

# On Soft Limits of Large-Scale Structure Correlation Functions

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## Abstract

We study soft limits of correlation functions for the density and velocity fields in the theory of structure formation. First, we re-derive the (resummed) consistency conditions at unequal times using the eikonal approximation. These are solely based on symmetry arguments and are therefore universal. Then, we explore the existence of equal-time relations in the soft limit which, on the other hand, depend on the interplay between soft and hard modes. We scrutinize two approaches in the literature: The time-flow formalism, and a background method where the soft mode is absorbed into a locally curved cosmology. The latter has been recently used to set up (angular averaged) ‘equal-time consistency relations’. We explicitly demonstrate that the time-flow relations and ‘equal-time consistency conditions’ are only fulfilled at the linear level, and fail at next-to-leading order for an Einstein de-Sitter universe. While applied to the velocities both proposals break down beyond leading order, we find that the ‘equal-time consistency conditions’ quantitatively approximates the perturbative results for the density contrast. Thus, we generalize the background method to properly incorporate the effect of curvature in the density and velocity fluctuations on short scales, and discuss the reasons behind this discrepancy. We conclude with a few comments on practical implementations and future directions.

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# 1 Introduction

Soft limits, that link  $(n + 1)$ -point and  $n$ -point correlators of the density perturbations in the theory of structure formation, have recently received significant attention. The main appeal of these relations (for unequal times) is that they are solely based on the assumption of a single-field inflationary background, providing the seed for the initial conditions, together with the diffeomorphism invariance of General Relativity. Therefore, they lead to (quite generally) non-perturbative statements about the system on short scales that serve as a probe of basic aspects of the theory at hand [1–9]. This becomes a very powerful tool in the context of using the forthcoming large-scale structure (LSS) surveys to test single-field inflation as a theory of initial conditions for the seed of structure, as well as the equivalence principle in gravitational theories, especially since fluctuations enter the non-linear regime at small redshift.

Various approaches have been used to derive soft limits for correlation functions in LSS.<sup>1</sup> As stated, these relations are most meaningful for correlation functions at different times. For equal-time correlators they become degenerate, in the sense that they vanish at leading order in  $q$ , with  $q$  being the soft (or long) mode.<sup>2</sup> To extract information about equal-time correlators, one therefore has to study next-to-leading order (NLO) effects, where dynamical information, as opposite to gauge artifacts in General Relativity, start to become important [2, 3, 15]. It is then relevant to determine whether at equal times one may be able to write down expressions that are still valid even when the short modes are deep in the non-linear regime. In fact, allegedly non-perturbative relations in the soft limit at equal times have been recently advocated in the literature [16, 17].

The purpose of this paper is thus twofold. In section 2, we first (re-)derive the consistency conditions for the soft (squeezed) limit of density and velocity correlators at *unequal times*. Here we exploit the compact notation of the fluid equations in the Eulerian representation of perturbation theory, which simultaneously includes both fields, e.g [18, 19]. One of the crucial aspects in the derivation of the unequal-time relation is the factorization of soft and hard modes, which can be resummed into an eikonal phase.

Next, we explore under which circumstances relations between *equal-time* correlators in the soft limit may exist beyond a perturbative treatment of the hard modes, when the coupling between long and short fluctuations becomes important. For this purpose, we compute in section 3 the three-point function (or bispectrum) at NLO in the soft limit as a benchmark for comparison of different methods. We study two different approaches that have appeared in the literature. The reader may chose to concentrate on one or the other without disturbing the flow of the paper.

In section 4, we study the time-flow formalism [20] which relies on applying a ‘closure,’ or truncation, approximation to a hierarchical set of evolution equations. For the case of the bispectrum, we show that the connected piece of the four-point function (or trispectrum), often neglected in the literature, plays an important role in assessing the validity of an equal-time relation. In general, only perturbative statements may be derived in the time-flow approach for a given truncation. In section 5, we study the implementation of a map, discussed in [2, 3],

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<sup>1</sup>Soft limits of inflationary correlation functions have also been extensively studied in the literature, e.g. [10–14].

<sup>2</sup>Technically speaking, when dealing with the density field (as opposite to the potential) the leading-order term scales like  $q^{-1}$ .

between dynamics on short scales within a flat Friedmann-Robertson-Walker (FRW) universe in the presence of a long-wavelength perturbation and a locally curved FRW background. This equivalence was used in [16, 17] to propose angular averaged soft-limit relations for correlation functions at equal times. As we argue, the expression derived in [16, 17] does carry information that is non-perturbative in the short modes, for example coming from the ‘displacement’ term (or equivalently the eikonal phase). However, the part that accounts for the growth of structure cannot be extended beyond leading order in perturbation theory, not even in an Einstein de-Sitter (EdS) universe where these relations were claimed to be formally exact. In spite of this, while the proposal of [16, 17] fails when applied to the velocity beyond linear order, we find that it produces quantitatively accurate results for the density contrast compared to standard perturbation theory (SPT). For example, for the bispectrum of density fluctuations at one-loop order the error is only a few percent. We generalize the background method to properly incorporate the effect of curvature in the density and velocity fluctuations on short scales, which we show to react differently (by a factor of order one) to the presence of a local curvature, and discuss the reasons behind this (somewhat small) discrepancy. The accuracy of the ‘equal-time consistency conditions’ of [16, 17] in the non-perturbative regime, similar to what we find in SPT, remains an open question. Finally, in section 6, we conclude with a discussion and future directions.

## 2 Correlation Functions at Unequal Times in the Soft Limit

### 2.1 Fluid equations

If we ignore deviations from the perfect fluid approximation, which are required to account for the imprint of hard modes on the long-distance scales [21, 22], the non-linear fluid equations of cosmological perturbation theory (the continuity, Euler and Poisson equations) can be expressed in a compact form by writing the matter density contrast  $\delta$  and the divergence of the velocity field in Fourier space as doublet  $\psi_a(\mathbf{k}, \eta)$  with  $a \in \{1, 2\}$ ,

$$\psi_a(\mathbf{k}, \eta) \equiv \begin{pmatrix} \delta(\mathbf{k}, \eta) \\ \Theta(\mathbf{k}, \eta) \end{pmatrix}, \quad (2.1)$$

with  $\Theta(\mathbf{k}, \eta) \equiv -\nabla \cdot \mathbf{v}(\mathbf{k}, \eta) / \mathcal{H}$ . Thereby, the conformal time  $\tau$  has been replaced by the time variable  $\eta \equiv \ln a(\tau)$  in terms of the scale factor  $a(\tau)$  and  $\mathcal{H} \equiv \partial \ln a(\tau) / \partial \tau = a H$  denotes the conformal expansion rate. In general background cosmologies, the fluid equations then read [18, 19]

$$\partial_\eta \psi_a(\mathbf{k}, \eta) = -\Omega_{ab}(\mathbf{k}, \eta) \psi_b(\mathbf{k}, \eta) + \gamma_{abc}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \psi_b(\mathbf{p}, \eta) \psi_c(\mathbf{q}, \eta), \quad (2.2)$$

using the convention that repeated indices are summed and integration over internal momenta has to be performed whenever the vertex function  $\gamma_{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q})$  appears. The only independent, non-vanishing, elements of  $\gamma_{abc}(\mathbf{k}, \mathbf{p}, \mathbf{q})$  with  $a, b, c \in \{1, 2\}$ ,

$$\begin{aligned} \gamma_{121}(\mathbf{k}, \mathbf{p}, \mathbf{q}) &= \gamma_{112}(\mathbf{k}, \mathbf{q}, \mathbf{p}) = \frac{1}{2} \delta^D(\mathbf{k} + \mathbf{p} + \mathbf{q}) \cdot \alpha(\mathbf{p}, \mathbf{q}), \\ \gamma_{222}(\mathbf{k}, \mathbf{p}, \mathbf{q}) &= \delta^D(\mathbf{k} + \mathbf{p} + \mathbf{q}) \cdot \beta(\mathbf{p}, \mathbf{q}), \end{aligned} \quad (2.3)$$

arise as products of the Dirac delta-distribution, denoted by  $\delta^D(\mathbf{k} + \mathbf{p} + \mathbf{q})$ , and the functions

$$\alpha(\mathbf{p}, \mathbf{q}) = \frac{(\mathbf{p} + \mathbf{q}) \cdot \mathbf{p}}{p^2}, \quad \beta(\mathbf{p}, \mathbf{q}) = \frac{(\mathbf{p} + \mathbf{q})^2 \mathbf{p} \cdot \mathbf{q}}{2 p^2 q^2}, \quad (2.4)$$

which couple different modes of density and velocity perturbations.

The dependence of the fluid equations (2.2) on the cosmological model is encoded in the matrix  $\Omega_{ab}$ . For example, for the simplest case of a flat, pure dark matter cosmology with EdS background, it is given by

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2} \Omega_m & 1 + \frac{1}{\mathcal{H}^2} \frac{\partial \mathcal{H}}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}. \quad (2.5)$$

Generalization to other cosmologies is straightforward, e.g. [20]. However, in order to perform explicit computations, we will use (2.5) as a working example, in particular when including higher-order perturbative corrections. In this case, the linear propagator, which describes the time evolution of  $\psi_a(\mathbf{k}, \eta)$  at the linear level, reads

$$g_{ab}(\eta, \eta') = \left[ B e^{(\eta - \eta')} + A e^{-\frac{3}{2}(\eta - \eta')} \right]_{ab} \theta(\eta - \eta'), \quad (2.6)$$

where  $\theta(\eta)$  denotes the Heaviside step function and

$$B = \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}, \quad A = \frac{1}{5} \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}. \quad (2.7)$$

## 2.2 The eikonal approximation

The derivation of consistency relations in the squeezed limit is relatively straightforward for unequal times. Since soft and hard modes evolve independently at leading order in  $q$ , the soft effects can be resummed, yielding an eikonal phase [23, 24].<sup>3</sup> For the fluctuations of the hard modes  $\mathbf{k}$ , with  $k \gg q$ , one finds (we suppress indices in what follows for simplicity)

$$\psi(\mathbf{k}, \eta) \simeq \exp \left[ \int^\eta d\eta' \int_{\mathbf{p}}^{\Lambda_L} \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} \Theta_L(\mathbf{p}, \eta') \right] \times \psi_S(\mathbf{k}, \eta), \quad (2.8)$$

where  $\int_{\mathbf{p}} \equiv \int \frac{d^3 \mathbf{p}}{(2\pi)^3}$ . Here,  $\psi_S(\mathbf{k}, \eta)$  denotes the fluctuations of short scales, including interactions of the type short-short, and  $\Theta_L(\mathbf{p}, \eta)$  are the long-scale modes of the velocity field, in units of the conformal Hubble parameter according to its definition in (2.1). We have also introduced a cutoff  $\Lambda_L$  to emphasize the integral is performed over soft momenta. Moreover, at leading/linear order we have (for the growing mode)  $\Theta_L(\mathbf{p}, \eta) = \delta_L(\mathbf{p}, \eta)$ . Therefore, the impact of the long modes comes in the form of an exponential that involves the linear density field. In particular, the effect of soft physics completely factorizes at this order. Using the expression in (2.8), one can readily derive the consistency relations of large-scale structure at unequal times, noticing that

$$\begin{aligned} \langle \psi_L(\mathbf{q}, \eta_q) \psi(\mathbf{k}_1, \eta_1) \cdots \psi(\mathbf{k}_n, \eta_n) \rangle' &\xrightarrow{q \rightarrow 0} \left\langle \psi_L(\mathbf{q}, \eta_q) \exp \left[ \sum_i \int^{\eta_i} d\eta'_i \int_{\mathbf{p}}^{\Lambda_L} \frac{\mathbf{k}_i \cdot \mathbf{p}}{p^2} \delta_L(\mathbf{p}, \eta'_i) \right] \right\rangle' \\ &\times \langle \psi_S(\mathbf{k}_1, \eta_1) \cdots \psi_S(\mathbf{k}_n, \eta_n) \rangle', \end{aligned} \quad (2.9)$$

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<sup>3</sup>See also [25, 26] for related discussions.

and

$$\begin{aligned} \langle \psi(\mathbf{k}_1, \eta_1) \cdots \psi(\mathbf{k}_n, \eta_n) \rangle' &\xrightarrow{q \rightarrow 0} \left\langle \exp \left[ \sum_i \int^{\eta_i} d\eta'_i \int^{\Lambda_L} \frac{\mathbf{k}_i \cdot \mathbf{p}}{p^2} \delta_L(\mathbf{p}, \eta'_i) \right] \right\rangle' \\ &\times \langle \psi_S(\mathbf{k}_1, \eta_1) \cdots \psi_S(\mathbf{k}_n, \eta_n) \rangle' , \end{aligned} \quad (2.10)$$

following the evaluation of the cumulants

$$\frac{\left\langle \psi_L(\mathbf{q}, \eta_q) \exp \left[ \sum_i \int^{\eta_i} d\eta'_i \int^{\Lambda_L} \frac{\mathbf{k}_i \cdot \mathbf{p}}{p^2} \delta_L(\mathbf{p}, \eta'_i) \right] \right\rangle'}{\left\langle \exp \left[ \sum_i \int^{\eta_i} d\eta'_i \int^{\Lambda_L} \frac{\mathbf{k}_i \cdot \mathbf{p}}{p^2} \delta_L(\mathbf{p}, \eta'_i) \right] \right\rangle'} = -P_L(q, \eta_q) \sum_i \frac{D(\eta_i)}{D(\eta_q)} \frac{\mathbf{k}_i \cdot \mathbf{q}}{q^2}, \quad (2.11)$$

with growth factor  $D(\eta)$ . The angular brackets denote ensemble averages, while the prime  $\langle \dots \rangle'$  indicates that the momentum conserving delta-function has been removed. This result reproduces the previously derived relations, e.g. [8], including in addition the velocity field (see (2.1)). Furthermore, it can be easily extended to account for different background cosmologies.

Notice that this derivation did not require more than the leading-order factorization of long modes in the squeezed limit. However, when  $\eta_q = \eta_i$  for all  $i$ , the right-hand side in (2.11) vanishes at leading order in  $q$  due to momentum conservation, such that a calculation to second order becomes unavoidable to obtain the behavior in the soft- $q$  limit at equal times. This is a crucial point in determining the validity of relations between correlation functions beyond linear order in such case.

### 3 The Bispectrum at Next-to-Leading Order

In order to assess the validity of consistency relations between correlation functions at *equal times*, we compute the NLO contribution to the power spectrum and bispectrum of density perturbations in SPT. We will then explore, as a first non-trivial check, the existence of a soft-limit connection between the bispectrum and power spectrum beyond the linear approximation. Note also that SPT predictions should agree with the exact solution within its realm of validity.<sup>4</sup>

To set up our notation, we introduce the two- and three-point correlation functions in terms of the power spectrum  $P_{ab}(k, \eta)$  and bispectrum  $B_{abc}(\mathbf{k}, \mathbf{q}, \mathbf{p}, \eta)$  by defining

$$\begin{aligned} \langle \psi_a(\mathbf{k}, \eta) \psi_b(\mathbf{q}, \eta) \rangle &\equiv \delta^D(\mathbf{k} + \mathbf{q}) P_{ab}(k, \eta) , \\ \langle \psi_a(\mathbf{k}, \eta) \psi_b(\mathbf{q}, \eta) \psi_c(\mathbf{p}, \eta) \rangle &\equiv \delta^D(\mathbf{k} + \mathbf{q} + \mathbf{p}) B_{abc}(\mathbf{k}, \mathbf{q}, \mathbf{p}, \eta) . \end{aligned} \quad (3.1)$$

Moreover, we write the linear power spectrum as  $P_{ab}^L(k, \eta) = u_a u_b P_L(k, \eta)$  with growing mode initial conditions  $u_a = (1, 1)$ . Thus, at leading order the relation between the angular averaged

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<sup>4</sup>The loop expansion in SPT requires techniques such as the effective field theory framework for LSS in order to properly account for the imprint of the short-distance physics. Therefore integrals that appear in SPT computations need to be regularized by introducing counter-terms [21, 22, 27–32]. This is even more relevant when dealing with the velocity field, which is a composite operator [22, 27, 30]. Nevertheless, one can always choose fictitious initial conditions for the power spectrum such that SPT converges quickly. The comparisons here may be then understood as mathematical statement at the level of the integrands, while judiciously choosing initial conditions such that the integrals are dominated by the modes well within the perturbative regime.

bispectrum in the soft limit and the power spectrum is given by [3]

$$B_{111}^L(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \left( \frac{47}{21} - \frac{1}{3} k \partial_k \right) P_L(k, \eta), \quad (3.2)$$

At NLO, or one-loop order, the bispectrum may be obtained in a straightforward manner using the standard techniques. However, the resulting expressions are too lengthy to allow for a meaningful analytic comparison. For ease of use, we then restrict ourselves to the limit where the loop momentum,  $l$ , is much larger than the external momenta,  $k$  or  $q$ . In this limit the angular averaged squeezed bispectrum may be thus written in the form

$$B_{111}^{1\text{-loop}}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \simeq P_L(q, \eta) \frac{k^2}{\pi^2} \left[ (\alpha k \partial_k + \beta) P_L(k, \eta) \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right) + \gamma k^2 \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^4} \right) \right], \quad (3.3)$$

with some numerical factors  $\alpha, \beta, \gamma$ . These coefficients can be obtained from the one-loop computation of the bispectrum in SPT after taking the squeezed limit and angular average, and we get

$$\alpha^{\text{SPT}} = \frac{61}{1890} \simeq 0.032, \quad \beta^{\text{SPT}} = -\frac{3719}{13230} \simeq -0.281, \quad \gamma^{\text{SPT}} = \frac{515}{5292} \simeq 0.097. \quad (3.4)$$

We collect a few more details on the SPT computation in appendix A. For completeness we also include the one-loop contribution to the power spectrum of density/velocity fluctuations in the limit of large loop momentum, which is given by

$$P_{ab}^{1\text{-loop}}(k, \eta) = \left( \frac{\frac{9}{196} \frac{19}{588}}{\frac{19}{588} \frac{61}{980}} \right) \frac{k^4}{\pi^2} \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^4} \right) - \left( \frac{\frac{61}{630} \frac{25}{126}}{\frac{25}{126} \frac{3}{10}} \right) \frac{k^2 P_L(k, \eta)}{\pi^2} \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right). \quad (3.5)$$

In the following sections, we will use these values as a benchmark for evaluating attempts to extend the result in (3.2).

## 4 Correlation Functions at Equal Times I: Time-Flow Approach

In this section, we study the time-flow approach, introduced in [20], to set up correlation functions in the squeezed limit. We will show that soft-limit relations may be derived, however, we will explicitly demonstrate for the case of the bispectrum that these are only valid at leading order. The reader may skip this section on a first read of the manuscript.

The time-flow approach consists of multiplying the fluid equations (2.2) by the fluctuations and take the statistical average. The  $\eta$ -evolution of the correlation functions can then be written as

$$\begin{aligned}
\partial_\eta \langle \psi_a \psi_b \rangle &= -\Omega_{ac} \langle \psi_c \psi_b \rangle - \Omega_{bc} \langle \psi_a \psi_c \rangle \\
&\quad + \gamma_{acd} \langle \psi_c \psi_d \psi_b \rangle + \gamma_{bcd} \langle \psi_a \psi_c \psi_d \rangle, \\
\partial_\eta \langle \psi_a \psi_b \psi_c \rangle &= -\Omega_{ad} \langle \psi_d \psi_b \psi_c \rangle - \Omega_{bd} \langle \psi_a \psi_d \psi_c \rangle - \Omega_{cd} \langle \psi_a \psi_b \psi_d \rangle \\
&\quad + \gamma_{ade} \langle \psi_d \psi_e \psi_b \psi_c \rangle + \gamma_{bde} \langle \psi_a \psi_d \psi_e \psi_c \rangle + \gamma_{cde} \langle \psi_a \psi_b \psi_d \psi_e \rangle, \\
\partial_\eta \langle \psi_a \psi_b \psi_c \psi_d \rangle &= \dots
\end{aligned} \tag{4.1}$$

To improve readability, we have omitted the momentum and time dependence of the correlation functions. All fields have to be evaluated at the same value of  $\eta$ . The procedure above produces an infinite hierarchy of evolution equations. Hence, the usefulness of the time-flow approach relies on finding a suitable ‘closure approximation’, as we will review next.

#### 4.1 Closure approximation

If we express the four-point correlation functions in terms of the power spectrum, bispectrum and trispectrum, defined as

$$\begin{aligned}
\langle \psi_a(\mathbf{k}, \eta) \psi_b(\mathbf{q}, \eta) \psi_c(\mathbf{p}, \eta) \psi_d(\mathbf{w}, \eta) \rangle &\equiv \delta^D(\mathbf{k} + \mathbf{q}) \delta^D(\mathbf{p} + \mathbf{w}) P_{ab}(k, \eta) P_{cd}(p, \eta) \\
&\quad + \delta^D(\mathbf{k} + \mathbf{p}) \delta^D(\mathbf{q} + \mathbf{w}) P_{ac}(k, \eta) P_{bd}(q, \eta) \\
&\quad + \delta^D(\mathbf{k} + \mathbf{w}) \delta^D(\mathbf{q} + \mathbf{p}) P_{ad}(k, \eta) P_{bc}(q, \eta) \\
&\quad + \delta^D(\mathbf{k} + \mathbf{p} + \mathbf{q} + \mathbf{w}) Q_{abcd}(\mathbf{k}, \mathbf{q}, \mathbf{p}, \mathbf{w}, \eta), \tag{4.2}
\end{aligned}$$

the closure approximation at this level consists of neglecting the trispectrum, namely setting  $Q_{abcd} \equiv 0$ . This then allows us to express the four-point correlation function in terms of power spectra. The flow equations (4.1) thus form a *closed* system, which can be formally solved as

$$\begin{aligned}
P_{ab}(k, \eta) &= g_{ac}(\eta, \eta_0) g_{bd}(\eta, \eta_0) P_{cd}(k, \eta_0) \\
&\quad + \int_{\eta_0}^{\eta} d\eta' \int d^3q g_{ae}(\eta, \eta') g_{bf}(\eta, \eta') \\
&\quad \times [\gamma_{ecd}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}) B_{fcd}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta') \\
&\quad + \gamma_{fcd}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}) B_{ecd}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta')], \tag{4.3}
\end{aligned}$$

and

$$\begin{aligned}
B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta) &= g_{ad}(\eta, \eta_0) g_{be}(\eta, \eta_0) g_{cf}(\eta, \eta_0) B_{def}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta_0) \\
&\quad + 2 \int_{\eta_0}^{\eta} d\eta' g_{ad}(\eta, \eta') g_{be}(\eta, \eta') g_{cf}(\eta, \eta') \\
&\quad \times [\gamma_{dgh}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}) P_{eg}(q, \eta') P_{fh}(|\mathbf{q} - \mathbf{k}|, \eta') \\
&\quad + \gamma_{egh}(-\mathbf{q}, \mathbf{q} - \mathbf{k}, \mathbf{k}) P_{fg}(|\mathbf{q} - \mathbf{k}|, \eta') P_{dh}(q, \eta') \\
&\quad + \gamma_{fgh}(\mathbf{q} - \mathbf{k}, \mathbf{k}, -\mathbf{q}) P_{dg}(k, \eta') P_{eh}(q, \eta')], \tag{4.4}
\end{aligned}$$

where  $\eta_0$  corresponds to the initial time.

## 4.2 The soft limit of the bispectrum

Provided the closure approximation holds, one may derive an angular averaged consistency relation for the soft limit  $q \rightarrow 0$  of the bispectrum  $B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}}$ , and in principle for general classes of cosmologies. We take Gaussian initial conditions such that the initial bispectrum vanishes. Under these assumptions the expression in (4.4) simplifies to

$$\begin{aligned} B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta) &= 2 \int_{\eta_0}^{\eta} d\eta' g_{ad}(\eta, \eta') g_{be}(\eta, \eta') g_{cf}(\eta, \eta') \\ &\times [\gamma_{dgh}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}) P_{eg}(q, \eta') P_{fh}(|\mathbf{q} - \mathbf{k}|, \eta') \\ &+ \gamma_{egh}(-\mathbf{q}, \mathbf{q} - \mathbf{k}, \mathbf{k}) P_{fg}(|\mathbf{q} - \mathbf{k}|, \eta') P_{dh}(k, \eta') \\ &+ \gamma_{fgh}(\mathbf{q} - \mathbf{k}, \mathbf{k}, -\mathbf{q}) P_{dg}(k, \eta') P_{eh}(q, \eta')] . \end{aligned} \quad (4.5)$$

Next, we expand all quantities in (4.5) which depend on the difference between the soft and the hard modes,  $|\mathbf{q} - \mathbf{k}|$ , in a perturbative series up to first order in  $q$ . Inserting the series expansion of the power spectrum,

$$P_{ab}(|\mathbf{q} - \mathbf{k}|, \eta) \simeq P_{ab}(k, \eta) - q \mu \partial_k P_{ab}(k, \eta) + \mathcal{O}(q^2) , \quad (4.6)$$

with  $\mu \equiv (\mathbf{q} \cdot \mathbf{k}) / (qk)$ , we then obtain<sup>5</sup>

$$\begin{aligned} B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta) &= 2 \int_{\eta_0}^{\eta} d\eta' g_{ad}(\eta, \eta') g_{be}(\eta, \eta') g_{cf}(\eta, \eta') P_{eh}(q, \eta') \\ &\times \left[ \gamma_{dhg}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}) \left( P_{fg}(k, \eta') - \mu q \partial_k P_{fg}(k, \eta') \right) + \gamma_{fgh}(\mathbf{q} - \mathbf{k}, \mathbf{k}, -\mathbf{q}) P_{dg}(k, \eta') \right] . \end{aligned} \quad (4.7)$$

Furthermore, in the limit  $q \rightarrow 0$  the bracket becomes

$$\begin{aligned} &\left[ \delta_{h1} \left( M_{dg}^A P_{gf}(k, \eta') + P_{dg}(k, \eta') M_{gf}^A \right) \right. \\ &\left. + \delta_{h2} \left( M_{dg}^B P_{gf}(k, \eta') + P_{dg}(k, \eta') M_{gf}^B - \frac{1}{2} (1 + \mu^2 k \partial_k) P_{df}(k, \eta') \right) \right] , \end{aligned} \quad (4.8)$$

with

$$M^A = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} , \quad M^B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \mu^2 \end{pmatrix} . \quad (4.9)$$

Performing the angular average and taking the linear approximation for the long modes, i.e.  $P_{ab}(q, \eta) \simeq u_a u_b P_L(q, \eta)$  with  $u_a = (1, 1)$ , we obtain

$$\begin{aligned} B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} &\xrightarrow{q \rightarrow 0} u_b P_L(q, \eta) \int_{\eta_0}^{\eta} d\eta' g_{ad}(\eta, \eta') g_{cf}(\eta, \eta') e^{-(\eta - \eta')} \\ &\times \left[ M_{dg}^{\text{av}} P_{gf}(k, \eta') + P_{dg}(k, \eta') M_{gf}^{\text{av}} - \left( 1 + \frac{1}{3} k \partial_k \right) P_{df}(k, \eta') \right] , \end{aligned} \quad (4.10)$$

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<sup>5</sup>Notice that the second term in (4.5) is linear in  $q$  and may be ignored.

and

$$M^{\text{av}} = 2 [M^A + M^B]^{\text{av}} = \begin{pmatrix} 1 & 1 \\ 0 & \frac{2}{3} \end{pmatrix}, \quad (4.11)$$

provided  $Q_{abcd} = 0$ . We would like to emphasize that up to this point we neither specified the background cosmology nor did we perform any perturbative expansion.

This expression readily reproduces the linear results. For example, using an EdS background and choosing  $\eta_0 = 0$ , we get at leading order<sup>6</sup>

$$B_{abc}^L(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) u_b \left( \frac{1}{21} \begin{pmatrix} 47 & 39 \\ 39 & 31 \end{pmatrix}_{ac} - \frac{1}{3} u_a u_c k \partial_k \right) P_L(k, \eta), \quad (4.12)$$

which coincides with the known result [3] for the component  $B_{111}^L$  quoted in (3.2). Note, at the same time, (4.12) is a generalization of [3] which includes the velocity field.

While the result above is a generalization of the expression found in the literature at leading order, our actual aim was an expression of soft-limit relations which are valid in the non-perturbative regime. The previous relations, however, relies on the closure approximation. Although at linear order this is guaranteed by factorization, there is in principle no reason to believe this is a reasonable approach at higher orders or, more ambitiously, at non-perturbative level. In fact, the trispectrum obeys an evolution equation that cannot be consistently set to zero at all times, despite what is often done in the literature, e.g. [20]. Therefore, it is important to assess to what extent the result (4.7) contains non-linear information. Using our results in section 3, we can already check these relations at NLO. For that purpose, we insert the expression for the power spectrum at one-loop order in (3.5) into the right hand side of (4.10) for the short modes. Consequently, we find that the time-flow approach leads to

$$\alpha^{\text{TF}} = \frac{103}{6930} \simeq 0.015, \quad \beta^{\text{TF}} = -\frac{233}{1890} \simeq -0.123, \quad \gamma^{\text{TF}} = \frac{271}{19404} \simeq 0.014. \quad (4.13)$$

These numbers differ significantly from the SPT results in (3.4). Hence, we conclude that the trispectrum cannot be ignored when computing the bispectrum beyond linear perturbation theory, which was the sole assumption on the way to derive (4.10).

## 5 Correlation Functions at Equal Times II: Locally Curved Universe

We now move to an alternative approach to set up equal-time relations in the soft limit based on ideas introduced in [2, 3], as well as the proposal for (angular averaged) ‘equal-time consistency conditions’ of [16] and [17]. We follow closely the analysis in [2, 3], to which we urge the reader to consult for further details. At the same time, we present a shortened derivation of the relevant transformation(s) that avoids using the Fermi coordinates and directly matches a flat coordinate system with a soft mode to a (locally) curved one.

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<sup>6</sup>The linearized form of the expression in (4.10) also allows for different background cosmologies.

## 5.1 Newtonian mapping

We start from a FRW cosmology in global coordinates that includes a long-wavelength (soft) perturbation,  $\Phi_L$ . In the Newtonian approximation, we then have

$$ds^2 = -[1 + 2\Phi_L(\mathbf{x}, t)] dt^2 + a^2(t)[1 - 2\Phi_L(\mathbf{x}, t)] d\mathbf{x}^2. \quad (5.1)$$

Assuming that the perturbation is spherically symmetric, the Newton potential is related to the density contrast by Poisson's equation,<sup>7</sup>

$$\Phi_L(\mathbf{x}, t) \simeq \frac{1}{4} H^2 a^2 \delta_L \mathbf{x}^2 \ll 1. \quad (5.2)$$

We will search for a coordinate transformation of the form

$$\begin{aligned} t &= t_K + f(t_K, \mathbf{x}_K), \\ \mathbf{x} &= \mathbf{x}_K(1 + g(t_K, \mathbf{x}_K)), \end{aligned} \quad (5.3)$$

with coordinate dependent functions  $f$  and  $g$ , such that one can transform to a locally curved system where the metric (in isotropic coordinates) takes the form

$$ds^2 = -dt_K^2 + a_K^2(t_K) \frac{d\mathbf{x}_K^2}{\left(1 + \frac{1}{4} K \mathbf{x}_K^2\right)^2}, \quad (5.4)$$

and  $K$  correspond to the curvature. The time-time component of the metric enforces

$$\dot{f} = -\Phi_L, \quad (5.5)$$

while the vanishing time-space part yields

$$a^2 \dot{g} \mathbf{x} = \nabla f. \quad (5.6)$$

Furthermore, the scale factor also transforms<sup>8</sup>

$$a(t) = a(t_K) (1 + H f), \quad (5.7)$$

and contributes to the curvature in the new coordinate system. In total, we find

$$\begin{aligned} a_K &= a(1 + g), \\ K \mathbf{x}^2 &= 4\Phi_L + 4H \int dt \Phi_L. \end{aligned} \quad (5.8)$$

---

<sup>7</sup>As it was shown in [6, 8], the constant and gradient pieces of the potential can be removed by a change of coordinates to a free-falling frame. Such transformation also leads to the consistency conditions we re-derived previously in section 2. Since we are interested in the *physical* squeezed limit, in this section we will only deal with the quadratic part of the potential,  $\Phi_L \propto \mathbf{x}^2$ , and the equivalence with a locally curved universe discussed in [2].

<sup>8</sup>Notice that we are dealing with a subclass of possible perturbations, i.e. soft and spherically symmetric. Our ansatz is valid provided  $\delta_L \ll 1$  and  $H^2 \mathbf{x}^2 \ll 1$ , such that  $f \simeq \mathbf{x}_K^2$  and  $g \simeq \mathbf{x}_K^0$ . Otherwise, spatial gradients of  $\delta_L$  are involved and the analysis has to be modified. However, these type of perturbations are sufficient for our purposes to study the squeezed limit of correlators.

In particular, the Hubble parameters in the two systems are related by

$$H_K \equiv \frac{\dot{a}_K}{a_K} \simeq H + \dot{g} = H - \frac{1}{2a^2} \int dt a^2 H^2 \delta_L. \quad (5.9)$$

In EdS, the integrand is approximately constant in time (since  $H^2 \propto a^{-3}$ ,  $\delta_L \propto a$ ) and we get

$$H_K \simeq H \left( 1 - \frac{1}{3} \delta_L \right), \quad (5.10)$$

$$a_K \simeq a \left( 1 - \frac{1}{3} \delta_L \right), \quad (5.11)$$

$$K \simeq \frac{5}{3} H^2 a^2 \delta_L. \quad (5.12)$$

Note that these results are consistent with the Friedman equations in the curved coordinate system,

$$H_K^2 = H^2 \frac{a^3}{a_K^3} - \frac{K}{a_K^2}. \quad (5.13)$$

Moreover, while the energy density in the curved system is larger

$$\bar{\rho}_K = \bar{\rho} (1 + \delta_L) = \bar{\rho} \frac{a^3}{a_K^3}, \quad (5.14)$$

the physical Hubble rate  $H_K$  in (5.10) is not, since the curvature over-compensates the density increase.

## 5.2 The non-perturbative (physical) soft limit

We now turn on the short scale modes to study the imprint of a long-wavelength perturbation in their dynamics. These will be affected by the coordinate transformation through the change in expansion rate (5.11) and the additional ‘contraction’  $(1 + g)$  of the spatial coordinates (see (5.3)). In particular, the density contrast of a hard mode transforms in real space as

$$\delta_K(\mathbf{x}_K, a_K) = \delta(\mathbf{x}, a) (1 - \delta_L), \quad (5.15)$$

so that the corresponding two-point correlation function, defined as  $\xi(r) \equiv \langle \delta(\mathbf{x}) \delta(\mathbf{x} + \mathbf{r}) \rangle$ , reads

$$\begin{aligned} \xi_{\delta_L}(r, a) &= \left[ 1 + \delta_L \left( 2 + \frac{1}{3} r \partial_r \right) \right] \xi_K(r, a_K) \\ &= \left[ 1 + \delta_L \left( 2 + \frac{1}{3} r \partial_r - \frac{1}{3} \partial_\eta \right) \right] \xi_K(r, a). \end{aligned} \quad (5.16)$$

Here,  $\xi_K(r, \eta)$  is the correlation function in the curved coordinate system, and we used again  $r_K = r(1 - g)$ ,  $a_K = a(1 + g)$  together the EdS result  $g = -\delta_L/3$ .

The local curvature also affects the growth of structure. In the case of an EdS universe, for instance, the linear growing mode of the density contrast becomes

$$\begin{aligned} D(K, a_K) &\simeq D(0, a) + K \frac{d}{dK} D(K, a_K) \Big|_{K=0} \\ &= \left( 1 + \frac{13}{21} \delta_L \right) D(0, a). \end{aligned} \quad (5.17)$$

This means at linear order we may replace

$$\xi_K^{(1)}(r, a_K) \rightarrow \left(1 + 2 \times \frac{13}{21} \delta_L\right) \xi^{(1)}(r, a), \quad (5.18)$$

where  $\xi(r, \eta)$  represents the correlation function in the original (flat) frame without the soft mode.

The combination of all the factors in (5.16) and (5.18), when correlated with  $\delta_L$ , gives the relation in the soft limit of the bispectrum at leading order, which was originally derived by Sherwin and Zaldarriaga in [3] and coincides with (3.2) (see also (4.12)). However, as advocated in [2], absorbing the soft mode into a curved background is a correct procedure also at the non-perturbative level. For example, this has led to a physical squeezed limit for correlation functions in the inflationary case [15]. Following similar steps as in [15], one can then write for the density perturbations in an EdS cosmology,

$$B_{111}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \left[ \left(1 - \frac{1}{3} k \partial_k - \frac{1}{3} \partial_\eta\right) P(k, \eta) + \frac{5}{3} \frac{\partial}{\partial \kappa} P_K(k, \eta) \Big|_{K=0} \right], \quad (5.19)$$

where  $\kappa = K/(a^2 H^2)$  and  $P_K(k, \eta)$  is the power spectrum for the density contrast in the presence of (local) curvature. This expression follows simply from (5.16), using the (linear) map between  $K$  and  $\delta_L$  given by (5.12).

The relation in (5.19), although generic, is not the type one would expect to confront against data, since it involves a correlation function in a hypothetical case of a curved universe. Moreover, the impact of curvature on the fluctuations, i.e.  $\partial P_K(k, \eta)/\partial K$ , cannot be readily obtained in terms of quantities at  $K = 0$  without resorting to perturbation theory. The work of [16, 17], on the other hand, is an attempt at precisely achieving this. We will study their proposal next.

### 5.3 The ‘equal-time consistency relations’

A proposal by Valageas, and also Kehagias, Perrier and Riotto (VKPR) to extend the computation in [3] into the non-linear regime appeared in [16] and [17], where it was coined the (angular averaged) ‘equal-time consistency conditions’ for large-scale structure. In practice, it consists of replacing each growth function for the short modes with a factor of (5.17), yielding

$$\xi_{\delta_L}^{\text{VKPR}}(r, \eta) = \left[1 + \delta_L \left(2 + \frac{1}{3} r \partial_r + \frac{13}{21} \partial_\eta\right)\right] \xi(r, \eta), \quad (5.20)$$

or in Fourier space, after correlating with a long-wavelength mode,

$$B_{111}^{\text{VKPR}}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \left[2 - \frac{1}{3} (3 + k \partial_k) + \frac{13}{21} \partial_\eta\right] P(k, \eta), \quad (5.21)$$

with the shortened notation  $P(k, \eta) \equiv P_{11}(k, \eta)$  for the density power spectrum. (Recall  $\eta \equiv \ln a(\tau)$ , and therefore the logarithmic derivative serves as a counter.) The claim in [16, 17] is that this expression is exact in EdS, hence also valid to all orders in SPT, while for other cosmologies its accuracy would rely only on the validity of replacing  $a(\eta) \rightarrow D(\eta)$  in the SPT expressions.

As a first step, we can check the validity of (5.21) at NLO by inserting the one-loop power spectrum (see (3.5)) on the right hand side of (5.21). For the coefficients defined in section 3 we consequently find

$$\alpha^{\text{VKPR}} = \frac{61}{1890} \simeq 0.032, \quad \beta^{\text{VKPR}} = -\frac{3599}{13230} \simeq -0.272, \quad \gamma^{\text{VKPR}} = \frac{135}{1372} \simeq 0.098, \quad (5.22)$$

which overall differ from the SPT results. We notice, nonetheless, that  $\alpha^{\text{VKPR}} = \alpha^{\text{SPT}}$ . This must be indeed the case since it comes from the eikonal phase in (2.8), which we argued is universal.

It is clear from our previous analysis that the subtle step in the derivation is the replacement

$$\xi_K^{\text{VKPR}}(r, a_K) = \left(1 + \frac{13}{21} \delta_L \partial_\eta\right) \xi(r, a), \quad (5.23)$$

that aims at generalizing (5.18). As we see in the comparison, this replacement is not valid at higher orders even in an EdS background, since it fails to capture relevant dynamics.

We can already start to see the seed of the problem in the perturbative expansion for the density field itself before computing the correlation with the long mode. As discussed in [2], a simple exercise consists of using SPT to compute the density perturbation to a given order and check whether the above replacement correctly accounts for the effect of the long mode.

This was performed directly in [3] up to second order, i.e.  $\delta^{(2)}(\mathbf{x}, \eta)$ , where one finds

$$\delta_{S, \delta_L}(\mathbf{x}, \eta) = \delta_S^{(1)}(\mathbf{x} + \mathbf{d}(\mathbf{x}, \eta), \eta) + \frac{34}{21} \delta_L^{(1)}(\mathbf{x}, \eta) \delta_S^{(1)}(\mathbf{x}, \eta) + \frac{4}{7} K_{ij}^L(\mathbf{x}, \eta) K_{ij}^S(\mathbf{x}, \eta) + \dots, \quad (5.24)$$

where the ellipses represent higher-order terms in SPT. The displacement and anisotropy terms are defined as

$$\mathbf{d}(\mathbf{x}, \eta) \equiv - \int_{\mathbf{q}} \frac{i\mathbf{q}}{q^2} \delta^{(1)}(\mathbf{q}, \eta) e^{i\mathbf{q} \cdot \mathbf{x}}, \quad K_{ij}(\mathbf{x}, \eta) \equiv \int_{\mathbf{q}} \left( \frac{\mathbf{q}_i \mathbf{q}_j}{q^2} - \frac{1}{3} \delta_{ij} \right) \delta^{(1)}(\mathbf{q}, \eta) e^{i\mathbf{q} \cdot \mathbf{x}}, \quad (5.25)$$

with  $\int_{\mathbf{q}} \equiv \int \frac{d^3\mathbf{q}}{(2\pi)^3}$ . Note that we have resummed the effects of the displacement to all orders, which follows from the eikonal approximation in (2.8), or can be also shown directly in Lagrangian space [27]. From here it is straightforward to derive the soft limit of the bispectrum by computing

$$\langle \xi_{\delta_L}(r, \eta) \delta_L \rangle = \langle \delta_{S, \delta_L}(\mathbf{x} + \mathbf{r}, \eta) \delta_{S, \delta_L}(\mathbf{x}, \eta) \delta_L \rangle, \quad (5.26)$$

and averaging over angles. The piece from the displacement reproduces the factor of  $\frac{1}{3} r \partial_r \xi(r)$  in (5.20), while the  $(1 + \frac{13}{21})$  (twice) gives the correct factor of  $\frac{68}{21}$  found in [3]. It was crucial in this derivation that the anisotropy term did not contribute upon angular averaging,  $\langle \delta_L(\mathbf{x}, \eta) K_{ij}^L(\mathbf{x}, \eta) \rangle^{\text{av}} \rightarrow 0$ . This is in fact what follows for spherically symmetric density perturbations and is accomplished by the angular integral.

At NLO, the validity of the proposal in [16, 17] requires that the density perturbation may be written as

$$\delta_{S, \delta_L}(\mathbf{x}, \eta) \supset \delta_S^{(2)}(\mathbf{x} + \mathbf{d}(\mathbf{x}, \eta), \eta) + \frac{47}{21} \delta_L^{(1)}(\mathbf{x}, \eta) \delta_S^{(2)}(\mathbf{x}, \eta) + \frac{4}{7} K_{ij}^L(\mathbf{x}, \eta) A_{ij}^S(\mathbf{x}, \eta) + \dots, \quad (5.27)$$

where the ellipses would include higher-order terms. The factor 47/21 would be a consequence of  $\delta^{(2)} \propto D(\eta)^2$ , as dictated by the expression in [16, 17]. We also collected the anisotropy piece into

the term containing the matrix  $A_{ij}^S$ , which vanishes once the soft mode is averaged over angles. However, as expected from the one-loop check of the explicit bispectrum calculation, this relation is not fulfilled. In fact, after averaging, we find an additional piece (see (5.35)) that leads to the (small) discrepancy between the  $\beta, \gamma$  coefficients computed in SPT and the proposal of [16, 17].

Albeit formally not valid at one-loop order, the errors in the coefficients  $\beta$  and  $\gamma$  are small. For instance including up to fourth order, i.e.  $\delta^{(4)}(\mathbf{x}, \eta)$ , the calculation shows that the claim in [16, 17] for the form of the density perturbation is quantitatively (very) close, but not quite the same as the one we expect from the SPT result.<sup>9</sup> We will elaborate on the reasons behind the discrepancy in section 5.5. However, we can already start by identifying plausible roots, in particular the impact of the velocity field, as we discuss next.<sup>10</sup>

#### 5.4 The impact of the velocity

To be precise, the relation (5.17) only holds for the growing mode of the density contrast, but not for the velocity. As we will discuss next, the response of the velocity fields to the presence of a long-wavelength mode is different.

The dependence of the growing mode on curvature comes from two sources

$$\frac{dD(K, a_K)}{d\delta_L} = \frac{\partial D}{\partial K} \frac{dK}{d\delta_L} + \frac{\partial D}{\partial a_K} \frac{da_K}{d\delta_L}. \quad (5.28)$$

While the first term contributes 20/21 and describes how the growing mode as a function of  $a$  (or  $\eta$ ) is modified by curvature, the second term gives  $-1/3$  and accounts for the change in the scale factor when the growing mode is compared at the same proper time. In total, the growing mode of the density contrast increases by a factor  $(1 + 13/21\delta_L)$  in the locally curved system [2, 3]. Transforming back using (5.15) yields that the density growth is enhanced by a factor  $(1 + 34/21\delta_L)$  in presence of a soft mode.

Repeating the same calculation for the velocity field, one finds that the growing mode of  $\Theta$ , defined in (2.1), responds differently to curvature. In particular,  $\partial D_\Theta / \partial K$  is twice as large. (Note that this is also required by the continuity equation.) Hence, the velocity field grows faster by a factor  $(1 + 33/21\delta_L)$  in the curved universe. In order to obtain the final expression for the change in  $\Theta$  we also need to include the effect of the coordinate transformation. First, we note that the physical velocity  $\mathbf{v}$  is the same in the two systems, and therefore  $\Theta$  transforms as

$$\Theta_K(\mathbf{r}_K, a_K) = \Theta(\mathbf{r}, a) (1 + \delta_L/3). \quad (5.29)$$

In total, the velocity then grows faster by a factor  $(1 + 26/21\delta_L)$  in presence of a soft mode, which is consistent with the explicit SPT calculation (for example  $2 \times 26/21 - 1 = 31/21$ , c.f. (4.12)).

<sup>9</sup>Notice that although the errors in the coefficients  $\beta$  and  $\gamma$  are small, they are still multiplied by an integral which, as we emphasized, needs to be regularized. This means that the discrepancy may be ultimately large (or even diverge) for initial conditions where the integrals are dominated by the hard part of the spectrum.

<sup>10</sup>Regarding the assumption of spherical symmetry, one may be worried that anisotropy terms could survive after angular averaging. Even though we are taking the soft limit  $q \rightarrow 0$ , the angular dependence coming from the soft mode remains because of the  $1/q^2$  enhancement from the eikonal phase, e.g.  $\frac{(\mathbf{k} \cdot \mathbf{q})^2}{q^2}$ . However, due to factorization, at equal times the anisotropy terms vanish upon averaging. Or in other words, we do not encounter singularities of the form  $1/q^4$  at equal times, since the contributions from the eikonal phase cancel each other and the remaining terms are analytic in  $q$ . This can be explicitly checked up to NNLO.

This result has important consequences. First of all, it means that a relation similar to (5.21) which includes the velocities may be (naively) written down through an extension of the linear result (including the other components of (4.12)) by use of the above transformations. Following the steps in [16, 17] translated to the velocity one would then write

$$\Theta_{S,\delta_L}(\mathbf{x},\eta) \supset \Theta_S^{(n)}(\mathbf{x} + \mathbf{d}(\mathbf{x},\eta),\eta) + \delta_L^{(1)}(\mathbf{x},\eta) \left( -\frac{1}{3} + \frac{33}{21} \partial_\eta \right) \Theta_S^{(n)}(\mathbf{x},\eta) + \dots \quad (5.30)$$

Here the ellipses would represent terms which either vanish when an angular average is performed or are higher order.<sup>11</sup> However, one can show that the expression in (5.30) dramatically fails beyond linear order (see below). This is already a signal that one has to carefully account for the impact of the velocities in the SPT computations.

### 5.5 Fluid perturbations in a curved background

In the standard SPT manipulations, one replaces  $\delta^{(1)}$  by  $\Theta^{(1)}$ . However, as we just showed, the velocity and density fields respond differently to the presence of a long-wavelength perturbation. Unfortunately, there is no easy way to deduce the dependence of the non-linear power spectrum on the two growing and decaying modes of  $\psi$ , separately. This would be necessary to account for the different curvature effects in each component. In principle, one would then expect a significant departure from the relation in (5.21) beyond linear order. In practice, on the other hand, what the above computations unravel (at least perturbatively) is that, for the case of the density field, the relation in (5.21) qualitatively reproduces the SPT result. In order to gain some intuition behind this small discrepancy, we will next inspect in more detail the dynamics of fluid fluctuations in a curved background in perturbation theory.

Consider the fluid equations in (2.2) for the short modes in a curved background. Adding curvature to the cosmology modifies the matrix  $\Omega_{ab}$ , and at leading order in  $K \propto \delta_L$  the corresponding contribution for an EdS cosmology reads

$$\Omega_{K,ab} \simeq \Omega_{K=0,ab} + K \frac{\partial}{\partial K} \Omega_{K,ab} \Big|_{K=0} = \Omega_{ab} + \kappa \begin{pmatrix} 0 & 0 \\ -3/2 & -1/2 \end{pmatrix}. \quad (5.31)$$

The additional piece will modify the propagator in a fully non-perturbative treatment. However, since the long-wavelength mode may be treated within perturbation theory, at leading order in  $\delta_L$  it suffices to treat the effect of curvature as an extra interaction. Note that this extra term is time dependent since  $\kappa \propto a(\eta)$ . (The curvature,  $K$ , is on the other hand time independent.)

At linear order in the fluctuations for the short modes we consequently have

$$\begin{aligned} \psi_K^{(1)}(\mathbf{k}_1, \eta) &\simeq \psi_{K=0}^{(1)}(\mathbf{k}_1, \eta) + K \frac{\partial}{\partial K} \psi_K^{(1)}(\mathbf{k}_1, \eta) \Big|_{K=0} \\ &= \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{4\kappa}{7} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right] e^{\eta-\eta_0} \delta^{(1)}(\mathbf{k}_1, \eta_0). \end{aligned} \quad (5.32)$$

Note, as we mentioned before, the impact of curvature on the velocity is twice as large as the response of the density field.

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<sup>11</sup>Notice we could also replace  $\delta_L^{(1)} \rightarrow \Theta_L^{(1)}$ .

In general there are two contributions to the  $n$ -th order solution,  $\psi_K^{(n)}$ , at linear order in  $K$ . The first arises from the additional interaction (just as the leading-order term)

$$K \int d\eta' g_{ab}(\eta, \eta') \frac{\partial}{\partial K} \Omega_{bc}(\eta') \psi_c^{(n)}(\mathbf{k}_1, \eta'), \quad (5.33)$$

while the second contribution stems from the  $K$ -dependence in the lower order solutions that enter in the SPT recursion relation

$$K \int d\eta' g_{ab}(\eta, \eta') \gamma_{bcd}(\mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2) \frac{\partial}{\partial K} \left[ \psi_{K,c}^{(n-m)}(\mathbf{k}_1, \eta') \psi_{K,d}^{(m)}(\mathbf{k}_2, \eta') \right]. \quad (5.34)$$

It is important to stress that, already at NLO,  $\partial \psi_K^{(2)}(\mathbf{k}, \eta) / \partial K$  is not *proportionally* related to  $\psi_{K=0}^{(2)}(\mathbf{k}_1, \mathbf{k}_2)$ . This is due to the different momentum dependence and the SPT kernels. Nonetheless, we find the following relation,

$$\begin{aligned} K \frac{\partial}{\partial K} \psi_K^{(2)}(\mathbf{k}, \eta) \Big|_{K=0} &= \frac{4\kappa}{7} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \psi_{K=0}^{(2)}(\mathbf{k}, \eta) \\ &+ \frac{\kappa}{147} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \int d\mathbf{k}_1 d\mathbf{k}_2 \delta^3(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 - k_1^2 k_2^2}{k_1^2 k_2^2} e^{2(\eta - \eta_0)} \delta^{(1)}(\mathbf{k}_1, \eta_0) \delta^{(1)}(\mathbf{k}_2, \eta_0). \end{aligned} \quad (5.35)$$

Hence, the overall difference with the expression in (5.21) for the density contrast (first entry) is rather small. At the same time, we see that the extension for the velocities in (5.30) fails (compare with second entry). Notice that the argument presented here in terms of an SPT calculation is quite different than the original claims in [16, 17]. As we discussed previously, their proposal assumes, in practice, that density and velocity will react similarly to a background curvature. We noted, however, that this ignores the additional time dependence in the  $\kappa$ -dependent term in (5.34), as well as the dependence on curvature in (5.33).

Even though we conclude that the curvature dependence cannot be naturally reformulated in terms of a derivative with respect to  $\eta$ , as suggested by the proposal in [16, 17], we find that (extending the result in (5.35))

$$K \frac{\partial}{\partial K} \psi_K^{(n)}(\mathbf{k}_K, \eta) \Big|_{K=0} \simeq \frac{4\kappa}{7} \begin{pmatrix} \partial_\eta & 0 \\ 0 & \partial_\eta + 1 \end{pmatrix} \psi_{K=0}^{(n)}(\mathbf{k}_K, \eta), \quad (5.36)$$

gives a reasonable *empirical* perturbative approximation, which translates into

$$\frac{\partial}{\partial \delta_L} \psi_{\delta_L}^{(n)}(\mathbf{k}, \eta) \Big|_{\delta_L=0} \simeq \frac{1}{21} \begin{pmatrix} 21 + 13 \partial_\eta & 0 \\ 0 & 13 + 13 \partial_\eta \end{pmatrix} \psi_{K=0}^{(n)}(\mathbf{k}_K, \eta). \quad (5.37)$$

This expression agrees with the proposal in [16, 17] *only* for the density field, which we already argued approximates the SPT result with an error of a few percent. On the other hand, for the velocities, while an extension along the arguments in [16, 17] fails, the above empirical relation is accurate roughly up to an error of 25% in SPT. Unfortunately, at this point we cannot assess the validity of this expression at the non-perturbative level.

## 6 Discussion

In this work, we studied correlators of the density and velocity fields in the soft limit. We first (re-)derived the well known consistency conditions at unequal times using the eikonal approximation, which naturally accounts for the resummation of the soft mode. This provides a compact expression that also generalizes to include the velocity field.

Afterwards, we explored under which circumstances equal-time relations exist. For that purpose we computed the bispectrum beyond leading order in SPT to assess two approaches that aim at deriving (allegedly non-perturbative) expressions. The first is based on the time-flow approach in the Eulerian representation. In this scheme, the connected four-point function, namely the trispectrum, is neglected with questionable success in computations involving the power spectrum [20, 33]. An attempt at including the information from the trispectrum appeared in [34], which points towards a non-negligible contribution already in the mildly non-linear regime. In principle, one could have hoped that the trispectrum was less relevant in the soft limit, and equal-time relations from the time-flow approach may be approximately accurate. Unfortunately, this is not the case and we found large deviations from SPT already at one-loop order for the bispectrum. Including the trispectrum and truncating the hierarchy at higher orders would reproduce the one-loop SPT result, however, it will fail at some given loop order depending on the truncation. Therefore, a truly non-perturbative result seems out of reach in the time-flow formalism.

Overall, in spite of not holding up to the same status as the unequal-time relations, perturbative statements between correlation functions at equal times in the squeezed limit may be still useful in special circumstances in which the short modes may be kept in the mildly non-linear regime. The perturbative relations of the time-flow approach are well suited, for instance, to study baryonic acoustic oscillations in a background cosmology that requires numerical input, e.g. models including massive neutrinos or quintessence. Compared to SPT, the time-flow approach only deals with equal-time quantities, so that the soft effects cancel from the outset. In such case, these relations may improve numerical stability and aid the computational treatment of the fluctuations.

We then moved to the curved background method and the (angular averaged) ‘equal-time consistency conditions’ of Valageas, and also Kehagias, Perrier and Riotto [16, 17]. Their proposal was an attempt to use an equivalence discussed in [2], between physics on short scales in the presence of a long-wavelength perturbation and a locally curved cosmology, to extend the leading result in [3] into the non-linear regime. In [2] it was argued that the physical equivalence applies even when the short modes are deep in the non-linear regime. As we discussed, this equivalence relates, for example, the bispectrum of density fluctuations in the soft limit to the variation of the power spectrum on short scales in the presence of local curvature  $K$ ,

$$B(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \left[ \left( 1 - \frac{1}{3} k \partial_k - \frac{1}{3} \partial_\eta \right) P(k, \eta) + \frac{5}{3} \frac{\partial}{\partial \kappa} P_K(k, \eta) \right]_{K=0}, \quad (6.1)$$

with  $\kappa = K/(a^2 H^2)$  (see (5.19)). The relation (6.1) can be shown to be universal, hence valid non-perturbatively. The term involving  $(1 - \frac{1}{3} k \partial_k)$  is a combination of two effects, namely, the

difference in the density contrast between the two cosmologies plus the shift induced by the displacement term [27] (or the eikonal phase); while the contribution  $\frac{1}{3}\partial_\eta$  follows from the change in the scale factor. It is not, however, an expression that can be directly confronted with data, since it involves the dependence of the power spectrum on a hypothetically (locally) curved universe through the last term. The proposal in [16, 17] may be then rephrased as an attempt to replace the variation with respect to curvature with quantities which can be computed when  $K = 0$ , more specifically

$$\text{VKPR : } \left. \frac{\partial}{\partial \kappa} P_K(k, \eta) \right|_{K=0} = \frac{4}{7} \partial_\eta P_{K=0}(k, \eta). \quad (6.2)$$

However, we explicitly demonstrated that the expression (6.2), which can be inferred from [16, 17], does not properly account for the effect of local curvature on the growth of structure. As we argued, this is due to the impact of the velocity field, which responds to curvature differently than the density contrast. On the other hand, albeit significantly off when applied to the velocities, we found that (6.2) for the density fluctuations is quantitatively accurate in the realm of SPT, to the few-percent level. This was confirmed by a one-loop check of the bispectrum, and also by explicitly computing the density fluctuations in SPT (see (5.36).) We find that the small discrepancy may be then associated with almost-cancelations in the SPT computations in the (angular averaged) soft limit. We cannot assess at this point whether the approximate accuracy of (6.2) may be translated into the non-perturbative regime.<sup>12</sup>

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<sup>12</sup>As discussed in [16], exact relations can be found in a simplified (1+1 dimensional) toy model, whose background equations resemble an EdS cosmology. However, in this example it is easy to see that the response to a soft mode is given by  $\psi_{\delta_L} \simeq [1 + \delta_L(1 + \partial_\eta)] \psi$ ; and moreover, the Zel’dovich approximation [35] is exact with fluctuations always remaining in the growing mode. Hence, both density and velocity respond equally to curvature. Unfortunately this does not provide any additional hindsight for the approximate validity of (6.2). On the other hand, an attempt at testing this proposal against numerical simulations in the non-linear regime appeared in [36]. There, small departures are also found which are consistent with our findings. However, other sources of errors appear due to the breakdown of the approximation of an EdS cosmology ( $\Omega_m/f^2 = 1$ ).

## A The Bispectrum at One-loop Order in SPT

The NLO correction to the linear bispectrum of density perturbations, i.e. the one-loop contribution  $B_{111}^{1-\text{loop}}$ , is given as the sum of four diagrams involving density correlations [19],

$$B_{111}^{1-\text{loop}}(k_1, k_2, k_3, \eta) = B^{222} + B_I^{321} + B_{II}^{321} + B^{411}. \quad (\text{A.1})$$

Taking the UV-limit for the loop momenta,  $l \gg k_{1,2,3}$ , these contributions take the form [31]

$$B^{222}(k_1, k_2, k_3, \eta) \simeq -\frac{1}{4802\pi^2} \left[ 30k_1^6 - 30k_1^4(k_2^2 + k_3^2) + k_1^2(-30k_2^4 + k_2^2k_3^2 - 30k_3^4) \right. \\ \left. + 30(k_2^2 - k_3^2)^2(k_2^2 + k_3^2) \right] \int dl l^2 \left( \frac{P_L^3(l, \eta)}{l^6} \right), \quad (\text{A.2})$$

$$B_I^{321}(k_1, k_2, k_3, \eta) \simeq \frac{1}{35280\pi^2 k_3^2} \left[ 170k_1^6 + k_1^4(83k_2^2 + 190k_3^2) + 2k_1^2(67k_2^4 + 256k_2^2k_3^2 \right. \\ \left. - 445k_3^4) - (387k_2^2 - 530k_3^2)(k_2^2 - k_3^2)^2 \right] P_L(k_3, \eta) \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^4} \right) \\ + 5 \text{ perm.}, \quad (\text{A.3})$$

$$B_{II}^{321}(k_1, k_2, k_3, \eta) \simeq -\frac{61}{105} F_2(\mathbf{k}_2, \mathbf{k}_3) P_L(k_2, \eta) P_L(k_3, \eta) k_3^2 \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right) \\ + 5 \text{ perm.}, \quad (\text{A.4})$$

and

$$B^{411}(k_1, k_2, k_3, \eta) \simeq -\frac{1}{226380} \frac{1}{k_2^2 k_3^2} \left[ 12409k_1^6 + 20085k_1^4(k_2^2 + k_3^2) \right. \\ \left. + k_1^2(-44518k_2^4 + 76684k_2^2k_3^2 - 44518k_3^4) \right. \\ \left. + 12024(k_2^2 - k_3^2)^2(k_2^2 + k_3^2) \right] P_L(k_2, \eta) P_L(k_3, \eta) \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^2} \right) \\ + 2 \text{ perm.} \quad (\text{A.5})$$

Thereby,  $F_2(\mathbf{k}_2, \mathbf{k}_3)$  denotes the symmetrized second-order kernel in SPT and the permutations have to be taken with respect to the external momenta.

We now rewrite (A.2)-(A.6) in terms of the momenta  $\mathbf{k}$  and  $\mathbf{q}$ , average over the respective angles and finally take the soft- $q$  limit. While

$$B^{222}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} 0, \quad (\text{A.6})$$

the other resulting expressions are given by

$$B_I^{321}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \frac{k^4}{\pi^2} \gamma^{\text{SPT}} \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^4} \right), \quad (\text{A.7})$$

$$B_{II}^{321}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \frac{k^2}{\pi^2} (\alpha_1^{\text{SPT}} k \partial_k + \beta_1^{\text{SPT}}) P_L(k, \eta) \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right), \quad (\text{A.8})$$

$$B^{411}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \xrightarrow{q \rightarrow 0} P_L(q, \eta) \frac{k^2}{\pi^2} (\alpha_2^{\text{SPT}} k \partial_k + \beta_2^{\text{SPT}}) P_L(k, \eta) \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right), \quad (\text{A.9})$$

with

$$\gamma^{\text{SPT}} = \frac{515}{5292}, \quad \alpha_1^{\text{SPT}} = \alpha_2^{\text{SPT}} = \frac{61}{3780}, \quad \beta_1^{\text{SPT}} = -\frac{671}{8820}, \quad \beta_2^{\text{SPT}} = -\frac{155}{756}. \quad (\text{A.10})$$

Finally, the sum of these contributions gives the total SPT one-loop correction to the bispectrum, yielding

$$B_{111}^{\text{1-loop}}(\mathbf{k}, -\mathbf{q}, \mathbf{q} - \mathbf{k}, \eta)^{\text{av}} \simeq P_L(q, \eta) \frac{k^2}{\pi^2} \left[ (\alpha^{\text{SPT}} k \partial_k + \beta^{\text{SPT}}) P_L(k, \eta) \int dl l^2 \left( \frac{P_L(l, \eta)}{l^2} \right) + \gamma^{\text{SPT}} k^2 \int dl l^2 \left( \frac{P_L^2(l, \eta)}{l^4} \right) \right], \quad (\text{A.11})$$

where we defined

$$\beta^{\text{SPT}} = \beta_1^{\text{SPT}} + \beta_2^{\text{SPT}} = -\frac{3719}{13230}, \quad (\text{A.12})$$

and

$$\alpha^{\text{SPT}} = \alpha_1^{\text{SPT}} + \alpha_2^{\text{SPT}} = \frac{61}{1890}. \quad (\text{A.13})$$

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