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On the $B^{*'} \to B$ transition

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We present a first $N_\mathrm{f} = 2$ lattice estimate of the hadronic coupling g_{12} which parametrises the strong decay of a radially excited B^* meson into the ground state B meson at zero recoil. We work in the static limit of Heavy Quark Effective Theory (HQET) and solve a Generalised Eigenvalue Problem (GEVP), which is necessary for the extraction of excited state properties. After an extrapolation to the continuum limit and a check of the pion mass dependence, we obtain $g_{12} = -0.17(4)$.

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I. INTRODUCTION

Questions have been raised recently on the poor handling of excited states in the analyses of experimental data and their comparison with theoretical predictions. For instance, it has been advocated that the $\sim 3\sigma$ discrepancy observed between V_{cb}^{excl} and V_{cb}^{incl} may be reduced if the transition $B \to D'$ were large. This attractive hypothesis implies a suppression of the $B \to D^{(*)}$ hadronic form factors, as a study in the OPE formalism suggests [1]. On the other hand, it has been argued that the light-cone sum rule determination of the $g_{D^* D\pi}$ coupling, which parametrises the $D^* \to D\pi$ decay, likely fails to reproduce the experimental measurement unless one explicitly includes the contribution from the first radial excited $D^{(*)'}$ state on the hadronic side of the three-point Borel sum rule [2]. Comparison with sum rules is of particular importance because the heavy mass dependence of $\hat{g}_Q \equiv \frac{g_{H^* H\pi} f_\pi}{2\sqrt{m_H m_{H^*}}}$ deduced from recent lattice simulations [3–8] and experiment [9] seems much weaker than expected from analytical methods [10], as shown in Figure 1.

Techniques have been developed to study excited states of mesons using lattice QCD [11], especially to extract the spectrum [12–15]. Similar techniques can now be applied to three-point correlation functions to perhaps illuminate the phenomenological issues discussed above. In this letter we will report on the lattice computation of $g_{12} \equiv \langle B^{*'} | \mathcal{A}_i | B \rangle$ in the static limit of HQET, where $\mathcal{A}_i$ is the axial vector bilinear of light quarks and $B^{*'}$ is polarised along the ith direction. As a by-product of our work, we will also report on the computation of $g_{11} \equiv \langle B^* | \mathcal{A}_i | B \rangle$ and $g_{22} \equiv \langle B^{*'} | \mathcal{A}_i | B' \rangle$.

The Heavy Quark Symmetry of leading order HQET is well suited for our qualitative study. As the spectra of excited B and B^* mesons are degenerate, it is enough to solve a single Generalized Eigenvalue Problem (GEVP) while degrees of freedom $\sim m_b$, that are somehow irrelevant for the dynamics of the cloud of light quarks and gluons that governs the process we examine, are integrated out. The plan of the letter is the following: in Sec. II we describe our approach while in Sec. III we present our lattice set-up and discuss results before concluding in Sec. IV.

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* On a leave of absence from CERN

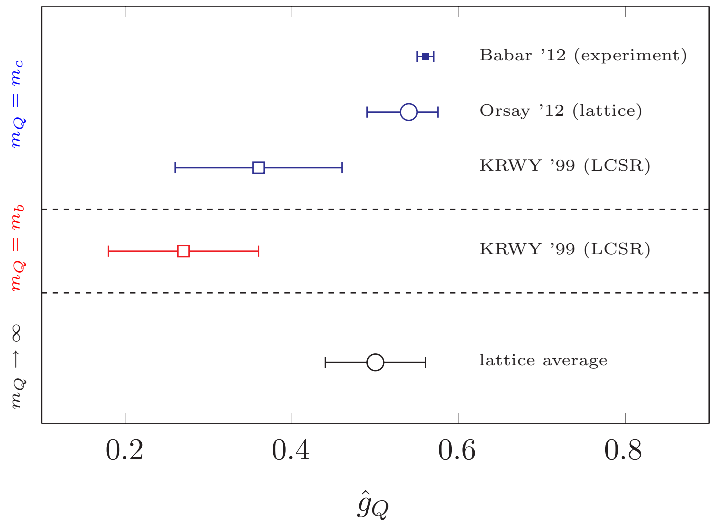

FIG. 1: Experimental measurement [9], lattice computations [3–7] and sum rules estimates [10] of $\hat{g}_c$, $\hat{g}_b$ and $\hat{g} \equiv \hat{g}_\infty$. We have performed a weighted average of recent $\hat{g}$ lattice results at $\mathrm{N_f} = 2$ with respect to the error quoted in [3–6].

II. EXTRACTION OF $\langle B^{*'}|\mathcal{A}_i|B\rangle$

The transition amplitude of interest is parametrised by

$$\begin{aligned}\langle B^{*'}(p', \epsilon_\lambda)|\mathcal{A}^\mu|B(p)\rangle &= 2m_{B^{*'}} A_0(q^2) \frac{\epsilon^{(\lambda)} \cdot q}{q^2} q^\mu + (m_B + m_{B^{*'}}) A_1(q^2) \left(\epsilon^{(\lambda)\,\mu} - \frac{\epsilon^{(\lambda)} \cdot q}{q^2} q^\mu \right) \\ &+ A_2(q^2) \frac{\epsilon^{(\lambda)} \cdot q}{m_B + m_{B^{*'}}} \left[(p_B + p_{B^{*'}})^\mu + \frac{m_B^2 - m_{B^{*'}}^2}{q^2} q^\mu \right], \end{aligned} \tag{1}$$

with $q = p' - p$. In the zero recoil kinematic configuration where $\vec{p} = \vec{p}' = \vec{0}$, one has $q^2_{\rm max} = (m_{B^{*'}} - m_B)^2$ so that

$$\langle B^{*'}(p', \epsilon_\lambda)|\mathcal{A}^i|B(p)\rangle = (m_B + m_{B^{*'}}) A_1(q^2_{\rm max}) \epsilon^{(\lambda)\,i}\,. \tag{2}$$

At that stage it is useful to introduce the HQET normalisation of states: $|H\rangle = \sqrt{2m_H}|H\rangle_{\rm HQET}$, with $\langle H(p)|H(p')\rangle = 2E(p)\delta^3(\vec{p} - \vec{p}')$:

$$\langle B^{*'}(p', \epsilon_\lambda)|\mathcal{A}^i|B(p)\rangle_{\rm HQET} = \frac{m_B + m_{B^{*'}}}{2\sqrt{m_B m_{B^{*'}}}} A_1(q^2_{\rm max}) \epsilon^{(\lambda)\,i}\,. \tag{3}$$

In the static limit we are left with $\langle B^{*'}(p', \epsilon_\lambda)|\mathcal{A}^i|B(p)\rangle_{\rm HQET} = A_1(q^2_{\rm max})\epsilon^{(\lambda)\,i}$. Choosing the quantization axis along the z direction and the polarisation vector $\epsilon^\mu(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, with the metric (+,-,-,-), we get finally $A_1(q^2_{\rm max}) = \langle B^{*'}(p', \epsilon_0)|\mathcal{A}^3|B(p)\rangle_{\rm HQET}$. Of course, extracting $g_{11} \equiv \hat{g}$ and g_{22} is similar, except that the relevant axial form factors are defined at $q^2 = 0$.

GEVP methods [16–18] are a very efficient tool to study excited states on the lattice. We consider $N \times N$ matrices of two-point correlation functions together with the corresponding matrices of three-point correlation functions $C^{(')(2)}_{ij}(t) \equiv \langle O^{(')}_i(t) O^{(')\dagger}_j(0)\rangle$ and $C^{(3)}_{ij}(t, t_s) \equiv \langle O'_i(t_s) O_\Gamma(t) O^\dagger_j(0)\rangle$, where i, j represent different wave functions and Dirac structures with quantum numbers generically denoted (h). More explicitly, the O_i are interpolating fields of pseudoscalar static-light mesons, the O'_i interpolating fields of vector static-light mesons and O_Γ the axial vector light-light bilinear of quarks.

In HQET the spectral decomposition reads $C^{(')(2)}_{ij}(t) = \sum_n \psi^{*(h^{(')})}_{ni} \psi^{(h^{(')})}_{nj} e^{-E_n t}$, $\psi^{(h^{(')})}_{ni} = \langle M^{(h^{(')})}_n | \hat{O}^{\dagger(')}_i |0\rangle$. The purpose of solving GEVP is to construct quantities which tend toward the desired excited state properties asymptotically in time. In practice we solve

$$\sum_j C^{(2)}_{ij}(t) v^{(n)}_j(t,t_0) = \sum_j \lambda^{(n)}(t,t_0) C^{(2)}_{ij}(t_0) v^{(n)}_j(t,t_0), \quad \lambda^{(n)}(t,t_0) = e^{-E^{\rm eff}_n(t,t_0)(t-t_0)} . \qquad (4)$$

We will use two ratio methods, GEVP and sGEVP, to extract the matrix element $M_{mn} \equiv \langle M^{(h)}_n | \hat{O}_\Gamma | M^{(h')}_m \rangle$. Those ratios converge quickly as the contribution of higher excited states is strongly suppressed [19][1] and read:

$$R^{\rm GEVP}_{mn}(t,t_s) = \frac{\langle v^{(n)}(t_s-t,t_0), C^{(3)}_\Gamma(t,t_s) v^{(m)}(t,t_0)\rangle \lambda^{(m)}(t_0+a,t_0)^{-t/2} \lambda^{(n)}(t_0+a,t_0)^{t-t_s/2}}{\sqrt{\langle v^{(n)}(t_s-t,t_0), C^{(2)}(t_s-t) v^{(n)}(t_s-t,t_0)\rangle \langle v^{(m)}(t,t_0), C'^{(2)}(t) v^{(m)}(t,t_0)\rangle}}$$
$$\overset{t/a\gg 1,(t_s-t)/a\gg 1}{\sim} M_{mn} + \mathcal{O}(e^{-\Delta_{N+1,m}t}, e^{-\Delta_{N+1,n}(t_s-t)}) , \qquad (5)$$

$$\Delta_{N+1,n} = E_{N+1} - E_n , \quad \langle a,b \rangle \equiv \sum_i a_i b_i .$$

$$R^{\rm sGEVP}_{mn}(t) = \partial_t \left[\frac{\langle v^{(m)}(t,t_0), [K^{mn}(t,t_0)/\lambda^{(n)}(t,t_0) - K^{mn}(t_0,t_0)] v^{(n)}(t,t_0)\rangle}{\sqrt{\langle v^{(m)}(t,t_0), D^{mn}(t,t_0) v^{(m)}(t,t_0)\rangle} \sqrt{\langle v^{(n)}(t,t_0), C^{(2)}(t_0) v^{(n)}(t,t_0)\rangle}} \right]$$
$$\overset{t/a\gg 1,(t_s-t)/a\gg 1}{\sim} M_{mn} + \mathcal{O}(\Delta t e^{-\Delta t_0}) , \qquad (6)$$

$$K^{mn}_{ij}(t,t_0) = \sum_{t_1} e^{-(t-t_1)\Sigma^{mn}(t,t_0)} C^{(3)}_{ij}(t_1,t) , \quad D^{mn}_{ij}(t,t_0) = e^{-t\Sigma^{mn}(t,t_0)} C'^{(2)}_{ij}(t),$$

In the appendix, we have calculated the time dependence of the corrections in $R^{\rm sGEVP}_{mn}(t)$ to first order in ϵ, where

$$C^{(2)}_{ij}(t) = C^{(2,0)}_{ij}(t) + \epsilon C^{(2,1)}_{ij}(t) = \sum_{n=1}^{N} e^{-E_n t} \psi_{ni}\psi_{nj} + \sum_{n=N+1}^{\infty} e^{-E_n t} \psi_{ni}\psi_{nj},$$

$$R^{\rm sGEVP}_{mn} = M_{mn} + \epsilon R^{\rm sGEVP,1}_{mn}$$

We have found that for $n > m$ the dominant contribution to $\epsilon R^{\rm sGEVP,1}_{mn}$ is $t e^{-(E_{N+1}-E_n)t}$ and for $n < m$ the leading contribution is in $e^{-(E_{N+1}-E_m)t}$.

The global phase is fixed by imposing the positivity of the 'decay constant' $f_{M^{(h)}_n} \equiv \langle M^{(h)}_n | O^\dagger_L |0\rangle = \frac{\sum_i C^{(2)}_{Li}(t) v^{(n)}_i(t,t_0) \lambda^{(n)}(t_0+a,t_0)^{-t/2}}{\sqrt{\langle v^{(n)}(t,t_0), C^{(2)}(t) v^{(n)}_j(t,t_0)\rangle}}$, where L refers to some local interpolating field.

III. LATTICE RESULTS

We have performed measurements on a subset of the $\mathrm{N_f} = 2$ CLS lattice ensembles, which employ the plaquette gauge action and non-perturbatively $\mathcal{O}(a)$ improved Wilson-Clover fermions. The parameters of the ensembles used in this work are collected in Table I. Three lattice spacings ($0.05\,\mathrm{fm} \lesssim a \lesssim 0.08\,\mathrm{fm}$)

[1] We give in the Appendix a hint of the proof of the t behaviour of $R^{\rm sGEVP}_{mn}(t)$, as it was not discussed in detail in [19].

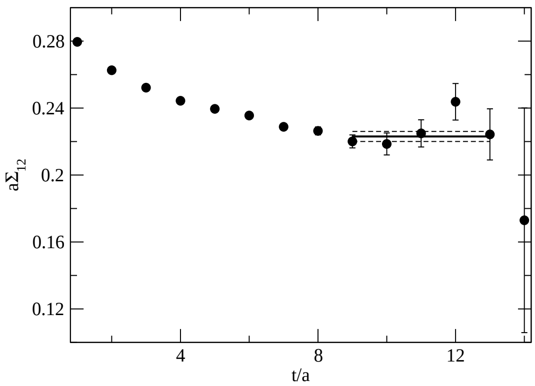

FIG. 2: Plateau of $E_2 - E_1$ for the CLS ensemble E5.

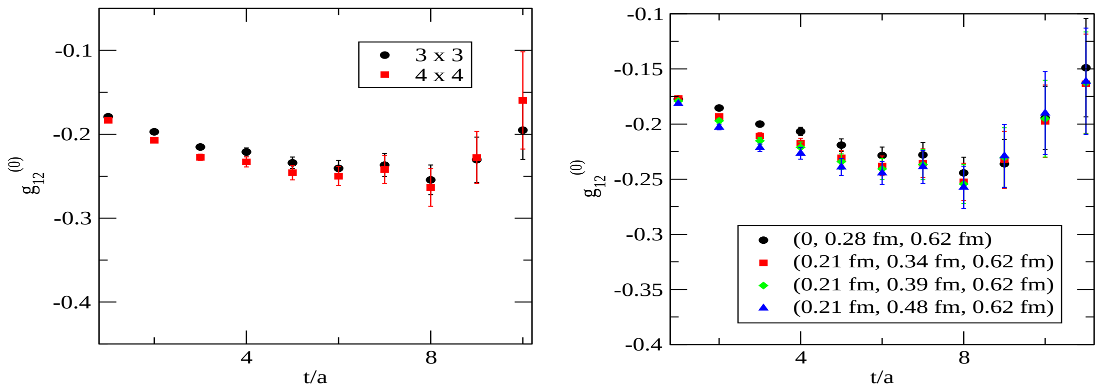

FIG. 3: Dependence of bare g_{12} on the size of the GEVP (left) and on the radius of wave functions (right) for the CLS ensemble E5.

are considered with pion masses in the range [310 MeV, 440 MeV]. The static-light correlation functions employ the 'HYP2' discretization of the static quark action [20, 21] and stochastically estimated all-to-all light quark propagators with full time dilution [22]. A single fully time-diluted stochastic source has been used on each gauge configuration, except for the ensemble E5 where we have four stochastic sources for each gauge configuration. We use interpolating fields for static-light mesons of the form [23]

$$O_i = \bar{\psi}_h \Gamma \, (1 + \kappa_G a^2 \Delta)^{R_i} \psi_l \, ,$$

where $\kappa_G = 0.1$, $r_i \equiv 2a\sqrt{\kappa_G R_i} \leq 0.6$fm, and Δ is a gauge covariant Laplacian made of 3 times APE-blocked links [24].

In order to reduce the statistical uncertainty in ratio (6), we have taken the asymptotic value of the energy splittings $\Sigma_\infty^{mn} = E_n - E_m$. We have shown in Figure 2 an example plateau for Σ_∞^{12}. In addition we have set t_s to $2t$ in (5). We have solved both 3×3 and 4×4 GEVP systems and checked the stability of the results when the local operator is included, as shown in Figure 3. To check the dependence on t_0, to which the contribution from higher excited states is sensitive, we have both fixed it at a small value (typically, 2a) and let it vary as $t - a$.

Though the uncertainty is a bit larger, we have confirmed the finding by [19] that using sGEVP (6) seems beneficial compared to the standard GEVP approach (5) to more strongly suppress contamination from higher excited states in the hadronic matrix element we measure. As illustrated in Figure 4, plateaux obtained from the GEVP and sGEVP are compatible: -0.25(1) for GEVP and -0.23(2) for sGEVP, with one additional point in the plateau of the sGEVP. Therefore, in the following we give results using the sGEVP only.

After applying a non-perturbative procedure to renormalise the axial light-light current [25, 26], we are ready to extrapolate to the continuum limit. Inspired by Heavy Meson Chiral Perturbation Theory

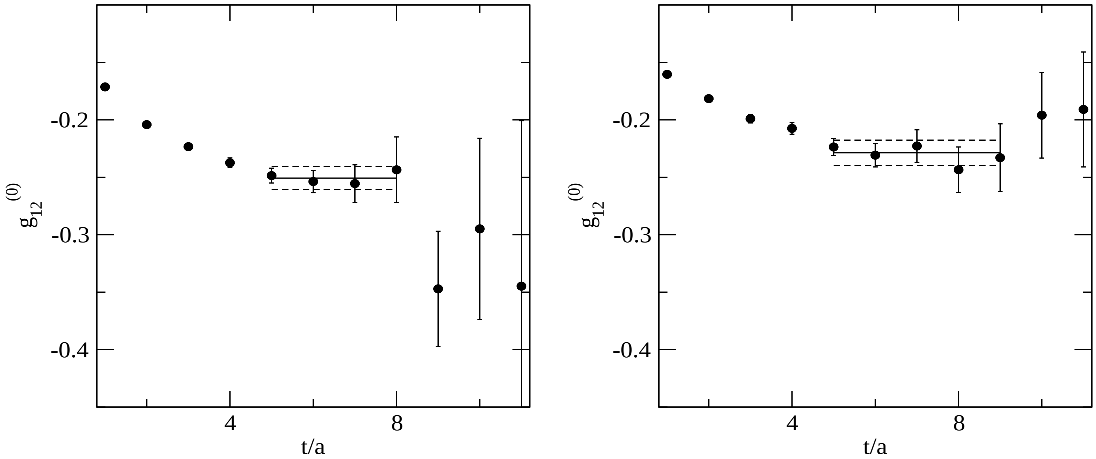

FIG. 4: Plateaus of bare g_{12} extracted by GEVP (left) and sGEVP (right) for the CLS ensemble E5.

at leading order [27, 28] and due to the $\mathcal{O}(a)$ improvement of the three-point correlation functions (the improved part of the axial current, $ac_A\partial_i\mathcal{P}$, is absent at zero momentum), we apply two fit forms:

$$g_{12} = C_0 + (a/a_{\beta=5.3})^2 C_1 + (m_\pi/m_\pi^0)^2 C_2 \,, \tag{7}$$

$$g_{12} = C_0' + (a/a_{\beta=5.3})^2 C_1' \,. \tag{8}$$

We show in Figure 5 the continuum extrapolation (7) of g_{12}. We observe quite large cut-off effects ($\sim 30\%$ at $\beta = 5.3$), it is thus crucial to have several lattice spacings. We obtain finally, using (7) as the best estimate of the central value,

$$g_{12} = -0.17(3)(2) \,, \tag{9}$$

where the first error is statistical, and the second error corresponds to the chiral uncertainty that we evaluate from the discrepancy between (7) and (8). We collect in Table II the value of g_{12} at each lattice point and at the physical point as well as the fit parameters for (7) and (8).

In simulations with light dynamical quarks, the onset of multi-hadron thresholds due to the emission of pions must be considered when examining excited B meson properties. Such thresholds significantly complicate the extraction of hadron-to-hadron matrix elements from the two- and three-point correlation functions considered here. However with the $L < 3\,\text{fm}$ volumes in this work, the P-wave decay $B^{*'}(\vec{0}) \to B(\vec{p})\pi(-\vec{p})$ is kinematically forbidden. The S-wave decay $B^{*'} \to B_1^*\pi$ is potentially more dangerous. Examining the mass splittings Σ_{12} in Table III, we notice that $630\,\text{MeV} \lesssim \Sigma_{12} \lesssim 710\,\text{MeV}$. If we assume that $400\,\text{MeV} \lesssim m_{B_1^*} - m_B \lesssim 500\,\text{MeV}$ in the pion mass range [310 MeV, 440 MeV], (as has been found in a recent lattice study of the static light meson spectrum [29]), we conclude that our analysis is safe from these threshold effects. Moreover the bare couplings g_{12} we obtain are similar to the quenched result of Ref. [19].

We show in Figure 6 a typical plateau of the bare coupling g_{11} and the extrapolation to the continuum and chiral limit. That extrapolation is smooth, with a negligible dependence on m_π, and we obtain from

CLS label	β	$L^3 \times T$	κ	a [fm]	m_π [MeV]	# of cnfgs
A5	5.2	$32^3 \times 64$	0.13594	0.075	330	500
E5	5.3	$32^3 \times 64$	0.13625	0.065	435	500
F6		$48^3 \times 96$	0.13635		310	600
N6	5.5	$48^3 \times 96$	0.13667	0.048	340	400

TABLE I: Parameters of the simulations.

	g_{12}
A5	-0.245(29)
E5	-0.186(8)
F6	-0.207(15)
N6	-0.181(12)
physical point	-0.173(28)(18)

	fit (7)	fit (8)
C_0	-0.178(29)	-0.155(26)
C_1	-0.063(32)	-0.040(29)
C_2	0.0053(29)	-

TABLE II: Value of g_{12} at the lattice points and at the physical point (left) as well as the fit parameters of Eq. (7) and (8) (right).

	$a\Sigma_{12}$
A5	0.255(8)
E5	0.222(8)
F6	0.216(12)
N6	0.173(7)

TABLE III: Mass splitting Σ_{12} in lattice units. The error we quote is the discrepancy between plateaux that we extract for different time ranges $\{[t_{\min}, t_{\max}], [t_{\min} \pm 0.2r_0, t_{\max} \pm 0.2r_0]\}$.

the fit form (7) $g_{11} = 0.52(2)$, in excellent agreement with a computation by the ALPHA Collaboration focused on that quantity [5]. We have added an error of 2% due to higher excited states which is estimated from plateaux at early times with a range ending at $\sim r_0$. Following the same strategy, we show in Figure 7 a typical plateau of the bare coupling g_{22} and the extrapolation to the continuum and chiral limit, once again quite smooth, with an almost absent dependence on the sea quark mass. We obtain from the fit form (8) $g_{22} = 0.38(4)$. Remarkably, the "diagonal" couplings g_{11} and g_{22} are significantly larger than the off-diagonal one g_{12}. This suggests that neglecting the contribution from B' mesons to the three-point light-cone sum rule used to obtain $g_{B^*B\pi}$ introduces uncontrolled systematics. Note that the decay constant $f_{B^{*\prime}}$ itself is large compared to f_B [30, 31]. For completeness we have collected in Table IV the value of g_{11} and g_{22} at each lattice point and at the physical point and the fit parameters of (7) and (8).

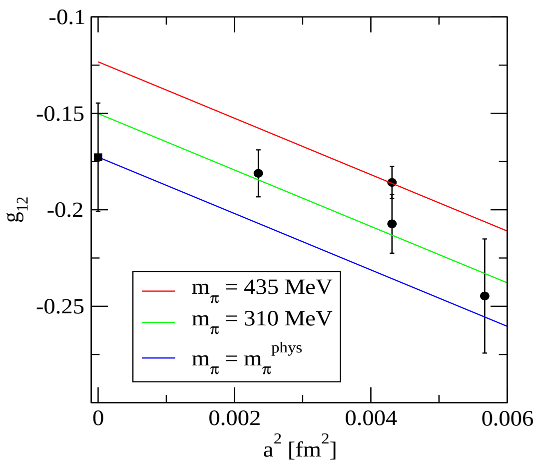

FIG. 5: Continuum and chiral extrapolation of g_{12}.

	g_{11}	g_{22}
A5	0.541(5)	0.492(19)
E5	0.535(8)	0.455(10)
F6	0.528(4)	0.474(26)
N6	0.532(6)	0.434(23)
physical point	0.516(12)(5)(10)	0.385(24)(28)

	g_{11}: fit (7)	g_{11}: fit (8)	g_{22}: fit (7)	g_{22}: fit (8)
C_0	0.515(13)	0.521(9)	0.416(27)	0.385(24)
C_1	0.012(9)	0.012(9)	0.074(25)	0.076(26)
C_2	0.0011(15)	-	-0.0033(33)	-

TABLE IV: Value of g_{11} and g_{22} at the lattice points and at the physical point (left) and fit parameters of eq. (7) and (8) (right). The third error on g_{11} is an estimate of the effects of higher excited states.

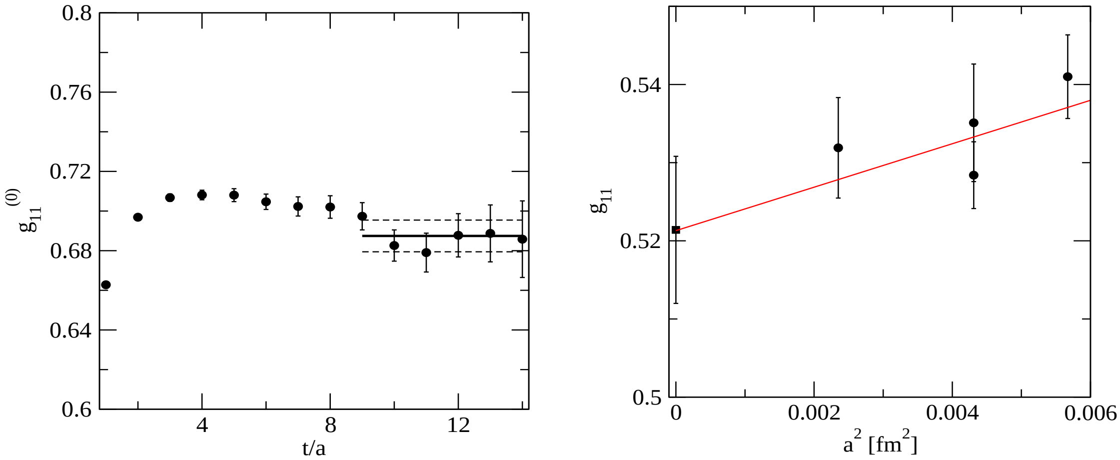

FIG. 6: Plateau of bare g_{11} for the CLS ensemble E5 (left) and its extrapolation to the continuum and chiral limit (right).

IV. CONCLUSION

We have performed a first estimate of the axial form factor $A_1(q^2_{\rm max}) \equiv g_{12}$ parametrising at zero recoil the decay $B^{*'} \to B$ in the static limit of HQET from $\rm N_f = 2$ lattice simulations. Assuming the positivity of decay constants f_B and $f_{B^{*\prime}}$, we have obtained a *negative* value for this form factor. It is almost three times smaller than the g_{11} coupling: $g_{12} = -0.17(4)$ while $g_{11} = 0.52(2)$. Moreover we find $g_{22} = 0.38(4)$, which is not strongly suppressed with respect to g_{11}. Our work is a first hint of confirmation of the statement made in Ref. [2] to explain the small value of $g_{D^*D\pi}$ computed analytically when compared to experiment. This computation using light-cone Borel sum rules may have been too naive. Following Ref. [32], a next step in our general study of excited static-light meson states would be the measurement of $A_1(0)$ by computing the distribution in r of the axial density $f_A(r) \equiv \langle B^{*'}|\bar{\psi}_l \gamma^i \gamma^5 \psi_l(r)|B\rangle$ and $A_1(0) = 4\pi \int_0^\infty r^2 f_A(r) e^{i\vec{q}\cdot\vec{r}}\, dr$.

Acknowledgements

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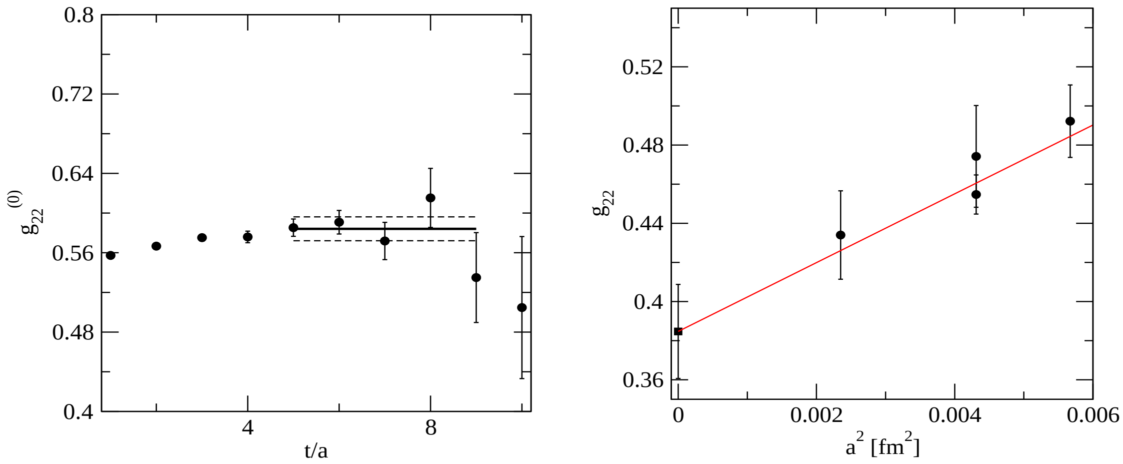

FIG. 7: Plateau of bare g_{22} for the CLS ensemble E5 (left) and its extrapolation to the continuum and chiral limit (right).

Appendix

In this section we discuss the time dependence of R_{mn}^{sGEVP} (6). To simplify notation, we have fixed a to 1. We have followed the strategy of Ref. [18] to treat in perturbation theory the full GEVP, with an exact computation of the N lowest states:

$$
\begin{aligned}
&C_{ij}^{(2)}(t) = \langle \mathcal{O}_i(t)\mathcal{O}_j(0)\rangle = C_{ij}^{(2,0)}(t) + \epsilon C_{ij}^{(2,1)}(t) = \sum_{n=1}^{N} e^{-E_n t}\psi_{ni}\psi_{nj} + \sum_{n=N+1}^{\infty} e^{-E_n t}\psi_{ni}\psi_{nj}\,,\\
&C^{(2)}(t)v_n(t,t_0) = \lambda_n(t,t_0)C^{(2)}(t_0)v_n(t,t_0)\,,\\
&v_n(t,t_0) = v_n^{(0)}(t,t_0) + \epsilon v_n^{(1)}(t,t_0)\,,\\
&\lambda_n(t,t_0) = \lambda_n^{(0)}(t,t_0) + \epsilon\lambda_n^{(1)}(t,t_0)\,.
\end{aligned}
$$

Vectors are normalised such that

$$
\begin{aligned}
\langle v_m^{(0)}, C^{(2,0)}(t_0)v_n^{(0)}\rangle &= \rho_n\delta_{nm}\,,\\
\langle v_n^{(1)}, C^{(2,0)}(t_0)v_n^{(0)}\rangle &= 0\,.
\end{aligned}
$$

where $\rho_n = e^{-E_n t_0}$. Introducing the dual vectors u_n defined by $\sum_{n=1}^{N} u_{ni}\psi_{mi} = \delta_{mn}\ \forall n \leq N$, we note that

$$
C^{(2,0)}(t)u_n = e^{-E_n t}\ \psi_n, \quad v_n^{(0)}(t,t_0) = u_n, \quad \lambda_n^{(0)}(t,t_0) = e^{-E_n(t-t_0)}.
$$

At first order in ϵ, we have

$$
\begin{aligned}
\lambda_n^{(1)} &= \rho_n^{-1}\left(v_n^{(0)}, \Delta_n v_n^{(0)}\right)\,,\\
v_n^{(1)} &= \sum_{m\neq n} v_m^{(0)}\rho_m^{-1}\frac{(v_m^{(0)}, \Delta_n v_n^{(0)})}{\lambda_n^{(0)} - \lambda_m^{(0)}} = \sum_{n\neq m}\alpha_{nm}\ v_m^{(0)},
\end{aligned}
$$

where $\Delta_n = C^{(2,1)}(t) - \lambda_n^{(0)}(t,t_0)C^{(2,1)}(t_0)$. With $c_{n,m,l} = \langle u_n, \psi_l\rangle\langle u_m, \psi_l\rangle$ we get:

$$\epsilon\frac{\lambda_n^{(1)}(t,t_0)}{\lambda_n^{(0)}(t,t_0)} = -\sum_{l>N} c_{n,n,l}e^{-(E_l-E_n)t_0}\left[1 - e^{-(E_l-E_n)(t-t_0)}\right],$$

$$\epsilon\ \alpha_{nm}(t,t_0) = -\sum_{l>N} c_{n,m,l}\frac{1-e^{-(E_l-E_n)(t-t_0)}}{1-e^{-(E_m-E_n)(t-t_0)}}e^{-(E_l-E_m)t_0}.$$

Finally the normalisation conditions read

$$\langle v_n(t,t_0), C(t_0)v_n(t,t_0)\rangle = \rho_n + \epsilon\langle v_n^{(0)}, C^{(2,1)}(t_0)v_n^{(0)}\rangle.$$

We are ready to develop (6) to first order in ϵ:

$$\begin{aligned}\mathcal{M}_{mn}^{\mathrm{eff},s} &= \partial_t\left\{\frac{\langle v_m(t,t_0), [K(t,t_0)/\lambda_n(t,t_0) - K(t_0,t_0)]\, v_n(t,t_0)\rangle}{\left[\langle v_m(t,t_0), C^{(2)}(t_0)v_m(t,t_0)\rangle\langle v_n(t,t_0), C^{(2)}(t_0)v_n(t,t_0)\rangle\right]^{1/2}}e^{\frac{t_0}{2}\Sigma(t_0,t_0)}\right\}\\ &= \mathcal{M}_{mn}^{\mathrm{eff},s,0} + \epsilon\mathcal{M}_{mn}^{\mathrm{eff},s,1},\end{aligned}$$

$$\mathcal{M}_{mn}^{\mathrm{eff},s,0} = \partial_t\left\{\frac{\langle u_m, \left[K(t,t_0)/\lambda_n^{(0)}(t,t_0) - K(t_0,t_0)\right]u_n\rangle}{\left[\langle u_m, C^{(2)}(t_0)u_m\rangle\langle u_n, C^{(2)}(t_0)u_n\rangle\right]^{1/2}}e^{\frac{t_0}{2}\Sigma(t_0,t_0)}\right\}.$$

With

$$\langle u_m, K(t,t_0)u_n\rangle = \sum_{t_1} e^{-\Sigma(t-t_1)}\langle u_m, C^{(3)}(t,t_1)u_n\rangle = h_{mn}\ te^{-E_nt}, \quad \langle u_n, C^{(2)}(t_0)u_n\rangle = \rho_n = e^{-E_nt_0}$$

we have at leading order

$$\mathcal{M}_{mn}^{\mathrm{eff},s,0} = h_{mn}.$$

The subleading order reads

$$\epsilon\mathcal{M}_{mn}^{\mathrm{eff},s,1} = \epsilon\partial_t\sum_{a=1}^{5}T_a.$$

$$T_1 = -\frac{\lambda_n^{(1)}(t,t_0)}{(\lambda_n^{(0)}(t,t_0))^2}\frac{\langle v_m^{(0)}(t,t_0), K(t,t_0)v_n^{(0)}(t,t_0)\rangle}{(\rho_n\rho_m)^{1/2}}e^{\frac{t_0}{2}\Sigma(t_0,t_0)} = -\frac{\lambda_n^{(1)}(t,t_0)}{\lambda_n^{(0)}(t,t_0)}\langle v_m^{(0)}(t,t_0), K(t,t_0)v_n^{(0)}(t,t_0)\rangle e^{E_nt}.$$

The first subleading contribution is given by

$$T_1 = -h_{mn}\ t\times\frac{\lambda_n^{(1)}(t,t_0)}{\lambda_n^{(0)}(t,t_0)} \sim c_{n,n,N+1}\ h_{mn}\times te^{-\Delta_{N+1,n}t_0}\left[1 - e^{-\Delta_{N+1,n}(t-t_0)}\right].$$

Defining the discrete derivative $\partial_t A = A(t+1) - A(t)$, and taking at the end of the computation $t_0 = t-1$, we get

$$\partial_t T_1 \sim c_{n,n,N+1}\ h_{mn}\left(1 - e^{-\Delta_{N+1,n}}\right)\times\left(t + 1 + e^{\Delta_{N+1,n}}\right)e^{-\Delta_{N+1,n}t}.$$

The second subleading contribution reads

$$\begin{aligned}T_2 = &\frac{\left(v_m^{(1)}(t,t_0), \left[K(t,t_0)/\lambda_n^{(0)}(t,t_0) - K(t_0,t_0)\right]v_n^{(0)}(t,t_0)\right)}{(\rho_n\rho_m)^{1/2}}e^{\frac{t_0}{2}\Sigma(t_0,t_0)} =\\ &\sum_{p\neq m}\alpha_{mp}\ \langle v_p^{(0)}(t,t_0), \left[K(t,t_0)e^{E_nt} - K(t_0,t_0)e^{E_nt_0}\right]v_n^{(0)}(t,t_0)\rangle.\end{aligned}$$

With some algebra, we deduce

$$\begin{aligned}
\langle v_p^{(0)}(t,t_0), K(t,t_0) v_n^{(0)}(t,t_0)\rangle e^{E_n t} &= \sum_{t_1} e^{-(t-t_1)(E_n-E_m)} \sum_{rs} \langle u_p, \psi_r\rangle \langle \psi_s, u_n\rangle h_{rs} e^{-E_r(t-t_1)} e^{-E_s t_1} e^{E_n t} \\
&= \sum_{t_1} e^{-(t-t_1)(E_n-E_m)} h_{pn} e^{-E_p(t-t_1)} e^{-E_n t_1} e^{E_n t} \\
&= \sum_{t_1} h_{pn} e^{-(E_p-E_m)t_1},
\end{aligned}$$

and

$$\langle v_p^{(0)}(t,t_0), \left[K(t,t_0)e^{E_n t} - K(t_0,t_0)e^{E_n t_0}\right] v_n^{(0)}(t,t_0)\rangle = \sum_{t_1=t_0+1}^{t} h_{pn} e^{-(E_p-E_m)t_1}.$$

Finally,

$$T_2 = \sum_{p\neq m} \left[\alpha_{mp}(t,t_0) \sum_{t_1=t_0+1}^{t} h_{pn} e^{-(E_p-E_m)t_1} \right],$$

$$\partial_t T_2 = \sum_{p\neq m} \left[(\alpha_{mp}(t+1,t_0) - \alpha_{mp}(t,t_0)) \sum_{t_1=t_0+1}^{t} h_{pn} e^{-(E_p-E_m)t_1} + \alpha_{mp}(t+1,t_0)\ h_{pn} e^{-(E_p-E_m)(t+1)} \right].$$

Setting $t_0 = t-1$, the first term reads

$$\begin{aligned}
&\sum_{p\neq m} (\alpha_{mp}(t+1,t_0) - \alpha_{mp}(t,t_0)) \times e^{-(E_p-E_m)t} \\
&\sim -\sum_{p\neq m} \left[c_{m,p,N+1} e^{-(E_{N+1}-E_p)(t-1)} \times \left(\frac{1-e^{-2(E_{N+1}-E_m)}}{1-e^{-2(E_p-E_m)}} - \frac{1-e^{-(E_{N+1}-E_m)}}{1-e^{-(E_p-E_m)}} \right) \right] \times h_{pn} e^{-(E_p-E_m)t} \\
&\quad \sim -e^{-(E_{N+1}-E_m)t} \sum_{p\neq m} \left[c_{m,p,N+1} h_{pn} e^{(E_{N+1}-E_p)} \times \left(\frac{1-e^{-2(E_{N+1}-E_m)}}{1-e^{-2(E_p-E_m)}} - \frac{1-e^{-(E_{N+1}-E_m)}}{1-e^{-(E_p-E_m)}} \right) \right],
\end{aligned}$$

and the second term reads

$$\begin{aligned}
&\sum_{p\neq m} \alpha_{mp}(t+1,t_0)\ h_{pn} e^{-(E_p-E_m)(t+1)} \\
&\qquad \sim -\sum_{p\neq m} e^{-(E_{N+1}-E_p)(t-1)} \frac{1-e^{-2(E_{N+1}-E_m)}}{1-e^{-2(E_p-E_m)}} c_{m,p,N+1}\ h_{pn} \times e^{-(E_p-E_m)(t+1)} \\
&\qquad\qquad \sim -e^{-(E_{N+1}-E_m)t} \sum_{p\neq m} e^{(E_{N+1}+E_m-2E_p)} \frac{1-e^{-2(E_{N+1}-E_m)}}{1-e^{-2(E_p-E_m)}} c_{m,p,N+1}\ h_{pn}.
\end{aligned}$$

We find

$$\partial_t T_2 \sim\ e^{-(E_{N+1}-E_m)t} \sum_{p\neq m} c_{m,p,N+1} h_{pn} \frac{1-e^{-(E_{N+1}-E_m)}}{1-e^{-(E_m-E_p)}}.$$

The third contribution

$$T_3 = \frac{\left(v_m^{(0)}(t,t_0), \left[K(t,t_0)/\lambda_n^{(0)}(t,t_0) - K(t_0,t_0)\right] v_n^{(1)}(t,t_0)\right)}{(\rho_n\rho_m)^{1/2}} e^{\frac{t_0}{2}\Sigma(t_0,t_0)}$$

is obtained similarly to $\partial_t T_2$, permuting m and n.
The fourth subleading contribution reads

$$T_4 \;=\; \frac{1}{\lambda_n^{(0)}(t,t_0)}\frac{\langle v_m^{(0)}(t,t_0),K^{(1)}(t,t_0)v_n^{(0)}(t,t_0)\rangle}{(\rho_n\rho_m)^{1/2}}e^{\frac{t_0}{2}\Sigma(t_0,t_0)} = \langle v_m^{(0)}(t,t_0),K^{(1)}(t,t_0)v_n^{(0)}(t,t_0)\rangle e^{E_n t}\,.$$

With some algebra we deduce

$$\begin{aligned}
\langle v_m^{(0)}(t,t_0),K^{(1)}(t,t_0)v_n^{(0)}(t,t_0)\rangle &= \sum_{t_1} e^{-(E_n-E_m)(t-t_1)} \sum_{(r\,\mathrm{or}\,s)>N} \langle u_m,\psi_r\rangle\langle\psi_s,u_n\rangle h_{rs}e^{-E_r(t-t_1)}e^{-E_s t_1} \\
&= \sum_{t_1} e^{-(E_n-E_m)(t-t_1)}\langle u_m,\psi_{N+1}\rangle h_{N+1,n}e^{-E_{N+1}(t-t_1)}e^{-E_n t_1} \\
&+ \sum_{t_1} e^{-(E_n-E_m)(t-t_1)}\langle u_n,\psi_{N+1}\rangle h_{N+1,m}e^{-E_m(t-t_1)}e^{-E_{N+1}t_1} \\
&+ \sum_{t_1} e^{-(E_n-E_m)(t-t_1)} \sum_{(r,s)>N} \langle u_n,\psi_r\rangle\langle u_m,\psi_s\rangle h_{r,s}e^{-E_r(t-t_1)}e^{-E_s t_1} \\
&\sim \sum_{t_1} e^{-(E_n-E_m)t_1}\langle u_m,\psi_{N+1}\rangle h_{N+1,n}e^{-E_{N+1}t_1}e^{-E_n(t-t_1)} \\
&+ \sum_{t_1} e^{-E_n(t-t_1)}\langle u_n,\psi_{N+1}\rangle h_{N+1,m}e^{-E_{N+1}t_1} \\
&+ \sum_{t_1} e^{-(E_n-E_m)(t-t_1)}\langle u_n,\psi_{N+1}\rangle\langle u_m,\psi_{N+1}\rangle h_{N+1,N+1}e^{-E_{N+1}t} \\
&\sim e^{-E_n t}\langle u_m,\psi_{N+1}\rangle h_{N+1,n}\sum_{t_1} e^{-(E_{N+1}-E_m)t_1} \\
&+ e^{-E_n t}\langle u_n,\psi_{N+1}\rangle h_{N+1,m}\sum_{t_1} e^{-(E_{N+1}-E_n)t_1} \\
&+ c_{n,m,N+1}\; h_{N+1,N+1}\; e^{-E_{N+1}t}\sum_{t_1} e^{-(E_n-E_m)t_1}\,,
\end{aligned}$$

and we obtain

$$\begin{aligned}
\partial_t T_4 \sim &+ \langle u_m,\psi_{N+1}\rangle h_{N+1,n}e^{-(E_{N+1}-E_m)(t+1)} \\
&+ \langle u_n,\psi_{N+1}\rangle h_{N+1,m}e^{-(E_{N+1}-E_n)(t+1)} \\
&- c_{n,m,N+1}h_{N+1,N+1}\frac{e^{-(E_{N+1}-E_n)}-1}{e^{-(E_n-E_m)}-1}e^{-(E_{N+1}-E_n)t} \\
&- c_{n,m,N+1}h_{N+1,N+1}\frac{e^{-(E_{N+1}-E_m)}-1}{e^{-(E_m-E_n)}-1}e^{-(E_{N+1}-E_m)t}\,.
\end{aligned}$$

The last subleading contribution reads

$$\begin{aligned}
T_5 \;&=\; -t\;h_{mn}\times\left(\frac{\langle v_m^{(0)},C^{(2,1)}(t_0)v_m^{(0)}\rangle}{2\rho_m}+\frac{\langle v_n^{(0)},C^{(2,1)}(t_0)v_n^{(0)}\rangle}{2\rho_n}\right) \\
&\sim\; -t\;h_{mn}\times\left(\frac{1}{2}c_{m,m,N+1}e^{-(E_{N+1}-E_m)t_0}+\frac{1}{2}c_{n,n,N+1}e^{-(E_{N+1}-E_n)t_0}\right).
\end{aligned}$$

With $t_0=t-1$, we get

$$\partial_t T_5 \sim -\frac{h_{mn}}{2}\times\left(c_{m,m,N+1}e^{-(E_{N+1}-E_m)(t-1)}+c_{n,n,N+1}e^{-(E_{N+1}-E_n)(t-1)}\right)$$

We see that for $n>m$ the dominating contribution T_1 to $\epsilon M_{mn}^{\mathrm{eff},s,1}$ is in $te^{-\Delta_{N+1,n}t}$ with subleading terms T_2-T_5 while for $n<m$ the leading contribution is in $e^{-(E_{N+1}-E_m)t}$.

We have tested numerically our finding in the toy model of Ref. [19], with $r_0 E_n = n$, $r_0 = 0.3$, the 3×5 matrix of couplings

$$\psi = \langle 0|\mathcal{O}_i|n\rangle = \begin{pmatrix} 0.92 & 0.03 & -0.10 & -0.01 & -0.02 \\ 0.84 & 0.40 & 0.03 & -0.06 & 0.00 \\ 0.56 & 0.56 & 0.47 & 0.26 & 0.04 \end{pmatrix},$$

and the hadronic matrix elements $M_{nn} = 0.7\frac{6}{n+5}$, $M_{n,n+m} = \frac{M_{nn}}{3m}$. The comparison between the analytical formulae and the numerical solution is plotted in Figure 8. It is encouraging to obtain such good agreement after $t = 8$.

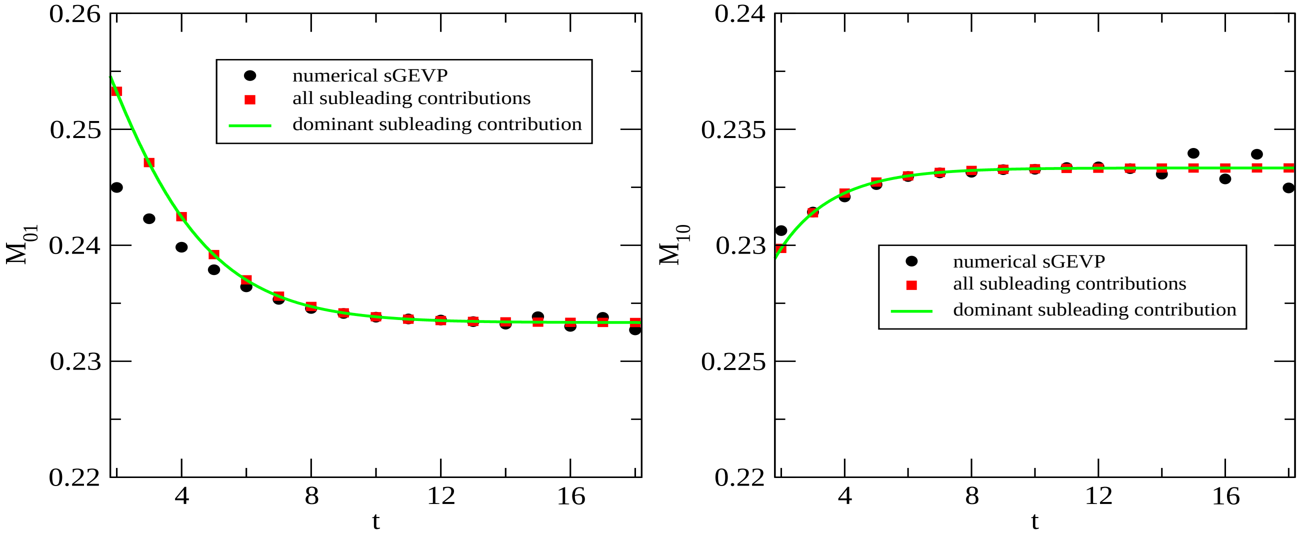

FIG. 8: Analytical formulae for R^{sGEVP}_{mn} compared to the numerical solution of our toy model.

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