]
\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
F_2 = (\alpha-\Delta+1)\Bigg[&(\beta-\Delta+h+1)\Big(uJ^{(1)}_{2,2,2}(-\beta+h-1)+J^{(2)}_{1,1,1}\big((2h-1)(v-1)+u\big)\Big)\\
&+(\beta-h+1)\,v\Bigg((-\beta+h-2)\Big((\alpha+1)J^{(0)}_{2,3,1}+uJ^{(1)}_{2,3,2}\Big)\\
&-J^{(2)}_{1,2,1}\big((1-2h)(v-1)+u\big)-(\alpha+1)J^{(1)}_{1,2,0}\Bigg)\Bigg]\\
&+(\alpha+1)\Bigg[(\beta-\Delta+h+1)\Bigg((\beta-h+1)\Big(uJ^{(1)}_{1,2,2}-J^{(0)}_{2,2,1}(\alpha-\Delta+1)-\alpha J^{(0)}_{1,2,1}\Big)\\
&-J^{(2)}_{0,1,1}\big((1-2h)(v-1)+u\big)+J^{(1)}_{1,1,0}(\alpha-\Delta+1)+\alpha J^{(1)}_{0,1,0}\Bigg)\\
&+(\beta-h+1)\Bigg(-\alpha\Big(J^{(0)}_{1,3,1}(\beta-h+2)+J^{(1)}_{0,2,0}\Big)+uJ^{(1)}_{1,3,2}(\beta-h+2)\\
&+J^{(2)}_{0,2,1}\big((2h-1)(v-1)+u\big)\Bigg)\Bigg]
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
F_3 = (\alpha-\Delta+1)\Bigg[&(\beta-\Delta+h+1)\Bigg(J^{(2)}_{1,1,1}\big((1-2h)(v-1)-u\big)\\
&-(2h-1)\Big((\alpha+1)J^{(0)}_{2,2,1}(-\beta+h-1)+J^{(1)}_{2,1,1}(-\alpha+\beta+h-2)\Big)\Bigg)\\
&+(\beta-h+1)v\Bigg((2h-1)(\beta-h+2)\Big((\alpha+1)J^{(0)}_{2,3,1}-vJ^{(1)}_{2,3,1}\Big)\\
&-(2h-1)J^{(1)}_{2,2,1}(-\alpha+2\beta-\Delta+2h)+J^{(2)}_{1,2,1}\big((1-2h)(v-1)+u\big)\Bigg)\Bigg]\\
&+(\alpha+1)\Bigg[(\beta-\Delta+h+1)\Bigg(J^{(2)}_{0,1,1}\big((1-2h)(v-1)+u\big)\\
&-(2h-1)\Big(J^{(1)}_{1,1,1}(-2\alpha+\beta+\Delta+h-2)-\alpha J^{(0)}_{1,2,1}(\beta-h+1)-\alpha J^{(1)}_{0,1,1}\Big)\Bigg)\\
&+(\beta-h+1)\Bigg((2h-1)(\beta-h+2)\Big(\alpha J^{(0)}_{1,3,1}-vJ^{(1)}_{1,3,1}\Big)+\alpha(2h-1)J^{(1)}_{0,2,1}\\
&+J^{(1)}_{1,2,1}(\beta-\alpha+h)\big((1-2h)(v+1)+2u\big)-J^{(2)}_{0,2,1}\big((2h-1)(v-1)+u\big)\Bigg)\Bigg]
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
F_4 = (\alpha-\Delta+1)\Bigg[&(\beta-\Delta+h+1)\Bigg((2h-1)\Big(J^{(1)}_{2,1,0}(-\alpha+\Delta-2)-2(\alpha+1)J^{(1)}_{1,1,0}\Big)\\
&+uJ^{(1)}_{2,2,1}(-\beta+h-1)+J^{(2)}_{1,1,0}\big((2h-1)(v-1)+u\big)\Bigg)\\
&+(\beta-h+1)\Bigg((\alpha+1)J^{(1)}_{1,2,0}\big((1-2h)(v+1)+2u\big)+uvJ^{(1)}_{2,3,1}(-\beta+h-2)\\
&-vJ^{(2)}_{1,2,0}\big((1-2h)(v-1)+u\big)+(2h-1)vJ^{(1)}_{2,2,0}(-\alpha+\Delta-2)\Bigg)\Bigg]
\end{aligned}
$$

$$+(\alpha+1)\Bigg[(\beta-\Delta+h+1)\Big(\alpha(1-2h)J^{(1)}_{0,1,0}+J^{(2)}_{0,1,0}\big((2h-1)(v-1)-u\big)\Big)$$
$$+(\beta-h+1)\Big(\alpha(1-2h)J^{(1)}_{0,2,0}+uJ^{(1)}_{1,3,1}(\beta-h+2)$$
$$+J^{(2)}_{0,2,0}\big((2h-1)(v-1)+u\big)+uJ^{(1)}_{1,2,1}(\beta-\Delta+h+1)\Big)\Bigg] \tag{A.8}$$

$$F_H=\frac{1}{h+1}\Bigg\{(\alpha-\Delta+1)\Bigg[(\beta-\Delta+h+1)\Bigg(\big((1-2h)(v-1)-u\big)$$
$$\times\left(J^{(2)}_{1,1,1}+J^{(2)}_{1,0,0}+J^{(2)}_{2,0,1}+uJ^{(2)}_{2,1,2}\right)\Bigg)$$
$$+(\beta-h+1)\Bigg(v\big((1-2h)(v-1)+u\big)\left(J^{(2)}_{1,1,1}+uJ^{(2)}_{2,2,2}+vJ^{(2)}_{2,2,1}+J^{(2)}_{1,1,0}\right)\Bigg)$$
$$+vJ^{(2)}_{2,1,1}\Bigg((1-2h)\Big(u-(v-1)\big(\Delta-2(\beta+1)\big)\Big)+(\Delta-1)u\Bigg)\Bigg]$$
$$+(\alpha+1)\Bigg[(\beta-\Delta+h+1)\big(u-(2h-1)(v-1)\big)\left(J^{(2)}_{1,1,1}v+J^{(2)}_{0,0,0}+J^{(2)}_{0,0,1}+uJ^{(2)}_{1,1,2}\right)$$
$$+(\beta-h+1)\big((1-2h)(v-1)-u\big)\left(uJ^{(2)}_{1,2,2}+J^{(2)}_{0,1,0}+J^{(2)}_{0,2,1}+J^{(2)}_{1,1,1}\right)$$
$$+J^{(2)}_{0,1,1}\Bigg((2h-1)\Big(u+(v-1)\big(\Delta-2(\beta+1)\big)\Big)-(\Delta-1)u\Bigg)\Bigg]$$
$$+J^{(2)}_{1,0,1}(\beta-\Delta+h+1)\Big((2h-1)(v-1)\big(\Delta-2(\alpha+1)\big)+\Delta u\Big)$$
$$-vJ^{(2)}_{1,2,1}(-\beta+h-1)\Big((2h-1)(v-1)\big(\Delta-2(\alpha+1)\big)-\Delta u\Big)\Bigg\}$$
$$+2J^{(0)}_{2,2,1}(\alpha+1)(-\alpha+\Delta-1)(-\beta+h-1)(\beta-\Delta+h+1)\big((2h-1)(v+1)-2u\big)$$
$$+(2h-1)\Bigg\{(\alpha-\Delta+1)\Bigg[(\beta-\Delta+h+1)\Bigg(2(\alpha+1)J^{(0)}_{2,1,1}(\beta-\Delta+h)$$
$$-(\alpha-\Delta+2)\Big(J^{(0)}_{3,1,1}(\beta-\Delta+h)+2vJ^{(0)}_{3,2,1}(\beta-h+1)\Big)\Bigg)$$
$$+(\beta-h+1)\Bigg(v(-\beta+h-2)\Big(vJ^{(0)}_{3,3,1}(-\alpha+\Delta-2)-2(\alpha+1)J^{(0)}_{2,3,1}\Big)\Bigg)\Bigg]$$
$$+(\alpha+1)\Bigg[(\beta-\Delta+h+1)\Big(-2\alpha J^{(0)}_{1,2,1}(-\beta+h-1)+\alpha J^{(0)}_{1,1,1}(\beta-\Delta+h)$$
$$+J^{(0)}_{1,3,1}(-\beta+h-2)(-\beta+h-1)\Big)\Bigg]\Bigg\} \tag{A.9}$$

A.3 Relations between the integrals $J^{(i)}_{j,k,l}$

In this appendix some relations among the integrals $J^{(i)}_{j,k,l}$ defined in (3.69) are derived. The definition is repeated here for convenience

$$J^{(i)}_{j,k,l} = \frac{\Gamma(h+i-f)\,\Gamma(f)\sin(\pi f)}{\sin\bigl(\pi(e+f-h-i)\bigr)} \int\limits_0^\infty \frac{\mathrm{d}x}{x} \int\limits_{x+1}^\infty \frac{\mathrm{d}y}{y} \frac{x^b y^e}{(y+vxy-ux)^{h+i-f}(y-x-1)^f}\,, \tag{A.10}$$

with

$$\begin{aligned} b &= \alpha+i+j-1\,, \\ e &= \beta-\Delta+h+i+k-l\,, \\ f &= 1-\beta+h-k\,. \end{aligned} \tag{A.11}$$

They satisfy

$$\frac{\partial}{\partial v} J^{(i)}_{j,k,l} = -J^{(i+1)}_{j,k,l}\,, \qquad \frac{\partial}{\partial u} J^{(i)}_{j,k,l} = -J^{(i+1)}_{j,k,l+1}\,. \tag{A.12}$$

Substituting $x \to \frac{x'}{v}$ in (A.10), taking the derivative by v on both sides and using (A.12) on the left hand side leads to the relation

$$(\alpha+i+j-1) J^{(i)}_{j,k,l} = v J^{(i+1)}_{j,k,l} + J^{(i+1)}_{j,k-1,l} + u J^{(i+1)}_{j,k,l+1}. \tag{A.13}$$

Similarly, one can substitute $y \to \frac{y'}{v}$ and derive by v again to find

$$(\beta-\Delta+h+i+k-l) J^{(i)}_{j,k,l} = v J^{(i+1)}_{j,k,l} + J^{(i+1)}_{j-1,k,l} + J^{(i+1)}_{j-1,k-1,l-1}. \tag{A.14}$$

More relations can derived by going back to the derivation of $J^{(i)}_{j,k,l}$ starting from eq. (3.3) of [14]. The equation and subsequent discussion is obviously unchanged if one exchanges the coordinates $X_3 \leftrightarrow X_4$ and at the same time $e \leftrightarrow f$. This implies

$$J^{(i)}_{j,k,l} = \frac{\Gamma(h+i-e)\,\Gamma(e)\sin(\pi e)}{\sin\bigl(\pi(e+f-h-i)\bigr)} \int\limits_0^\infty \frac{\mathrm{d}x}{x} \int\limits_{x+1}^\infty \frac{\mathrm{d}y}{y} \frac{\left(\frac{x}{v}\right)^b y^f}{\left(y+\frac{1}{v}xy-\frac{u}{v}x\right)^{h+i-e}(y-x-1)^e}\,. \tag{A.15}$$

Similarly, one can exchange $X_1 \leftrightarrow X_2$ and let $b \to a = 2h-b-e-f$ to find

$$J^{(i)}_{j,k,l} = \frac{\Gamma(h+i-f)\,\Gamma(f)\sin(\pi f)}{\sin\bigl(\pi(e+f-h-i)\bigr)} \int\limits_0^\infty \frac{\mathrm{d}x}{x} \int\limits_{x+1}^\infty \frac{\mathrm{d}y}{y} \frac{x^a y^f}{(vy+xy-ux)^{h+i-f}(y-x-1)^f}\,. \tag{A.16}$$

Now one can substitute $y \to uy'$ in (A.15) and $x \to ux'$ in (A.16) and take the derivative by u in both cases to find the relations

$$\begin{aligned} -(\beta-h-1+k) J^{(i)}_{j,k,l} &= J^{(i+1)}_{j,k-1,l} + J^{(i+1)}_{j-1,k-1,l} + J^{(i+1)}_{j-1,k-1,l-1}\,, \\ -(\alpha-\Delta+l-j) J^{(i)}_{j,k,l} &= J^{(i+1)}_{j-1,k,l} + J^{(i+1)}_{j-1,k-1,l} + u J^{(i+1)}_{j,k,l+1}\,. \end{aligned} \tag{A.17}$$

Let us collect all four relations we found

$$
\begin{aligned}
(\alpha+i+j-1)J^{(i)}_{j,k,l} &= vJ^{(i+1)}_{j,k,l} + J^{(i+1)}_{j,k-1,l} + uJ^{(i+1)}_{j,k,l+1}\,, \\
-(\alpha-\Delta+l-j)J^{(i)}_{j,k,l} &= J^{(i+1)}_{j-1,k,l} + J^{(i+1)}_{j-1,k-1,l} + uJ^{(i+1)}_{j,k,l+1}\,, \\
(\beta-\Delta+h+i+k-l)J^{(i)}_{j,k,l} &= vJ^{(i+1)}_{j,k,l} + J^{(i+1)}_{j-1,k,l} + J^{(i+1)}_{j-1,k-1,l-1}\,, \\
-(\beta-h-1+k)J^{(i)}_{j,k,l} &= J^{(i+1)}_{j,k-1,l} + J^{(i+1)}_{j-1,k-1,l} + J^{(i+1)}_{j-1,k-1,l-1}\,.
\end{aligned}
\tag{A.18}
$$

These relations are already well suited for eliminating all prefactors containing α, β or Δ from the functions that multiply the different tensor structures in the conformal blocks. Note that there exist also three-term relations that can be obtained from (A.18) by adding any two relations and subtracting the other two

$$
\begin{aligned}
(\Delta+l-h-1)J^{(i)}_{j,k,l} &= uJ^{(i+1)}_{j,k,l+1} - J^{(i+1)}_{j-1,k-1,l-1}\,, \\
(\alpha+\beta-\Delta+i+j+k-l-1)J^{(i)}_{j,k,l} &= vJ^{(i+1)}_{j,k,l} - J^{(i+1)}_{j-1,k-1,l}\,, \\
(\alpha-\beta+j-k)J^{(i)}_{j,k,l} &= J^{(i+1)}_{j,k-1,l} - J^{(i+1)}_{j-1,k,l}\,.
\end{aligned}
\tag{A.19}
$$

Finally, adding all four relations from (A.18) yields

$$
(h+i)J^{(i)}_{j,k,l} = J^{(i+1)}_{j-1,k-1,l-1} + J^{(i+1)}_{j-1,k-1,l} + J^{(i+1)}_{j-1,k,l} + J^{(i+1)}_{j,k-1,l} + vJ^{(i+1)}_{j,k,l} + uJ^{(i+1)}_{j,k,l+1}\,. \tag{A.20}
$$

A.4 Multi-residues of tachyon amplitudes from the worldsheet

This appendix contains an explicit derivation of the multiple residue of the Koba-Nielsen amplitude. With the conventional gauge fixing

$$
z_1 = 0, \quad z_{N-1} = 1, \quad z_N \to \infty\,, \tag{A.21}
$$

the Koba-Nielsen formula reads

$$
\mathcal{A}_N = \prod_{u=2}^{N-2} \int_0^{z_{u+1}} \mathrm{d}z_u \prod_{v=2}^{N-2} z_v^{k_{1v}} \prod_{l=2}^{N-2} (1-z_l)^{k_{l,N-1}} \prod_{\substack{i,j \\ 1<i<j<N-1}} (z_j - z_i)^{k_{ij}}. \tag{A.22}
$$

Using binomial expansion[2]

$$
(z_j - z_i)^{k_{ij}} = \sum_{a_{ij}=0}^{\infty} \binom{k_{ij}}{a_{ij}} (-1)^{a_{ij}} z_i^{a_{ij}} z_j^{k_{ij}-a_{ij}}, \tag{A.23}
$$

[2]Alternatively, one could use Mellin-Barnes representations here, see e.g. [92] for a systematic approach.

the amplitude becomes

$$\mathcal{A}_N = \sum_{a_{23},\ldots,a_{N-2,N-1}=0}^{\infty} \prod_{\substack{i,j \\ 1<i<j<N}} (-1)^{a_{ij}} \binom{k_{ij}}{a_{ij}} \prod_{u=2}^{N-2} \int_0^{z_{u+1}} \mathrm{d}z_u z_i^{a_{ij}} \prod_{v=2}^{N-2} z_v^{k_{1v}} \prod_{\substack{s,t \\ 1<s<t<N-1}} z_t^{k_{st}-a_{st}}. \tag{A.24}$$

Doing the integrals one by one, one finds that the (l+1)th integral gives the factor

$$\frac{1}{\alpha'(k_1+\ldots+k_l)^2 + \sum\limits_{\substack{1<u\leq l \\ l<v<N}} a_{uv} - 1}. \tag{A.25}$$

Now compute the N-point tachyon amplitudes with all internal particles on-shell $-\alpha'(k_1 + \ldots + k_l)^2 = A_l - 1$. In our other notation these mass levels correspond to $A_2 = A, A_3 = B$ and so on. Doing the integral using binomial expansion, one can see a way to write the result in general

$$\left(\prod_{l=2}^{N-2} \mathrm{Res}_{s_{1\ldots l}\to A_l-1}\right) \mathcal{A}_N = (-1)^{N-3} \sum_{a_{23},\ldots,a_{N-2,N-1}=0}^{\infty} \prod_{\substack{i,j \\ 1<i<j<N}} (-1)^{a_{ij}} \binom{k_{ij}}{a_{ij}} \prod_{l=2}^{N-2} \delta_{A_l, \sum\limits_{\substack{1<u\leq l \\ l<v<N}} a_{uv}}. \tag{A.26}$$

For example the double residue of the 5-tachyon amplitude is

$$\begin{aligned} &\mathrm{Res}_{s_{12}\to A_2-1}\mathrm{Res}_{s_{123}\to A_3-1}\mathcal{A}_5 \\ &= \sum_{a_{23},a_{24},a_{34}=0}^{\infty} (-1)^{a_{23}+a_{24}+a_{34}} \binom{k_{23}}{a_{23}}\binom{k_{24}}{a_{24}}\binom{k_{34}}{a_{34}} \delta_{A_2,a_{23}+a_{24}}\delta_{A_3,a_{24}+a_{34}} \\ &= \sum_{a_{24}=0}^{\min(A_2,A_3)} (-1)^{A_2+A_3-a_{24}} \binom{k_{23}}{A_2-a_{24}}\binom{k_{24}}{a_{24}}\binom{k_{34}}{A_3-a_{24}}. \end{aligned} \tag{A.27}$$

Analogously for $N = 6$

$$\begin{aligned} &\mathrm{Res}_{s_{12}\to A_2-1}\mathrm{Res}_{s_{123}\to A_3-1}\mathrm{Res}_{s_{1234}\to A_4-1}\mathcal{A}_6 \\ &= -\sum_{a_{23},\ldots,a_{45}=0}^{\infty} (-1)^{a_{23}+a_{24}+a_{25}+a_{34}+a_{35}+a_{45}} \binom{k_{23}}{a_{23}}\binom{k_{24}}{a_{24}}\binom{k_{25}}{a_{25}}\binom{k_{34}}{a_{34}}\binom{k_{35}}{a_{35}}\binom{k_{45}}{a_{45}} \\ &\cdot \delta_{A_2,a_{23}+a_{24}+a_{25}}\delta_{A_3,a_{24}+a_{25}+a_{34}+a_{35}}\delta_{A_4,a_{25}+a_{35}+a_{45}} \\ &= -\sum_{a_{24}=0}^{\min(A_2,A_3)} \sum_{a_{25}=0}^{\min(A_2,A_3,A_4)} \sum_{a_{35}=0}^{\min(A_3,A_4)} (-1)^{A_2+A_3+A_4-a_{24}-2a_{25}-a_{35}} \\ &\cdot \binom{k_{23}}{A_2-a_{24}-a_{25}}\binom{k_{24}}{a_{24}}\binom{k_{25}}{a_{25}}\binom{k_{34}}{A_3-a_{24}-a_{25}-a_{35}}\binom{k_{35}}{a_{35}}\binom{k_{45}}{A_4-a_{25}-a_{35}}. \end{aligned} \tag{A.28}$$

A.5 On-shell space of kinematic variables

In Section 4.3 the Koba-Nielsen amplitudes were factored into 3-point amplitudes by putting all the Mandelstams $s_{12\ldots}$ on the mass shell. It is shown in this appendix how in this configuration any k_{ij} can be expressed in terms of the remaining $\frac{(N-2)(N-3)}{2}$ independent variables

k_{ij} with $1 < i < j < N$. After obtaining a result which can contain any k_{ij} the following rules can be used to remove spurious kinematic variables. First momentum conservation is used to remove the variables k_{iN}

$$k_{iN} = \sum_{j=1}^{N-1} -k_{ij}. \tag{A.29}$$

The $N-3$ conditions to put $k_1 + k_2$ up to $k_1 + k_2 + \ldots + k_{N-2}$ on-shell can be used to eliminate $k_{12}, \ldots, k_{1,N-2}$

$$\begin{aligned}
-\alpha'(k_1 + k_2)^2 &= A_2 - 1, \\
-\alpha'(k_1 + k_2 + k_3)^2 &= A_3 - 1, \\
&\vdots \\
-\alpha'(k_1 + k_2 + \ldots + k_{N-2})^2 &= A_{N-2} - 1.
\end{aligned} \tag{A.30}$$

Finally, there is always one additional condition that eliminates $k_{1,N-1}$. This condition is found by removing the 1 using momentum conservation and then $k_{N-1,N}$ using the last on-shell condition again (using momentum conservation to get $-\alpha'(k_{N-1} + k_N)^2 = A_{N-2} - 1$)

$$k_{1,N-1} = \sum_{j=2}^{N} -k_{j,N-1} = \sum_{j=2}^{N-1} -k_{j,N-1} + A_{N-2} + 1. \tag{A.31}$$

Now only those invariants are left that appear in the residues of the N-tachyon amplitude (5.100).

A.6 Cyclicity as alternative input for fixing the residue coefficients

For the 4 and 5 point tachyon amplitudes in the bosonic string the coefficients for the basis elements can also be fixed just by the assumption that the amplitudes are cyclic. It is likely that a generalization to N points is possible.

4 points

With the residues derived before, the 4-point amplitude is

$$\sum_{A=0}^{\infty} \binom{k_{23}}{A} \frac{h_A}{k_{12} + A + 1}, \tag{A.32}$$

and the coefficients h_A are to be determined. Due to momentum conservation $k_{34} = k_{12}$ holds and so cyclic invariance yields

$$\sum_{A=0}^{\infty} \binom{k_{23}}{A} \frac{h_A}{k_{12} + A + 1} = \sum_{B=0}^{\infty} \binom{k_{12}}{B} \frac{h_B}{k_{23} + B + 1}. \tag{A.33}$$

to calculate the constants h_A consider the line $k_{12} = k_{23} - A'$ (with $A' \in \mathbb{N}$) in the space of kinematic variables and multiply both sides with $(k_{23} + 1)$

$$(k_{23} + 1) \sum_{A=0}^{\infty} \binom{k_{23}}{A} \frac{h_A}{k_{23} - A' + A + 1} = (k_{23} + 1) \sum_{B=0}^{\infty} \binom{k_{23} - A'}{B} \frac{h_B}{k_{23} + B + 1}. \tag{A.34}$$

Now set $k_{23} = -1$ and obtain

$$(-1)^{A'} h_{A'} = h_0. \tag{A.35}$$

5 points

The 5-point amplitude is in terms of the basis derived above (5.74)

$$\sum_{A,B=0}^{\infty} \sum_{a=0}^{\min(A,B)} \binom{k_{23}}{A-a} \binom{k_{34}}{B-a} \binom{k_{24}}{a} \frac{h_{A,B,a}}{(k_{12}+A+1)(k_{45}+B+1)}. \tag{A.36}$$

It is useful to change to a set of variables that is mapped to itself under a cyclic relabelling of the external particles. For this, exchange k_{24} for k_{51} using $k_{24} = k_{51} - k_{23} - k_{34} - 1$

$$\sum_{A,B=0}^{\infty} \sum_{a=0}^{\min(A,B)} \binom{k_{23}}{A-a} \binom{k_{34}}{B-a} \binom{k_{51}-k_{23}-k_{34}-1}{a} \frac{h_{A,B,a}}{(k_{12}+A+1)(k_{45}+B+1)}. \tag{A.37}$$

Consider the cyclic permutation by two positions

$$\sum_{A,B,a} h_{A,B,a} \frac{\binom{k_{23}}{A-a}\binom{k_{34}}{B-a}\binom{k_{51}-k_{23}-k_{34}-1}{a}}{(k_{12}+A+1)(k_{45}+B+1)} = \sum_{C,D,b} h_{C,D,b} \frac{\binom{k_{45}}{C-b}\binom{k_{51}}{D-b}\binom{k_{23}-k_{45}-k_{51}-1}{b}}{(k_{34}+C+1)(k_{12}+D+1)}. \tag{A.38}$$

This time restrict to $k_{45} = k_{34} - B'$ and multiply by $(k_{12}+A'+1)(k_{34}+C'+1)$

$$\begin{aligned} &(k_{12}+A'+1)(k_{34}+C'+1) \sum_{A,B} \sum_{a=0}^{\min(A,B)} h_{A,B,a} \frac{\binom{k_{23}}{A-a}\binom{k_{34}}{B-a}\binom{k_{51}-k_{23}-k_{34}-1}{a}}{(k_{12}+A+1)(k_{34}-B'+B+1)} \\ &= (k_{12}+A'+1)(k_{34}+C'+1) \sum_{C,D} \sum_{b=0}^{\min(C,D)} h_{C,D,b} \frac{\binom{k_{34}-B'}{C-b}\binom{k_{51}}{D-b}\binom{k_{23}-k_{34}+B'-k_{51}-1}{b}}{(k_{34}+C+1)(k_{12}+D+1)}. \end{aligned} \tag{A.39}$$

Set $k_{12} = -A' - 1$ and $k_{34} = -C' - 1$

$$\begin{aligned} &\sum_{a=0}^{\min(A',B'+C')} h_{A',B'+C',a} \binom{k_{23}}{A'-a} \binom{-C'-1}{B'+C'-a} \binom{k_{51}-k_{23}+C'}{a} \\ =& \sum_{b=0}^{\min(C',A')} h_{C',A',b} \binom{-B'-C'-1}{C'-b} \binom{k_{51}}{A'-b} \binom{k_{23}+B'+C'-k_{51}}{b}. \end{aligned} \tag{A.40}$$

Now set $C' = 0$

$$\sum_{a=0}^{\min(A',B')} h_{A',B',a} \binom{k_{23}}{A'-a} \binom{-1}{B'-a} \binom{k_{51}-k_{23}}{a} = h_{0,A',0} \binom{k_{51}}{A'}. \tag{A.41}$$

One can choose $k_{23} = k_{51} \notin \mathbb{Z}$ (where the integers are avoided to make sure not to hit a zero) to gain

$$h_{A',B',0} \binom{-1}{B'} = h_{0,A',0}. \tag{A.42}$$

Applying this formula twice to $h_{0,0,0}$ gives us all coefficients with $a = 0$

$$h_{A',B',0} = (-1)^{A'+B'} h_{0,0,0}. \tag{A.43}$$

To calculate the other coefficients, go back to (A.41) and set $k_{23} = -1$ and $k_{51} = a' - 1$ with $a' \in \mathbb{N}, 1 \leq a' \leq \min(A', B')$

$$0 = \sum_{a=0}^{a'} h_{A',B',a} \binom{-1}{A'-a} \binom{-1}{B'-a} \binom{a'}{a} = (-1)^{A'+B'} \sum_{a=0}^{a'} h_{A',B',a} \binom{a'}{a}. \tag{A.44}$$

These are enough equations to fix all $h_{A',B',a'}$ and given that the alternating sum of binomial coefficients vanishes, the solution is

$$h_{A',B',a'} = (-1)^{a'} h_{A',B',0}. \tag{A.45}$$

Together with (A.43) the result is

$$h_{A',B',a'} = (-1)^{A'+B'+a'} h_{0,0,0}. \tag{A.46}$$

A.7 Couplings of two tachyons and one massive particle

In this appendix (4.26) is used to compute the general 3-point coupling of two tachyons and one arbitrary on-shell particle. The right-hand side of (4.26) consists of contractions of the terms $(k_1 - k_2)$ and $(k_3 - k_4)$, i.e.

$$\frac{\alpha'}{2}(k_1 - k_2) \cdot (k_3 - k_4)\Big|_{s_{12}=A-1} = s_{23} + \frac{A+3}{2}, \tag{A.47}$$

and

$$\frac{\alpha'}{2}(k_1 - k_2) \cdot (k_1 - k_2)\Big|_{s_{12}=A-1} = \frac{\alpha'}{2}(k_3 - k_4) \cdot (k_3 - k_4)\Big|_{s_{12}=A-1} = \frac{A+3}{2}. \tag{A.48}$$

Start by writing the residues of the Veneziano amplitude as a function of the polynomial (A.47). Then all couplings $c_{TT[A,[a]]}$ that appear as part of the 3-point amplitudes on the right-hand side of (4.26) are computed by matching up the coefficients of these polynomials on both sides. In this appendix the shorthand notation $c_{A,a} \equiv c_{TT[A,[a]]}$ is used.

The residues of the Veneziano amplitude at mass level $A \in \mathbb{N}_0$ are

$$-\operatorname{Res}_{s_{12}\to A-1} \mathcal{A}_4(s_{12}, s_{23}) = \frac{1}{A!} \prod_{i=1}^{A} (s_{23} + 1 + i). \tag{A.49}$$

These residues can be expressed as linear combinations of the terms (A.47). For even A,

$$\begin{aligned} -\operatorname{Res}_{s_{12}\to A-1} \mathcal{A}_4(s_{12}, s_{23}) &= \frac{1}{A!} \prod_{i=1}^{\frac{A}{2}} \left\{ \left(s_{23} + \frac{A+3}{2}\right)^2 - \left(i - \frac{1}{2}\right)^2 \right\} \\ &= \sum_{k=0}^{\frac{A}{2}} V_{k,A\,\text{even}} \left(s_{23} + \frac{A+3}{2}\right)^{A-2k}, \end{aligned} \tag{A.50}$$

holds, with

$$V_{0,A\,\mathrm{even}} = \frac{1}{A!}, \quad V_{k,A\,\mathrm{even}} = \frac{(-1)^k}{A!} \sum_{j_1=1}^{\frac{A}{2}} \left(j_1 - \frac{1}{2}\right)^2 \sum_{j_2=j_1+1}^{\frac{A}{2}} \left(j_2 - \frac{1}{2}\right)^2 \cdots \sum_{j_k=j_{k-1}+1}^{\frac{A}{2}} \left(j_k - \frac{1}{2}\right)^2. \tag{A.51}$$

Similarly, for odd A

$$\begin{aligned} -\mathrm{Res}_{s_{12}\to A-1}\mathcal{A}_4(s_{12}, s_{23}) &= \frac{1}{A!}\left(s_{23} + \frac{A+3}{2}\right) \prod_{i=1}^{\frac{A-1}{2}} \left\{ \left(s_{23} + \frac{A+3}{2}\right)^2 - i^2 \right\} \\ &= \sum_{k=0}^{\frac{A-1}{2}} V_{k,A\,\mathrm{odd}} \left(s_{23} + \frac{A+3}{2}\right)^{A-2k}, \end{aligned} \tag{A.52}$$

with

$$V_{0,A\,\mathrm{odd}} = \frac{1}{A!}, \qquad V_{k,A\,\mathrm{odd}} = \frac{(-1)^k}{A!} \sum_{j_1=1}^{\frac{A-1}{2}} j_1^2 \sum_{j_2=j_1+1}^{\frac{A-1}{2}} j_2^2 \cdots \sum_{j_k=j_{k-1}+1}^{\frac{A-1}{2}} j_k^2. \tag{A.53}$$

$V_{k,A\,\mathrm{even}}$ and $V_{k,A\,\mathrm{odd}}$ are essentially the central factorial numbers $t(A,k)$ and $t_2(A,k)$

$$V_{k,A\,\mathrm{even}} = \frac{(-1)^k}{A!4^k} t_2\left(\frac{A}{2}, k\right), \qquad 0 \le k \le \left\lfloor \frac{A}{2} \right\rfloor, \tag{A.54}$$

$$V_{k,A\,\mathrm{odd}} = \frac{(-1)^k}{A!} t\left(\frac{A-1}{2}, k\right), \qquad 0 \le k \le \left\lfloor \frac{A}{2} \right\rfloor, \tag{A.55}$$

where

$$\begin{aligned} t(n,0) &= 1, \\ t(n,n) &= (n!)^2, \\ t(n,k) &= n^2 t(n-1,k-1) + t(n-1,k), \end{aligned} \tag{A.56}$$

$$\begin{aligned} t_2(n,0) &= 1, \\ t_2(n,n) &= ((2n-1)!!)^2, \\ t_2(n,k) &= (2n-1)^2 t_2(n-1,k-1) + t_2(n-1,k). \end{aligned} \tag{A.57}$$

To compute the right-hand side of (4.26) the projectors to the traceless symmetric tensors given in (2.38) are needed. Since the 3-point amplitudes with two tachyons are already fully symmetric (4.25), it is not necessary to perform the symmetrizations depicted by birdtrack symbols in (2.38) again. The projector without the symmetrizations, lifted to $\mathbb{R}^{d,1}$ as explained in Section 2.8.1, can be written as

$$\Pi^{A_1\ldots A_a, B_1\ldots B_a}_{\square\square\square\cdots\square \text{ non-symmetrized}} = \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} W_{a,k} \prod_{i=1}^{k} \eta^{A_{2i-1},A_{2i}} \eta^{B_{2i-1},B_{2i}} \prod_{j=2k+1}^{a} \eta^{A_j,B_j}, \tag{A.58}$$

where $W_{a,k}$ is defined in (2.39). Inserting the projector into (4.26) and using (A.47) and

(A.48) yields

$$\begin{aligned}
&\sum_{a=0}^{A} c^2_{A,a} \sum_{k=0}^{\lfloor \frac{a}{2} \rfloor} W_{a,k} \left(\frac{A+3}{2}\right)^{2k} \left(s_{23} + \frac{A+3}{2}\right)^{a-2k} \\
&= c^2_{A,A} \left(s_{23} + \frac{A+3}{2}\right)^{A} + \left[c^2_{A,A} W_{A,1} \left(\frac{A+3}{2}\right)^{2} + c^2_{A,A-2}\right] \left(s_{23} + \frac{A+3}{2}\right)^{A-2} \\
&\quad + \left[c^2_{A,A} W_{A,2} \left(\frac{A+3}{2}\right)^{4} + c^2_{A,A-2} W_{A,1} \left(\frac{A+3}{2}\right)^{2} + c^2_{A,A-4}\right] \left(s_{23} + \frac{A+3}{2}\right)^{A-4} + \dots .
\end{aligned} \tag{A.59}$$

Here it was already used that only even or odd powers of the polynomial in s_{23} appear exclusively in (A.50,A.52). This implies that all $c^2_{A,a}$ with $(A-a)$ odd are zero which is expected (4.1). Now a recursive formula for $c^2_{A,a}$ can be read off by matching up (A.50,A.52) and (A.59)

$$c^2_{A,A} = V_{0,A}, \qquad c^2_{A,A-2k} = V_{k,A} - \sum_{l=1}^{k} c^2_{A,A-2k+2l} W_{A,l} \left(\frac{A+3}{2}\right)^{2l} . \tag{A.60}$$

Observing that each term which multiplies the number $V_{k,A}$ in (A.60) contains the same power of $\left(\frac{A+3}{2}\right)$, the recursion relation can be cast into the form

$$c^2_{A,A-2k} = \sum_{l=0}^{k} V_{k-l,A} \left(\frac{A+3}{2}\right)^{2l} M^{A,k}_{l}, \tag{A.61}$$

with

$$M^{A,k}_0 = 1, \qquad M^{A,k}_l = -\sum_{j=1}^{l} W_{A-2k+2l,j} M^{A,k}_{l-j}. \tag{A.62}$$

This can be expressed in a closed form. Start simplifying with the observation that (with $a = A - 2k$ and $p = (p_1, \dots, p_m)$ denotes a partition of l) each term in $M^{A,k}_l$ consists of a product

$$\prod_{i=1}^{m} W_{a+2(l-\sum_{k<i} p_k),p_i}. \tag{A.63}$$

A common factor can be pulled out of all of these products

$$\begin{aligned}
\prod_{i=1}^{m} W_{a+2(l-\sum_{k<i} p_k),p_i} =& \prod_{i=1}^{m} \frac{(-1)^{p_i} (a + 2(l - \sum_{k<i} p_k))_{2p_i}}{2^{2p_i}} \tilde{W}_{a+2(l-\sum_{k<i} p_k),p_i} \\
=& \frac{(-1)^l (a+1)^{(2l)}}{2^{2l}} \prod_{i=1}^{m} \tilde{W}_{a+2(l-\sum_{k<i} p_k),p_i}, \\
\text{where} \quad \tilde{W}_{a,j} =& \frac{1}{j!(\frac{d}{2} + a - 2)_j},
\end{aligned} \tag{A.64}$$

and the raising and falling factorials $(x)^{(l)} = (x+l-1)_l = x(x+1)(x+2)\dots(x+l-1)$ were used. $M^{A,k}_l$ is proportional to this overall factor

$$M^{A,k}_l = \frac{(-1)^l (a+1)^{(2l)}}{2^{2l}} \tilde{M}^{A,k}_l, \qquad \tilde{M}^{A,k}_0 = 1, \qquad \tilde{M}^{A,k}_l = -\sum_{j=1}^{l} \tilde{W}_{a+2l,j} \tilde{M}^{A,k}_{l-j}. \tag{A.65}$$

Next by induction it can be proven that

$$\tilde{M}_l^{A,k} = \frac{(-1)^l}{l!(\frac{d}{2}+a)^{(l)}}. \tag{A.66}$$

The statement is true for $l = 0$. Plugging in $\tilde{M}_{l-j}^{A,k}$ into the recursive definition for the induction step yields

$$\begin{aligned} \tilde{M}_l^{A,k} &= -\sum_{j=1}^{l} \frac{1}{j!(\frac{d}{2}+a+2l-2)_j} \frac{(-1)^{l-j}}{(l-j)!(\frac{d}{2}+a)^{(l-j)}} \\ &= \frac{1}{(\frac{d}{2}+a)^{(2l-1)}} \sum_{j=1}^{l} \frac{(-1)^{l-j+1}(\frac{d}{2}+a+l-j)^{(l-1)}}{j!(l-j)!}. \end{aligned} \tag{A.67}$$

To show that this equals (A.66) use the identity

$$\sum_{j=0}^{l} \frac{(-1)^{l-j+1}(\frac{d}{2}+a+l-j)^{(l-1)}}{j!(l-j)!} = 0, \tag{A.68}$$

which can be proved using computer algebra. This yields the expression

$$M_l^{A,k} = \frac{(a+1)^{(2l)}}{2^{2l} l!(\frac{d}{2}+a)^{(l)}}, \tag{A.69}$$

which inserted into (A.61) yields the final result

$$c_{A,a}^2 = \begin{cases} \sum_{l=0}^{\frac{A-a}{2}} V_{\frac{A-a}{2}-l,A} \left(\frac{A+3}{4}\right)^{2l} \frac{(a+1)^{(2l)}}{l!(\frac{d}{2}+a)^{(l)}} & A-a \text{ even}, \\ 0 & A-a \text{ odd}. \end{cases} \tag{A.70}$$

This is the result quoted in the main text.

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