Properties of States of Low Energy on Cosmological Spacetimes

Dissertation
zur Erlangung des Doktorgrades
des Department Physik
der Universität Hamburg

vorgelegt von
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Hamburg
2013

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Datum der Disputation:	Montag, 28. Januar 2013
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Zusammenfassung

Die vorliegende Dissertation untersucht die Eigenschaften einer Klasse von physikalischen Zuständen des quantisierten Skalarfeldes auf FRW Raumzeiten, die sogenannten Zustände niedriger Energie (ZnE). Diese Zustände sind charakterisiert durch ihre Eigenschaft, die durch eine glatte Testfunktion f kompakten Trägers zeitlich verschmierte Energiedichte eines isotropen Beobachters zu minimieren und darüber hinaus alle räumlichen Symmetrien der Raumzeit zu besitzen. Da sie Hadamardzustände sind, können Erwartungswerte von Observablen wie der Energiedichte rigoros über die sogenannte Pointsplitting–Methode definiert werden. Zunächst wird dieses Programm auf die explizite Berechnung der Energiedichte in ZnE auf dem de Sitter Hintergrund mit räumlich flachen Cauchyflächen angewendet. Dabei wird insbesondere der Einfluss der Masse m und der Wahl der Testfunktion f untersucht. Die Ergebnisse führen dann zu der Frage, ob die ZnE gegen den ausgezeichneten Grundzustand der betrachteten Raumzeit (hier das Bunch–Davies Vakuum) konvergieren, wenn der Träger von f gegen die unendliche Vergangenheit strebt. Wir zeigen dass dies zutrifft und sogar auf die Klasse der asymptotischen de Sitter Raumzeiten verallgemeinert werden kann, auf denen ein Analogon zum Bunch–Davies Vakuum existiert. Dieses Resultat zeigt, dass dieser ausgezeichnete Grundzustand als Zustand niedriger Energie in der unendlichen Vergangenheit interpretiert werden kann, und zwar unabhängig von der genauen Form von f. Schließlich diskutieren wir die Rolle von Zuständen niedriger Energie für das Rückwirkungsproblem. Wir leiten die semiklassische Friedmanngleichung mittels eines Störungsansatzes um den Minkowskiraum her. Durch diese Gleichung kann die Stabilität des Minkowskiraumes untersucht werden, indem das asymptotische Langzeitverhalten von perturbativen Lösungen für den Skalenfaktor analysiert wird. Zum Schluss präsentieren wir ein numerisches Lösungsverfahren.

Abstract

The present thesis investigates properties of a class of physical states of the quantised scalar field in FRW spacetimes, namely the states of low energy (SLE's). These states are characterised by minimising the time-smeared energy density measured by an isotropic observer, where the smearing is performed with respect to a test function f of compact support. Furthermore, they share all spatial symmetries of the spacetime. Since SLE's are Hadamard states, expectations values of observables like the energy density can be rigorously defined via the so called point-splitting method. In a first step, this procedure will be applied to the explicit calculation of the energy density in SLE's for the case of de Sitter space with flat spatial sections. In particular, the effect of the choice of the mass m and the test function f will be discussed. The obtained results motivate the question whether SLE's converge to a distinguished state (namely the Bunch Davies state) when the support of f is shifted to the infinite past. It will be shown that this is indeed the case, even in the more general class of asymptotic de Sitter spacetimes, where an analogon of the Bunch Davies state can be defined. This result enables the interpretation of such distinguished states to be SLE's in the infinite past, independently of the form of the smearing function f. Finally, the role of SLE's for the semiclassical backreaction problem will be discussed. We will derive the semiclassical Friedmann equation in a perturbative approach over Minkowski space. This equation allows for a stability analysis of Minkowski space by the investigation of asymptotic properties of solutions. We will also treat this problem using a numerical method.

Contents

Introduction

Quantum field theory (QFT) has turned out to be the best framework for the description of matter which is available. It incorporates, on the one hand, the concept of locality, meaning that physical signals cannot travel faster than light. On the other hand, it accomodates the principle of Quantum Physics, which manifests itself in the familiar Heisenberg uncertainty relation. In the absence of gravity, QFT is formulated on Minkowski space. It is due to this special geometric structure that we can interpret the theory in terms of particles. In this setting, QFT makes predictions in high energy physics by means of perturbation theory which fit the experimental data with an astonishing accuracy. On the other hand, we know from Einstein's general relativity (GR) that the essence of gravity is a dynamical spacetime $(\mathcal{M}, g)$, subject to the interaction with matter and energy distribution living on $(\mathcal{M}, g)$. Being also a succesful physical theory in the regime of large scales, it is however in deep conflict with Quantum Theory. This incompatibility shows up when trying to measure the coordinates of a spacetime point with an accuracy of the Planck length. According to Quantum Theory, it would be neccessary to concentrate there that amount of energy which would suffice to create a black hole [DFR95]. Therefore, the concept of a spacetime point looses its operational meaning. This calls for a new understanding of spacetime structure at very small distances, which is the subject of current research. Since such a theory of quantum gravity is not in reach, it is worthwile to investigate a regime where a combination of GR and QFT –which is QFT on curved spacetimes (QFT on CST)– makes sense and leads to new predictions. This semiclassical regime is characterised by curvatures that are low enough for the neglection of quantum gravity effects, but high enough in order to contradict results from QFT on Minkowski space. Despite the fact that terrestrial accelerators do not yet allow for testing such a theory directly, there are other good reasons to study QFT on CST. Let us first comment on the conceptual ones.

The necessity of abandoning many concepts of Minkowskian QFT – such as particles, vacuum, Poincaré invariance – shifts the attention to the "F" in our theory, which stands for *field*. Spacetime symmetries and particle states are not the core of the theory, but rather secondary objects and hence disposable. The quantisation of a classical field theory happens first of all at the level of the *algebra of observables* $\mathcal{A}$, independently of some Hilbert space representation. It is the philosophy of the algebraic approach to quantum field theory (AQFT) that the physical content of a (quantum) field theory should be contained in the assignment of algebras to spacetime regions,

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}),$$

together with some axioms which represent basic physical requirements such as causality and stability. This framework was first proposed in [HK64] and, although it was formulated with focus on Minkowski space, it is general enough to cope with the situation of curved spacetimes. Namely, it disentangles the description of a physical system (which is the algebra of observables $\mathcal{A}$) and the concept of states of such a system, which is of great importance in QFT, where, due to the presence of infinitely many degrees of freedom, unitarily inequivalent Hilbert space representations occur generically. In the algebraic framework, states are nothing but normalised positive functionals ω on $\mathcal{A}$. In QFT one needs furthermore a characterisation of physically meaningful states. It turned out that such a criterion can be given by requiring the two–point distribution $\mathcal{W}_2^\omega(f,g) \doteq \omega(\Phi(f)\Phi(g))$ to be of Hadamard form, which basically means that $\mathcal{W}_2^\omega$ must have the same singularity structure like the Hadamard fundamental solution of the wave equation which characterises the underlying classical field theory [KW91]. Quite recently, Radzikowski [Rad96] has shown that this is equivalent with a certain condition on the wave front set of $\mathcal{W}_2^\omega$, a concept from microlocal analysis. This latter characterisation is more abstract than the position space formulation and has proven to be a useful tool for conceptual work such as the perturbative construction of interacting theories [BFK96; BF00]. Another nice example for the elegance of the microlocal approach is Fewster's proof of a quantum energy inequality [Few00], to which we will return later. The increasing progress in understanding QFT on a more fundamental level was finally crowned by the important work of [BFV03], which uses the language of categories in order to define a generally locally covariant QFT as a covariant functor from a suitable category of spacetimes to a category of algebras.

All this conceptual progress is, however, not only end in itself. The semiclassical approach of describing a quantum field on a general background permits the description of physical phenomena which are impossible to derive from Minkowskian QFT, such as particle creation in expanding universes [Par69] and the famous radiation of black holes [Haw75]. Such examples can be understood by considering free quantum fields propagating on a fixed curved spacetime. The quantum fields "feels" the influence of gravity via the equation of motion, where the partial derivatives ∂_a of the corresponding Minkowski space model are replaced by the Lévi-Cività connection ∇_a induced by the metric. In order for this to be a meaningful approximation, one assumes that the quantum field is in a state for which the expectation value of the energy momentum tensor T_{ab} is small and thus the backreaction to the metric is negligible. However, a more ambitious goal of QFT on CST is to investigate the backreaction of quantum matter to the spacetime metric via the semiclassical Einstein equation

$$G_{ab} = -8\pi G\omega(: T_{ab} :),$$

where ω is a suitable quantum state and $: . :$ stands for a regularisation prescription for pointwise products of fields. In contrast to ordinary QFT, there is a finite renormalisation freedom in the definition of $\omega(: T_{ab} :)$ already at the level of a free field. By making simplifying assumptions, the backreaction problem can lead to interesting results. Namely, assuming the simple model of a conformally coupled massive scalar field in a FRW spacetime and making use of the aforementioned renormalisation freedom, it can be shown that the semiclassical Einstein equation admits a stable de Sitter phase whose Hubble parameter can be fine tuned by using the renormalisation freedom in order to agree with the measured value of today [DFP08]. This provides a simple and elegant explanation for dark energy. However, it is far from clear how the choice of the state ω can affect the solution space of the semiclassical Einstein equation under more general conditions, since then the state dependent part of $\omega(: T_{ab} :)$ will in general de a functional of the metric. We will come back to this problem in the last chapter of this thesis. As we already mentioned, much progress has been made in the mathematically rigorous formulation of QFT. But in order to make contact with physical observations, e.g. cosmological data, one needs a better control over the state space of the theory, along with the necessity of providing examples of states for concrete calculations and their interpretation. While it is known that Hadamard states for the free scalar field always exist on globally hyperbolic spacetimes [FNW81], it is still neccessary to construct such states, which is a difficult task when spacetime symmetries are absent. Therefore, the largest part of rigorous approaches to the semiclassical Einstein equation is restricted to FRW models, since their symmetry group is large enough in order to enable analytical computations. Nevertheless this is not just a meaningless oversimplification, since the results might be relevant in the realm of cosmology.

In the present work we shall consider the free scalar field on spatially flat FRW spacetimes, although some preliminary discussions will hold in greater generality. The class of states that we will focus on are the so called states of low energy (SLE's), recently introduced by Olbermann [Olb07a]. They have the simple physical interpretation of minimising the energy density of an isotropic observer γ, smeared with a smooth weighting function f of compact support in the proper time of γ. SLE's are good candidates for reference vacuum states for the following reasons: They share all spacetime symmetries, carry low energy in the sense described above and are Hadamard states. Finally, they reduce to the Minkowski vacuum when $a(t) = \text{const}$. In a previous work [DV10], the particle picture induced by such states was already investigated. Since the meaning of particles is a rather fuzzy concept in general FRW spacetimes, it is worthwile to investigate expectation values of field theoretic quantities in SLE's, such as the Wick square $:\Phi^2(x):$ or the energy density $:\rho(x):$. The latter is of great importance for the backreaction problem in FRW spacetimes, where the semiclassical Einstein equation reduces to the semiclassical Friedmann equation

$$H^2 = \frac{8\pi G}{3}\omega(:\rho:).$$

After this short motivation, we shall briefly sketch the outline of this thesis: In chapter 1 we will collect some important facts and definitions pertaining to globally hyperbolic spacetimes. Chapter 2 is dedicated to the rigorous quantisation of the free scalar field on globally hyperbolic spacetimes. We will introduce the important notion of Hadamard states and construct the expectation value of the renormalised stress energy tensor in such states. This procedure will then be specialised to the case of isotropic and homogenous states on FRW spacetimes, reviewing the work done in [Sch10]. Finally, this chapter closes with the introduction of SLE's. In chapter 3 we will apply these concepts to the calculation of the energy density in SLE's on de Sitter space. We obtain explicit results which will motivate the investigation of a certain limit state of SLE's on asymptotic de Sitter space in chapter 4. This limit arises from shifting the support of the smearing function f to the infinite past. We will show that such a limit state exists and that it coincides with the preferred state on asymptotic de Sitter spacetimes, which was constructed in [DMP09a; DMP09b]. Finally, chapter 5 is concerned with the semiclassical Friedmann equation, taking SLE's as reference states. We will argue that SLE's play a distinguished role for the backreaction problem in general and that they may be used for a stability analysis of fixed backgrounds. We will apply these considerations to Minkowski space by deriving a semiclassical Friedmann equation in a perturbative approach over the fixed background. We obtain an integro-differential equation which governs the scale factor perturbation δa. We will then analyse the possible asymptotic behaviour of such a perturbative solution and present a numerical treatment.

1. Structure of Spacetime

The notion of a spacetime in General Relativity (GR) is, as usual in physics, a mathematical abstraction of something intuitive. Space and time play a basic role in the physical description of the world. An instant of time and a position in space (w.r.t. a given observer) constitute an event. In the context of GR, we assume that the determination of these two data can be given with arbitrarily high precision. Thus, the set of all events should form a "continuum" and look locally like $\mathbb{R}^4$. An appropriate mathematical model for this is a four dimensional topological paracompact smooth manifold $\mathcal{M}$. In order to incorporate the notion of Einstein causality (no signal can travel faster than the speed of light), we require $\mathcal{M}$ to be endowed with a smooth Lorentzian metric g, which basically tells us if a given event p can influence another given event q. Since all physical measurements are of local character, they are not able to say anything about the global structure of spacetime. Nevertheless, we require that the initial data of the state of the entire universe at one instant of time completely determine the entire future and past. This leads to the additional assumption of global hyperbolicity, which is believed to be fulfilled by all "physically reasonable" spacetimes. In the next section we will define the basic definitions pertaining to spacetimes and their causal structure, following the monograph [Wal84]. In section 1.3 we will give some special examples of spacetimes which will be important in the sequel of the thesis.

1.1. Basic Conventions and Definitions

Throughout this thesis, the notion of *spacetime* will refer to a four dimensional smooth Lorentzian manifold $(\mathcal{M}, g)$, which is connected, paracompact and Haussdorf. We use the sign conventions $(---)$ according to the classification system of [MTW73]. The natural volume element of $(\mathcal{M}, g)$ will be denoted by $\mathrm{d}\mu_g$, which in local coordinates reads $\sqrt{|g|}\mathrm{d}^4x$, where $|g|$ is the modulus of the determinant of g in this chart. A very nice introduction to the concepts of differential geometry which stays close to physical applications is [Wal84]. We will use the abstract index notation like in this reference, meaning that tensors decorated with *latin* indices do not refer to components with respect to a particular basis (which will be the case when using *greek* indices), but rather have the function to indicate the rank of the object in question. Thus ${T^{ab}}_c$ is a tensor field of rank $(2, 1)$ (i.e. a multilinear map whose first two arguments are covector fields and whose third argument is a vector field). This notation is also used to denote the basis independent operation of contraction. ${T^{ab}}_b$ is simply the vector obtained by contraction with respect to the last two arguments. Occasionally we will supress the abstract indices when the type of tensor is clear from the context, e.g. the metric tensor $g_{ab} = g$. Since our sign convention is $(+, -, -, -)$, we call a vector $v \in T_x\mathcal{M}$ *timelike* if $g(v, v) > 0$, *lightlike* if $g(v, v) = 0$ and *spacelike* if $g(v, v) < 0$. We call it *causal* if it is timelike or lightlike. The same classification applies to vector fields $V : \mathcal{M} \to T\mathcal{M}$ and C^1- curves $\gamma : \mathbb{R} \supseteq I \to \mathcal{M}$, if the relations given above hold for V and the tangent vector field of γ on the whole respective domain of definition. We call $\mathcal{M}$ *orientable* if there exists a nonvanishing continous four–form ϵ and $\mathcal{M}$ is said to be *time orientable* if there exists a globally defined vector field $t : \mathcal{M} \to T\mathcal{M}$ with $g(t,t) > 0$. t then provides a time orientation in the sense that we call a causal C^1- curve γ *future directed* if $g(\dot{\gamma}, t) > 0$ and *past directed* if $g(\dot{\gamma}, t) < 0$. Similarly, we call a causal vector field $k : \mathcal{M} \to T\mathcal{M}$ future-/past directed, if $g(k,t)| \gtrless 0$, for which we write $k \rhd 0$ and $k \lhd 0$, respectively. The same conventions hold for covector fields $l : \mathcal{M} \to T^*\mathcal{M}$ by applying the previous criteria to the vector field $g^{ab}l_a$. From now on we assume that both an orientation and a time orientation have been chosen for $\mathcal{M}$. For later use we need to define the following sets: For $p \in \mathcal{M}$ we define its *chronological future* $I^+(p)$ to consist of all points of $\mathcal{M}$ that can be reached by a future directed timelike curve starting at p. For subsets $S \subseteq \mathcal{M}$, the chronological future $I^+(S)$ is just the union

$$I^+(S) = \bigcup_{p \in S} I^+(p).$$

Analogously we define the *chronological past* $I^-(p)$ of p and $I^-(S)$ of S by inserting past directed timelike curves in the above definition. In the same way we define the *causal future* $J^+(p)$ and *causal past* $J^-(p)$ of p as the set of events which can be reached by a causal future– respectively past directed curve starting in p. $J^+(S)$ and $J^-(S)$ are again defined by the union

$$J^\pm(S) = \bigcup_{p \in S} J^\pm(p).$$

Finally we set $J(S) \doteq J^+(S) \cup J^-(S)$.

We call an open set $S \subseteq \mathcal{M}$ *geodesically starshaped* with respect to $x \in S$ if there exists $S' \subseteq T_x\mathcal{M}$, open, with the properties that S' is starshaped with respect to $0 \in T_x\mathcal{M}$ and the exponential map $\exp_x : S' \to S$ is a diffeomorphism. We define S to be *geodesically convex* if it is starshaped with respect to all its points. This implies in particular that any two points of S can be connected by a unique geodesic which is contained in S. It can be shown that $\mathcal{M}$ can be covered by geodesically convex sets [O'N83], ensuring the existence of a geodesically convex neighborhood for each $p \in \mathcal{M}$.

Now we come to the important notion of *global hyperbolicity*, which is a necessary requirement for our spacetime in order to have a well posed Cauchy problem for field theories. To this avail, we define a causal curve $\gamma : I \to \mathcal{M}$ to be *extendible* iff there exists a causal curve $\tilde{\gamma} : \tilde{I} \to \mathcal{M}$ and a subinterval $J \subset \tilde{I}$ together with a parameter transformation $\phi : J \to I$ such that $\gamma \circ \phi = \tilde{\gamma} \restriction_J$. With this notion we may state the following important definition:

Definition 1.1.1. *$\Sigma \subset \mathcal{M}$ is called a Cauchy surface for $(\mathcal{M}, g)$ iff every inextendible causal curve $\gamma : I \to \mathcal{M}$ intersects Σ exactly once.*

One may thus think of Σ as a spatial slice of $(\mathcal{M}, g)$, or the space at one instant of time. A spacetime $(\mathcal{M}, g)$ is then defined to be *globally hyperbolic* if it contains a Cauchy surface. It can be shown that for $(\mathcal{M}, g)$ globally hyperbolic, Σ can always chosen to be smooth and that such $(\mathcal{M}, g)$ are diffeomorphic to $\mathbb{R} \times \Sigma$ [BS03]. When speaking about a Cauchy surface Σ, we will from now on assume that it is smooth. In the sequel we will also need some concepts pertaining to smooth mappings between manifolds. Let $\phi : \mathcal{M} \to \mathcal{N}$ be smooth, where $\mathcal{M}$ and $\mathcal{N}$ can be of different dimension. The *pullback* of a function $f : \mathcal{N} \to \mathbb{R}$ is a function on $\mathcal{M}$, defined by the composition $f \circ \phi$. Since vectors at $p \in \mathcal{M}$ are derivations on functions, we may *push forward* a vector $v \in T_p\mathcal{M}$ to a vector $\phi^* v \in T_{\phi(p)}\mathcal{N}$ by

$$\phi^* v(f) \doteq v(f \circ \phi),$$

which defines the *tangential map* $\phi^* : T_p\mathcal{M} \to T_{\phi(p)}\mathcal{N}$, a linear vector space homomorphism. Similarly, ϕ can be used to *pull back* a covector $w \in T^*_{\phi(p)}\mathcal{N}$ to a covector $\phi_* w \in T^*_p\mathcal{M}$ via

$$\phi_* w(v) \doteq w(\phi^* v)$$

for $v \in T_p\mathcal{M}$. It is obvious how to extend ϕ_* and ϕ^* to tensors of rank $(0,l)$ and $(l,0)$, respectively. For the case that ϕ is a diffeomorphism, i.e. its inverse ϕ^{-1} exists and is smooth, we can both pull back *and* push forward functions, vectors and covectors and extend this operations to tensors of arbitrary rank. Now consider a one parameter group ϕ_t of such diffeomorphisms, generated by the vector field v. We can then define the *Lie derivative* of a smooth tensor field in the direction of v by

$$\mathcal{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \doteq \lim_{t \to 0} \left(\frac{\phi^*_{-t} T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{a_1 \dots a_k}{}_{b_1 \dots b_l}}{t} \right).$$

A diffeomorphism ϕ is called an *isometry*, if it is a symmetry transformation for the metric g, i.e. if there holds $\phi^* g = g$. A similar notion is that of an *isometric embedding* $\phi : \mathcal{N} \to \mathcal{M}$, where ϕ is an isometry between $\mathcal{N}$ and $\phi(\mathcal{N}) \subseteq \mathcal{M}$. If ϕ_t is a one parameter group of isometries, we call the corresponding generating vector field a *Killing vector field.* Finally, a diffeomorphism ϕ is called a *conformal* isometry if it satisfies $\phi^* g = \Omega^2 g$ for a nonvanishing function Ω. Correspondingly, the generator of a one parameter group of conformal isometries is called *conformal Killing vector field.*

1.2. General Relativity

The mathematical definition of a spacetime as given in the above section tries to capture the minimal requirements one would like to have for the arena of physics. The reason for this cautious approach is the fact that spacetime is not a fixed eternal structure, but rather subject to a dynamics in the presence of matter and energy, described by the stress–energy–tensor T_{ab}. This principle is expressed by the famous Einstein equation,

$$R_{ab} - \frac{1}{2} R g_{ab} = -8\pi G T_{ab}. \tag{1.1}$$

In local coordinates, it is a nonlinear second order partial differential equation for the components of the metric g_{ab}. It says nothing about the global structure of the spacetime. The geometrical tensors appearing on the left hand side of (1.1) are defined as follows. First we remark that on a Lorentzian manifold $(\mathcal{M}, g)$, we have a unique covariant derivative $\nabla_a : \mathcal{T}(k,l) \to \mathcal{T}(k, l+1)$ which is torsion–free and compatible with the metric. This object can now be used to define the *Riemann tensor* by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) t_c \doteq -R_{abc}{}^d t_d \tag{1.2}$$

for all covector fields t_d. By taking the trace we obtain the *Ricci tensor*

$$R_{ac} \doteq R_{abc}{}^{b}. \tag{1.3}$$

Taking again the trace, we define the *scalar curvature* by

$$R \doteq R_a{}^a. \tag{1.4}$$

The *Einstein tensor*

$$G_{ab} \doteq R_{ab} - \frac{1}{2} R g_{ab} \tag{1.5}$$

has the remarkable property of being covariantly conserved,

$$\nabla^a G_{ab} = 0. \tag{1.6}$$

While T_{ab} acts as a source term for the dynamics of the metric, the metric in turn tells the matter how to move in the absence of other forces. Freely falling test particles (observers) should move on timelike geodesics, i.e. curves of minimal length between two points w.r.t. g_{ab}. Given a starting point $x = \gamma(0)$ and initial velocity $v = \dot{\gamma}(0)$, the *geodesic equation*

$$t^a \nabla_a t^b = 0$$

has a unique solution for the geodesic γ, where t^a denotes the tangent vector of γ. In a similar manner we may generalise the equation of motion for field theories on Minkowski space, simply by replacing partial derivatives ∂_a by their counterpart ∇_a. Thus, the Klein Gordon equation with minimal coupling[1] reads on general backgrounds

$$(\nabla^a \nabla_a + m^2)\phi = 0.$$

Solutions ϕ can then be interpreted as "freely falling fields".

Some remarks about (1.1) are now in order. The first one pertains to its consistency. Since for all forms of matter described by T_{ab} there must hold $\nabla^a T_{ab} = 0$ due to local conservation of energy, eq. (1.6) shows that Einstein's equation is in fact consistent. The second remark concerns the uniqueness of (1.1). Namely, G_{ab} is not the only covariantly conserved tensor which can be built locally from g_{ab}. By definition of ∇_a we have of course $\nabla_a g_{bc} = 0$. Thus we may modify (1.1) by adding the term Λg_{ab} to the left hand side. The corresponding equation is known as Einstein's equation with *cosmological constant* Λ. Note that the presence of Λ in (1.1) conflicts with the Newtonian limit of GR; however this argument is not sufficient to discard such a modified theory of gravity since Λ could be chosen small enough. Depending on the point of view, the term Λg_{ab} can either be regarded as a type of matter which is part of T_{ab}, or as a one parameter freedom in the theory, thus giving Λ the status of a constant of nature. As we shall see later, in the context of QFT on curved spacetime Λ will play the role of a renormalisation parameter due to the ambiguity in the definition of a quantised version of T_{ab}. While linear combinations of g_{ab} and G_{ab} give the most general symmetric and covariantly conserved tensors of rank $(0, 2)$ constructed out of the metric and its derivatives up to second order, we finally mention that there are two other independent tensors,

$$I_{ab} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} R^2 \mathrm{d}\mu_g \tag{1.7}$$

[1] A more general version of the Klein Gordon equation contains also coupling to the curvature via the term $\xi R\phi$.

and

$$J_{ab} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} R_{ab} R^{ab} \mathrm{d}\mu_g, \tag{1.8}$$

which are also covariantly conserved, symmetric and built locally from the metric and its derivatives up to fourth order [BD82]. We will encounter them later in the context of renormalisation issues.

1.3. FRW Spacetimes

One of the most important applications for (1.1) is the description of the dynamics of the whole universe. The complicated interplay between matter and spacetime is the topic of cosmology. There is much experimental evidence that our universe is isotropic and homogenous on very large scales. A well known example is the isotropy of the cosmic microwave background together with its very small relative anisotropic fluctuations of order 10^{-5} [Wri03]. Thus, besides mathematical simplifications, it is physically highly motivated to seek solutions of (1.1) which are spatially homogenous and isotropic. Following [Wal84], we call a spacetime $(\mathcal{M}, g)$ *spatially homogenous* if there exists a one parameter family of spacelike hypersurfaces Σ_t with $\cup_{t\in\mathbb{R}}\Sigma_t = \mathcal{M}$ such that for all t and $p, q \in \Sigma_t$ there exists an isometry ϕ_t with $\phi_t(p) = q$. We call $(\mathcal{M}, g)$ *spatially isotropic* if it can be covered by pairwise disjoint timelike curves with tangent vector field u^a, such that for all $x \in \mathcal{M}$ and all spatial vectors $v^a, w^a \in T_x\mathcal{M}$ satisfying $g(u, v) = g(u, w) = 0$ there exists an isometry $\phi : \mathcal{M} \to \mathcal{M}$ such that $\phi(x) = x$ and $\phi^* v = w$.

Spacetimes which are spatially homogenous and isotropic are called Friedmann–Robertson–Walker (FRW) spacetimes. Topologically they have the form $\mathcal{M} = \mathbb{R} \times \Sigma^\kappa$, where κ labels the three possibilities of the Cauchy surfaces: $\Sigma^1 \doteq S^3$, $\Sigma^0 \doteq \mathbb{R}^3$ and $\Sigma^3 \doteq \{x \in \mathbb{R}^4 : x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1, x_0 > 0\}$. The metric takes the form

$$g_{ab} = \mathrm{d}t^2 - h_{ab}(t), \tag{1.9}$$

where the time dependent induced Riemannian metric on Σ^κ is given by

$$h_{ab} = a^2(t) \begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) & \psi, \theta \in [0, \pi); \phi \in [0, 2\pi) \\ dx^2 + dy^2 + dz^2 & x, y, z \in (-\infty, \infty) \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) & \psi \in (-\infty, \infty), \theta \in [0, \pi), \\ & \phi \in [0, 2\pi) \end{cases} \tag{1.10}$$

for $\kappa = 1, 0, -1$ and $a(t) : \mathbb{R} \supseteq I \to \mathbb{R}$ is a strictly positive smooth function. For all subsequent calculations we will restrict to flat spatial sections, i.e. $\kappa = 0$ and with respect to the preferred *standard coordinates* $(t, x, y, z) \in I \times \mathbb{R}^3$ (which are also global coordinates) the components of the metric tensor take the form

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2). \tag{1.11}$$

The coordinate t has the interpretation of global cosmological time (i.e. the time measured by an observer described by the geodesic worldline (t, x_0, y_0, z_0)). In this sense, such observers are distinguished and they can agree on the time t. In many calculations it will be useful to switch to another time coordinate τ, defined by

$$\tau(t) \doteq \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')}.$$

With the further definition $C(\tau(t)) \doteq a^2(t)$, the components of the metric tensor with respect to the new global coordinates (τ, x, y, z) read

$$g_{\mu\nu} = C(\tau)\mathrm{diag}(1, -1, -1, -1),$$

thus showing that spatially flat FRW spacetimes are conformally flat, which is the reason for calling τ conformal time.

We will briefly sketch which form (1.1) takes in FRW spacetimes with $\kappa = 0$. The most general form for T_{ab} which is compatible with the spacetime symmetries is

$$T_\mu{}^\nu = \mathrm{diag}(\rho, -p, -p, -p) \tag{1.12}$$

w.r.t. our standard coordinates, where ρ and p are energy density and pressure, respectively. Depending on the assumptions on the type of matter, ρ and p are usually related by an *equation of state*. For radiation we have $\rho = 3p$, whereas for dust there holds $p = 0$. When treating the above discussed term Λg_{ab} as a type of energy–matter (dark energy), we have $\rho = \Lambda$ and $p = -\Lambda$. It is a straightforward task to calculate G_{ab} as function of $a(t)$ and its derivatives. Einsteins equation (1.1) then reduces to the *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{1.13}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.14}$$

We can immediately draw some interesting conclusions: Given $\rho > 0$ and $p \geq 0$ (as it is the case for dust and radiation), (1.14) reveals that $\ddot{a} < 0$. In light of the observed expansion of our universe via redshift measurement of the light of distant galaxies, (1.14) predicts that $a(t_0) = 0$ for some finite time t_0 in the past, which is the theoretical reason for the Big Bang theory. Furthermore, the covariant conservation of T_{ab} reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0,$$

which also could have been inferred from eqs. (1.13) and (1.14). It follows that $\rho a^3 = \mathrm{const}$ for dust and $\rho a^4 = \mathrm{const}$ for radiation. Thus, for an expanding universe there must have been an early radiation–dominated epoch followed by a dust–dominated one, whereas the late time dynamics is driven by the cosmological constant.

At a later stage of this thesis, we will encounter a particular class of spacetimes, the so called asymptotic de Sitter spacetimes, which we will introduce here. When solving the Friedmann equations for T_{ab} consisting of dark energy alone, one obtains

$$a(t) = e^{H_0 t}, \quad H_0 \doteq \sqrt{\frac{8\pi G\Lambda}{3}}$$

as solution for the scale factor, where t ranges from $-\infty$ to ∞. It turns out that this spacetime can be viewed to be part of a larger spacetime, namely de Sitter spacetime $(\mathcal{M}_{dS}, g_{dS})$. It is defined as an embedded submanifold in the five dimensional Minkowski space $(\mathbb{R}^5, \eta)$ via

$$\mathcal{M}_{dS} \doteq \{x \in \mathbb{R}^5 : y_0^2 - \sum_{i=1}^{4} y_i^2 = H_0^{-2}\},$$

equipped with the induced metric $g_{dS} = \eta \restriction_{\mathcal{M}_{dS}}$. $(\mathcal{M}_{dS}, g_{dS})$ is a maximally symmetric globally hyperbolic spacetime, the isometry group being $O(1,4)$. There are different possibilities how to coordinatise $\mathcal{M}_{dS}$. Let us choose the map $\psi : \mathbb{R}^4 \to \mathcal{M}_{dS}$, given by

$$\begin{aligned}
y_0 &= H_0^{-1}\sinh(H_0 t) + \frac{H_0}{2} r^2 e^{H_0 t} \\
y_1 &= H_0^{-1}\cosh(H_0 t) - \frac{H_0}{2} r^2 e^{H_0 t} \\
y_i &= e^{H_0 t} x_{i-1},
\end{aligned}$$

where $r^2 \doteq x_1^2 + x_2^2 + x_3^2$. Clearly, ψ is not surjective since $y_0 + y_1$ is constrained to be positive. Thus, the coordinates $(t, \mathbf{x})$ cover only one half of $\mathcal{M}_{dS}$. With respect to the new coordinates we obtain

$$\psi_* g_{dS} = \mathrm{d}t^2 - e^{2H_0 t}\mathrm{d}^2\mathbf{x},$$

which verifies the above claim. The conformal time is then given by

$$\tau(t) = -\frac{1}{H_0} e^{-H_0 t},$$

and we have

$$a((\tau(t)) = -(H_0 \tau)^{-1}. \tag{1.15}$$

We want now to define a class of scale factors a which asymptotically approach the form of (1.15). To this avail we require the following asymptotic behaviour of a:

$$a = -\frac{1}{H\tau} + O(\tau^{-2}), \quad \frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{1}{H\tau^2} + O(\tau^{-3}) \tag{1.16}$$

The class of spatially flat FRW spacetimes whose scale factors exhibit this behaviour will be called asymptotic de Sitter spacetimes and henceforth denoted by $(\mathcal{M}, g_{adS})$. We shall briefly point out the reason for considering this class of spacetimes, following [DMP09a]. We start by recalling the notion of cosmological horizons, due to Rindler [Rin06]. Consider the complete geodesic of a comoving observer $\gamma \subset \mathcal{M}$, and its causal past $J^-(\gamma)$. If $J^-(\gamma) \neq \mathcal{M}$, then there exist events from which γ cannot *receive* information. This situation thus gives rise to a *cosmological horizon* $\partial J^-(\gamma)$. Conversely one may look at $J^+(\gamma)$. If this set in turn does not cover the whole of $\mathcal{M}$, then there are events to which γ cannot *send* any information. $\partial J^+(\gamma)$ then defines a *cosmological past horizon.* Both horizons have the structure of lightlike three–dimensional hypersurfaces. An easy way to check for the existence of these objects in spatially flat FRW spacetimes for isotropic observers γ is going over to conformal time τ. If its upper/lower bound is finite, then $J^{-/+}(\gamma)$ does not cover $\mathcal{M}$. We will be concerned with another type of horizons. In simple terms, this horizon will be outside, but arbitrarily close to $\mathcal{M}$. To get an idea of such an horizon, consider the above mentioned example $(\mathcal{M}, g) = (\mathbb{R}^4, \psi_* g_{dS})$. As we already know, it can be isometrically embedded in the larger spacetime $(\hat{\mathcal{M}}, \hat{g}) = (\mathcal{M}_{dS}, g_{dS})$. Thus we can regard its boundary $\partial\mathcal{M}$ as a submanifold in $\hat{\mathcal{M}}$. Clearly, this object will now play the role of a past cosmological horizon for $\mathcal{M}$, since all events in $\hat{\mathcal{M}} \setminus \mathcal{M}$ can never be reached by signals starting in any event in $\mathcal{M}$. $(\mathbb{R}^4, \psi_* g_{dS})$ is a prototype of a class of spacetimes called expanding universes with cosmological horizons. For the sake of completeness, we quote the general definition from [DMP09a]:

Definition 1.3.1. *A globally hyperbolic spacetime $(\mathcal{M}, g)$ equipped with a positive smooth function $\Omega : \mathcal{M} \to \mathbb{R}^+$, a future-oriented timelike vector X defined on $\mathcal{M}$, and a constant $\gamma \neq 0$, will be called an* ***expanding universe with (geodesically complete) cosmological (past) horizon*** *when the following facts hold:*

1. *$(\mathcal{M}, g)$ can be isometrically embedded as the interior of a submanifold with boundary of a larger spacetime $(\hat{\mathcal{M}}, \hat{g})$, the boundary $\mathcal{J}^- \doteq \partial\mathcal{M}$ fulfilling*
$$\mathcal{J}^- \cap J^+(\mathcal{M}; \hat{\mathcal{M}}) = \varnothing.$$

2. *Ω extends to a smooth function on $\hat{\mathcal{M}}$ such that $\Omega \restriction_{\mathcal{J}^-} = 0$ and $\mathrm{d}\Omega \neq 0$ everywhere on $\mathcal{J}^-$.*

3. *X is a conformal Killing vector for $\hat{g}$ in a neighborhood of $\mathcal{J}^-$ in $\mathcal{M}$, with*
$$\mathcal{L}_X(\hat{g}) = -2X(\ln\Omega)\hat{g}, \tag{1.17}$$
where $X(\ln\Omega) \to 0$ approaching $\mathcal{J}^-$ and X does not tend everywhere to the zero vector approaching $\mathcal{J}^-$.

4. *$\mathcal{J}^-$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ and the metric $\hat{g} \restriction_{\mathcal{J}^-}$ takes the Bondi form globally up to the constant factor $\gamma^2 > 0$:*
$$\hat{g} \restriction_{\mathcal{J}^-} = \gamma^2 \left(-\mathrm{d}l \otimes \mathrm{d}\Omega - \mathrm{d}\Omega \otimes \mathrm{d}l + \mathrm{d}\mathbb{S}^2(\theta, \phi)\right), \; l \in \mathbb{R}, \; (\theta, \phi) \in \mathbb{S}^2, \tag{1.18}$$
$\mathrm{d}\mathbb{S}^2$ being the standard metric on the unit 2-sphere. Hence $\mathcal{J}^-$ is a null 3-submanifold. Finally, the curves $\mathbb{R} \ni l \mapsto (l, \theta, \phi)$ are required to be complete null geodesics.

Remark 1.3.2. *The manifold $\mathcal{J}^-$ is called* ***cosmological (past) horizon*** *of $\mathcal{M}$. The integral parameter of X is called* ***conformal cosmological time****. There is a completely analogous definition of contracting universe referring to the existence of $\mathcal{J}^+$ in the future instead of $\mathcal{J}^-$.*

It is not difficult to show that the class of asymptotic de Sitter spacetimes fulfils all these requirements [DMP09a]. The scale factor a and the vector field ∂_τ play then the role of Ω and X, respectively. Note that the above definition may encompass much more general spacetimes, because neither homogeneity nor isotropy of $\mathcal{M}$ are required. We would also like to mention that the group of diffeomorphisms of $\mathscr{J}^-$ contains all Killing isometries of $(\mathcal{M}, g)$. The motivation for looking at spacetimes with a past cosmological horizon is twofold. On the one hand, particle horizons are not present and thus all events in $\mathcal{M}$ have been in causal contact with a common event in the past, giving a natural explanation for the homogeneity of our universe without assuming a special initial state. On the other hand, as we will explain in greater detail in chapter 4, such spacetimes allow for the construction of a distinguished ground state for QFT's on $\mathcal{M}$, which could be used to derive the almost scale free anisotropies of the CMB spectrum.

2. Quantum Field Theory on Curved Spacetimes

The goal of this chapter is the quantisation of the free scalar field on globally hyperbolic spacetimes $(\mathcal{M}, g)$ in the spirit of the algebraic approach. A nice introduction to this topic is provided by the monograph of Haag [Haa96]. We will start with discussing the corresponding classical free field theory. Since the Cauchy problem is well posed, we can construct the symplectic vector space of solutions and a certain subalgebra of observables, consisting of smeared fields. This algebra can then be quantised in a straightforward manner by constructing the so called Borchers–Uhlmann algebra $\mathcal{A}(\mathcal{M}, g)$. However, $\mathcal{A}(\mathcal{M}, g)$ does not contain important observables like the pointwise product $:\Phi^2:$, also known as Wick square, or the quantised energy–momentum tensor $:T_{ab}:$. It will turn out that expectation values of such objects can be defined for the class of Hadamard states, which will be introduced in section 2.3.1. Finally, we will devote an own section to the problem of defining the quantum expectation value of $:T_{ab}:$, which includes the discussion of its renormalisation freedom, and we will sketch an explicit procedure for calculating it on FRW spacetimes.

In the following we will denote the space of compactly supported real valued functions on the manifold $\mathcal{M}$ by $\mathcal{D}(\mathcal{M})$. It carries a natural locally convex topology which is determined by saying that a sequence $f_n \in \mathcal{D}(\mathcal{M})$ converges to $f \in \mathcal{D}(\mathcal{M})$ if there exists a compact subset $K \subset M$ with $\mathrm{supp} f_n \subseteq K$, $\mathrm{supp} f \subseteq K$ and f_n together with all its derivatives converges to f and its derivatives uniformly on K. Its topological dual space, consisting of continous linear functionals (distributions), is referred to as $\mathcal{D}'(\mathcal{M})$. Finally, we write $\mathcal{E}(\mathcal{M})$ for the real valued smooth functions on $\mathcal{M}$ and define its locally convex topology by saying that a sequence $f_n \in \mathcal{E}(\mathcal{M})$ converges to $f \in \mathcal{E}(\mathcal{M})$ if we have uniform convergence of f_n and all its derivatives to f on all compact subsets of $\mathcal{M}$.

2.1. Classical Free Scalar Field Theory on Curved Spacetimes

We consider the free scalar field on the globally hyperbolic spacetime $(\mathcal{M}, g)$. It obeys the following second order partial differential equation:

$$P\Phi \doteq (\Box_g + \xi R + m^2)\Phi = 0, \quad \Phi \in \mathcal{E}(\mathcal{M}). \tag{2.1}$$

Here, ξ is a real number and constitutes an additional free parameter of the theory, just like the mass m. In some parts of this thesis we will choose $\xi = 0$, which is referred to as *minimal coupling*. However, many constructions in this chapter work for an arbitrary choice of ξ and we will indicate when this should not be the case. Obviously, for vanishing curvature, (2.1) reduces to the relativistically invariant Klein–Gordon equation on Minkowski space, which describes, according to Wigners definition of elementary particles, a particle of mass m and spin 0 [Wig39].

The ambiguity due to the presence of ξ can be turned into an advantage in case of the massless field by choosing $\xi = 1/6$, known as *conformal coupling*[1]. In this case, Φ will be conformally invariant [Wal84, app.D].

The problem of establishing the solution theory is extensivly treated in the monograph [BGP07], where the following important theorem is proven:

Theorem 2.1.1.

1. *The Klein–Gordon operator* $P \doteq \Box_g + \xi R + m^2$ *on a globally hyperbolic spacetime* $(\mathcal{M}, g)$ *possesses unique retarded and advanced fundamental solutions* $E^{\pm}$*, which are continous linear maps*

$$E^{\pm} : \mathcal{D}(\mathcal{M}) \to \mathcal{E}(\mathcal{M})$$

with the properties

$$\begin{aligned} PE^{\pm}f = f = E^{\pm}Pf \\ \mathrm{supp}(E^{\pm}f) \subseteq J^{\pm}(\mathrm{supp}f) \end{aligned}$$

$\forall f \in \mathcal{D}(\mathcal{M})$.

2. *Let* Σ *be a Cauchy surface of* $(\mathcal{M}, g)$ *and* N^a *denote its normal vector field. Then for every choice of functions* $u, w \in \mathcal{D}(\Sigma)$ *there exists a unique solution* $\Phi \in \mathcal{E}(\mathcal{M})$ *of (2.1) satisfying* $\Phi \restriction_{\Sigma} = u$ *and* $N^a \nabla_a \Phi \restriction_{\Sigma} = w$*. Furthermore there holds*

$$\mathrm{supp}\Phi \subseteq J(\mathrm{supp}u \cup \mathrm{supp}w).$$

Such Φ *are called solutions with compact Cauchy data.*

Using $E^{\pm}$ we may define the *causal propagator*

$$E \doteq E^{+} - E^{-}.$$

It follows from the above theorem that E maps test functions $f \in \mathcal{D}(\mathcal{M})$ into the linear space $\mathcal{S}$ of smooth solutions of (2.1) with compact Cauchy data:

$$E : f \mapsto Ef \doteq \Phi_f \in \mathcal{S}.$$

Note that E is surjective, but has a nontrivial kernel. Thus we may identify the quotient space

$$\mathcal{L} \doteq \mathcal{D}(\mathcal{M})/\mathrm{Ker}E$$

with $\mathcal{S}$ by means of the map $\epsilon : \mathcal{L} \to \mathcal{S}$, given by

$$\epsilon : [f] \mapsto Ef = \Phi_f,$$

which is a vector space isomorphism. Finally, we denote by $\mathcal{C}_{\Sigma} \doteq \mathcal{D}(\Sigma) \times \mathcal{D}(\Sigma)$ the space of Cauchy data for Φ on Σ, which is by the preceding theorem also isomorphic to $\mathcal{S}$. The corresponding isomorphism will be called ρ and reads

$$\rho : \Phi_f \mapsto (\Phi_f \restriction_{\Sigma}, N^a \nabla_a \Phi_f \restriction_{\Sigma}) \doteq (\phi_f, \pi_f).$$

[1] In spacetime dimension $n = 4$.

But there is more structure on the solution space $\mathcal{S}$ and its isomorphic counterparts $\mathcal{C}_\Sigma$ and $\mathcal{L}$, apart from being infinite dimensional vector spaces. Namely there exists a *symplectic form* $\sigma : \mathcal{C}_\Sigma \times \mathcal{C}_\Sigma \to \mathbb{R}$ on $\mathcal{C}_\Sigma$, given by

$$\sigma\left((\phi_f, \pi_f), (\phi_g, \pi_g)\right) \doteq \int_\Sigma \mathrm{d}\mu_h (\phi_f \pi_g - \pi_f \phi_g),$$

where $\mathrm{d}\mu_h$ is the induced volume element on Σ. One can show that σ is *strongly nondegenerate*, which means that $\sigma\left((\phi_f, \pi_f), (\phi_g, \pi_g)\right) = 0$ for all $(\phi_f, \pi_f) \in \mathcal{C}_\Sigma$ implies $(\phi_g, \pi_g) = 0$. Furthermore, σ is independent of the Cauchy surface, since it can also be written as

$$\sigma\left((\phi_f, \pi_f), (\phi_g, \pi_g)\right) = \int_{\mathcal{M}} f E g \mathrm{d}\mu_g \doteq s(\Phi_f, \Phi_g),$$

as shown e.g. in [Dim80]. Thus, $\mathcal{S}, \mathcal{L}$ and $\mathcal{C}_\Sigma$ are symplectic vector spaces and the maps between them are symplectomorphisms. We would like to remark that the causal propagator E can also be viewed as an antisymmetric bidistribution on $\mathcal{D}(\mathcal{M})$, by setting

$$E(f, g) \doteq \int_{\mathcal{M}} f E g \mathrm{d}\mu_g.$$

Now let us turn to the observables of our classical field theory. Just like in classical mechanics, where observables are functions on the (finite dimensional) phase space, in classical field theory they are nothing but functionals on the infinite dimensional phase space, which is our solution space $\mathcal{S}$. Observables which are linear in Φ can be constructed using the symplectic form and test functions f via the definition

$$F_f \doteq s(\Phi_f, .) \doteq \Phi(f).$$

Note that also the assignment $f \mapsto \Phi(f)$ is linear and has to be distinguished from the map $f \mapsto \Phi_f$. If the field is in the (pure) state[2] Ψ, the outcome of a measurement of F_f will be given by $s(\Phi_f, \Psi) = \int_{\mathcal{M}} f \Psi \mathrm{d}\mu_g$, which corresponds to a spacetime average of the field configuration Ψ w.r.t. to the weighting function f. For $f \to \delta_x$ we obtain a sharp measurement of the field strength at the point x. We may also switch to the equal time picture via the symplectomorphism ρ, giving rise to the definition

$$\tilde{F}_f = \sigma\left((\phi_f, \pi_f), .\right)$$

of $\tilde{F}_f$. When choosing f in such a way that $\pi_f = 0$, then the evaluation of $\tilde{F}_f$ in the state Ψ is

$$\sigma\left((\phi_f, 0), \rho(\Psi)\right) = \int_\Sigma \phi_f N^a \nabla_a \Psi \mathrm{d}\mu_h,$$

which corresponds to the averaged momentum of Ψ on Σ w.r.t. the density ϕ_f. It can be shown (for a thorough discussion see [Wal94]) that for these fundamental observables the *Poisson bracket* of observables on $\mathcal{C}_\Sigma$ can be written in the form

$$\{\tilde{F}_f, \tilde{F}_g\} = \sigma((\phi_f, \pi_f), (\phi_g, \pi_g)). \tag{2.2}$$

[2] Recall that in classical physics, a state is a normalised positive measure on phase space. A pure state corresponds to an element of phase space which can be identified with a solution of the equations of motion.

Indeed, choosing this time the observables such that $\phi_f = 0$ and $\pi_g = 0$, and applying[3] (2.2) to generic Cauchy data (ϕ, π) on Σ we get

$$\left\{ \int_\Sigma \pi_f \phi \mathrm{d}\mu_h, \int_\Sigma \phi_g \pi \mathrm{d}\mu_h \right\} = \int_\Sigma \pi_f \phi_g \mathrm{d}\mu_h.$$

With $\pi_f \to \delta_x, \phi_g \to \delta_y$ we finally obtain

$$\{\phi(x), \pi(y)\} = \delta(x, y),$$

where the δ–distributions used here are defined w.r.t. the measure $\mathrm{d}\mu_h$ on Σ. Thus we recover (on a formal level) the expected Poisson bracket for the canonical conjugate variables (ϕ, π). To summarise, the observables carry the structure of an associative Poisson algebra, generated by the fundamental observables $\Phi(f)$, which can be viewed as compactly supported smooth distributions on $\mathcal{S}$.

2.2. Canonical Quantisation of the Free Scalar Field

Having discussed the structure of the classical free field theory in the last section, we turn now to the quantisation of the theory, which will be guided by the canonical formalism. This formalism basically consists in linearly mapping the commutative subalgebra of fundamental classical field observables into a non commutative algebra $\mathcal{A}(\mathcal{M}, g)$,

$$\Phi(f) \mapsto \hat{\Phi}(f),$$

where the commutation relations in $\mathcal{A}(\mathcal{M}, g)$ are derived by the Poisson bracket of the classical counterpart. Thus, our fundamental quantised field observables should fulfil

$$\begin{aligned} C_0^\infty(\mathcal{M}, \mathbb{R}) \ni f \mapsto \hat{\Phi}(f) &\in \mathcal{A}(\mathcal{M}, g) \text{ is } \mathbb{R} - \text{linear} \\ \hat{\Phi}((\Box_g + m^2)f) &= 0 \ \ \forall\ f \in C_0^\infty(M, \mathbb{R}) \\ \hat{\Phi}(f)^* &= \hat{\Phi}(f) \\ [\hat{\Phi}(f), \hat{\Phi}(g)] &= iE(f, g). \end{aligned}$$

Since from now on we deal with quantum observables, we will skip the hat over such objects for the sake of notational simplicity. The quantum field algebra $\mathcal{A}(\mathcal{M}, g)$ should now be a unital $*$–algebra generated by the objects $\Phi(f)$ and obeying the above relations. A straightforward way to construct $\mathcal{A}(\mathcal{M}, g)$ is given by going over to the *Borchers–Uhlmann–algebra*:

Definition 2.2.1. *The Borchers–Uhlmann–algebra of the neutral Klein Gordon field is defined as*

$$\mathcal{A}(\mathcal{M}, g) = \mathcal{A}_0(\mathcal{M}, g)/\mathcal{I}(\mathcal{M}, g).$$

$\mathcal{A}_0(\mathcal{M}, g)$ *is the free tensor algebra over* $\mathcal{D}(\mathcal{M})$*,*

$$\mathcal{A}_0(\mathcal{M}, g) = \bigoplus_{n=0}^{\infty} \mathcal{D}(\mathcal{M}^n)$$

[3]Note that the r.h.s. of (2.2) is a multiple of the constant function on $\mathcal{C}_\Sigma$.

with $\mathcal{D}(\mathcal{M}^0) \doteq \mathbb{C}$ and only elements of $\mathcal{A}_0(\mathcal{M}, g)$ with finitely many entries are allowed. On $\mathcal{A}_0(\mathcal{M}, g)$ we define an involution $*$ by antilinear extension of $f^*(x_1, ..., x_n) = f(x_n, ..., x_1)$ for $f \in \mathcal{D}(\mathcal{M}^n)$. $\mathcal{I}(\mathcal{M}, g)$ is the $*$–ideal generated by elements of the form $-iE(f, g) \oplus (f \otimes g - g \otimes f)$ and Pf.

We would like to add a few remarks on this definition. Elements of $\mathcal{A}_0(\mathcal{M}, g)$ can be thought of as vectors f, the nth entry $f^{(n)}$ being an element in $\mathcal{D}(\mathcal{M}^n)$. The multiplication of two elements in $\mathcal{A}_0(\mathcal{M}, g)$, $f = (\alpha, f^{(1)}, f^{(2)}, ...)$ and $g = (\beta, g^{(1)}, g^{(2)}, ...)$ reads

$$fg = (\alpha\beta, \alpha g^{(1)} + \beta f^{(1)}, f^{(1)} g^{(1)} + \alpha g^{(2)} + \beta f^{(2)}, ...).$$

Moreover, $\mathcal{A}_0(\mathcal{M}, g)$ carries a natural topology. A sequence f_k is said to converge to 0 in $\mathcal{A}_0(\mathcal{M}, g)$ if all entries converge to 0 in the locally convex topology of $\mathcal{D}(\mathcal{M}^n)$ and if there is an N such that $f_k^{(n)} = 0$ for all $n > N$ and for all k. $\mathcal{A}(\mathcal{M}, g)$ then inherits the corresponding quotient topology. We may also define a local algebra $\mathcal{A}(\mathcal{O}, g \restriction_{\mathcal{O}})$ by requiring $f^{(n)} \in \mathcal{D}(\mathcal{O}^n)$. Obviously, the unit element is given by $(1, 0, 0, ...)$. To make contact with the aim of generating an algebra out of the $\Phi(f)$, note that we can identify the element $(0, f, 0, ...)$ with $\Phi(f)$, $(0, 0, f \otimes g)$ with $\Phi(f)\Phi(g)$ and so forth.

We have thus constructed the free field algebra in a purely algebraic manner, i.e. without any recourse to some Hilbert space representation. So far, our fields $\Phi(f)$ are nothing but distributions with values in a non commutative topological $*$–algebra. For our purposes it is sufficient to have a $*-$algebra (i.e. without norm), whose Hilbert space representation yields unbounded field operators. We just mention here that one can also construct a C^*- algebra for the field in a unique manner, the so–called *Weyl algebra* (see e.g. [Dim80] for details). Its representation yields bounded operators on a Hilbert space $\mathcal{H}$, with the advantage of avoiding subtle domain questions. In the sequel, $\mathcal{A}(\mathcal{M}, g)$ will always refer to the Borchers–Uhlmann algebra defined above.

It is obvious that the construction of $\mathcal{A}(\mathcal{M}, g)$ works for every globally hyperbolic spacetime. Consider now an isometric embedding $\psi : \mathcal{N} \to \mathcal{M}$ of a globally hyperbolic spacetime $(\mathcal{N}, h)$ into $(\mathcal{M}, g)$, which is also globally hyperbolic. Assume furthermore that ψ preserves orientation and time orientation and is causal, i.e. if $x, x' \in \mathcal{N}$, then all causal curves connecting $\psi(x)$ and $\psi(x')$ in $\mathcal{M}$ are contained in $\psi(\mathcal{N})$. We can then define an algebra homomorphism α_ψ : $\mathcal{A}(\mathcal{N}, h) \to \mathcal{A}(\mathcal{M}, g)$ via $\alpha_\psi(\Phi'(f')) = \Phi(\psi_*^{-1} f')$, with $f' \in \mathcal{D}(\mathcal{N})$. Now, regarding the quantum field Φ as a map $\Phi_{(\mathcal{M},g)} : \mathcal{D}(\mathcal{M}) \to \mathcal{A}(\mathcal{M}, g)$ defined for all spacetimes $(\mathcal{M}, g)$, we require

$$\alpha_\psi \circ \Phi_{(\mathcal{N},h)}(f') = \Phi_{(\mathcal{M},g)}(\psi_*^{-1} f')$$

for all isometric causal embeddings $\psi : (\mathcal{N}, h) \to (\mathcal{M}, g)$. Such Φ is then called a *locally covariant* quantum field. Note that this notion of local covariance refers to the particular observable Φ. The observable $\Phi(f)$ itself is obviously locally covariant. The idea behind the notion of locally covariant observables is that they can be constructed in an arbitrarily small region $\mathcal{O}$, without the knowledge of the spacetime outside $\mathcal{O}$. We will return to this issue when discussing the energy–momentum tensor.

2.3. States

In order to do physics with our quantised field, we have to turn to the notion of states, which is defined as follows:

Definition 2.3.1. *A state on the $*$-algebra $\mathcal{A}$ is a normalised positive linear continous functional ω on $\mathcal{A}$, that is*

1. $\omega(\mathbb{1}) = 1$

2. $\omega(A^*A) \geq 0 \ \forall A \in \mathcal{A}$

Given a state ω on $\mathcal{A}(\mathcal{M}, g)$, we can define the *$n$–point–functions* $\mathcal{W}_n^\omega$ by

$$\mathcal{W}_n^\omega(f_1, ..., f_n) \doteq \omega(\Phi(f_1)...\Phi(f_n)),$$

which extend to distributions on $\mathcal{D}(\mathcal{M}^n)$ by continuity of ω and the Schwartz kernel theorem. Conversely, if all n–point–functions are given, ω is determined on the whole of $\mathcal{A}(\mathcal{M}, g)$. The following definition will provide some notions pertaining to states that will be worked with in the sequel:

Definition 2.3.2.

- *A state ω on $\mathcal{A}(\mathcal{M}, g)$ is called **quasifree** or **Gaussian** state if its n–point–functions satisfy*

$$\mathcal{W}_n^\omega(f_1, ..., f_n) = \begin{cases} \sum_{p \in P_n} \prod_{(i,j) \in p} \mathcal{W}_2^\omega(f_i, f_j) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

  where P_n denotes all possible partitions p of $\{1, ..., n\}$ in pairs (i, j) with $i < j$ and the product is performed over all pairs of such a partition.

- *ω is called **pure** if it cannot be written as a convex combination*

$$\lambda\omega_1 + (1 - \lambda)\omega_2, \ \lambda < 1$$

  *of two other states $\omega_i \neq \omega$. Otherwise it is called **mixed**.*

Quasifree states are convenient to work with, since they are entirely characterised by their two–point–function. Important examples are the Minkowski vacuum state and the states of low energy, which will be introduced later. A prototype for mixed states are thermal states, such as the Unruh state, obtained by restricting the Minkowski–vacuum to the Rindler wedge. We will only be concerned with pure states, which can also be thought of as states carrying maximal information about a physical system. A very important link to the usual formalism of QFT involving a Hilbert space structure is given by the famous GNS theorem:

Theorem 2.3.3. *Let ω be a state on $\mathcal{A}(\mathcal{M}, g)$. Then there exists a unique (up to unitary equivalence) representation π of $\mathcal{A}(\mathcal{M}, g)$ by linear operators on a dense domain $\mathcal{D}$ of some Hilbert space $\mathcal{H}$ and a normalised vector $\Omega \in \mathcal{D}$ such that*

$$\omega(A) = (\Omega, \pi(A), \Omega)$$

and $\mathcal{D} = \{\pi(A)\Omega, A \in \mathcal{A}(\mathcal{M}, g)\}$.

Proof. We will only sketch the important steps of the proof. The state ω induces a scalar product on $\mathcal{A}(\mathcal{M}, g)$ via

$$(A, B) \doteq \omega(A^* B),$$

which is hermitean, i.e. $(A, B) = \overline{(B, A)}$, and positive semidefinite due to the positivity of ω. Next we introduce the *Gelfand ideal* $\mathcal{G} \doteq \{A \in \mathcal{A}(\mathcal{M}, g) : \omega(A^* A) = 0\}$, which is a left ideal of $\mathcal{A}(\mathcal{M}, g)$, i.e. $\mathcal{G}$ is a subspace of $\mathcal{A}(\mathcal{M}, g)$ and $BA \in \mathcal{G}$ for $A \in \mathcal{G}$ and $B \in \mathcal{A}(\mathcal{M}, g)$. Thus, $(.,.)$ is positive on the quotient space $\mathcal{D} \doteq \mathcal{A}(\mathcal{M}, g)/\mathcal{G}$ and we can complete $\mathcal{D}$ w.r.t. the norm provided by $(.,.)$ to obtain the Hilbert space $\mathcal{H}$. The representation of $A \in \mathcal{A}(\mathcal{M}, g)$ on $\mathcal{D}$ is now defined by

$$\pi(A)[B] = [AB],$$

$[.]$ denoting equivalence classes. Defining $\Omega \doteq [\mathbb{1}]$ yields the desired identity $(\Omega, \pi(A)\Omega) = ([\mathbb{1}], [A]) = \omega(A)$. Finally, if $(\mathcal{H}', \mathcal{D}', \pi', \Omega')$ constitutes another such representation one checks that $U : \mathcal{D} \to \mathcal{D}'$, defined by

$$U\pi(A)\Omega \doteq \pi'(A)\Omega'$$

extends to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$. □

The GNS–construction leads quite naturally to the famous Fock space structure of our scalar bosonic theory. Note that due to the commutation relations incorporated in the ideal $\mathcal{I}(\mathcal{M}, g)$ in definition (2.2.1), all elements of $\mathcal{A}(\mathcal{M}, g)$ lie in the equivalence class of elements whose $n-$th entry are *symmetric* test functions in $\mathcal{D}(\mathcal{M}^n)$. If we consider the quasifree Minkowski vacuum ω_{Mink} on $\mathcal{A}(\mathbb{R}^4, \eta)$, the corresponding null space of the induced scalar product from the GNS construction will consist of elements f of $\mathcal{A}(\mathbb{R}^4, \eta)$, where the Fourier transform $\hat{f}^{(n)}(p_1, ..., p_n)$ of the $n-$th entry $f^{(n)}(x_1, ..., x_n)$ of f vanishes if one of its momenta satisfies $p_0^2 - \mathbf{p} = m^2; p_0 > 0$. Thus, vectors in the dense domain $\mathcal{D}$ of the corresponding GNS Hilbert space will be equivalence classes $[v]$ of vectors $v \in \mathcal{A}(\mathbb{R}^4, \eta)$ whose entries, when restricted to the upper mass shell, are nonzero. They can then be interpreted as $n-$ particle wave functions.

The GNS construction gives rise to the notion of the *folium* of a state ω. Namely, $\mathrm{Fol}(\omega)$ consists of all states which can be realised as vectors or density matrices on the GNS Hilbert space of ω. An important concept which links the GNS representation of a quasifree state with a representation of the field on a Fock space is the so called *one particle Hilbert space structure*:

Definition 2.3.4. *Let $\mathcal{K}$ be a Hilbert space and $\kappa : \mathcal{V} \to \mathcal{K}$ a real linear map on the symplectic vector space $(\mathcal{V}, \sigma)$ with the property*

$$\mathrm{Im}\langle \kappa(f), \kappa(g) \rangle = \frac{1}{2}\sigma(f, g)$$

for all $f, g \in \mathcal{V}$. Then $(\kappa, \mathcal{K})$ is a one particle Hilbert space structure for $(\mathcal{V}, \sigma)$.

Given such a one particle Hilbert space structure $(\kappa, \mathcal{K})$, we can construct a Fock representation of the field algebra $\mathcal{A}(\mathcal{M}, g)$ by taking the symmetric Fock space $\mathcal{F}_s(\mathcal{K})$ over $\mathcal{K}$ and setting $\pi(\Phi(f)) = a^\dagger(\kappa([f])) + a(\kappa([f]))$, where $a^\dagger$ and a denote the usual creation and annihilation operators on $\mathcal{F}_s(\mathcal{K})$. Moreover, the vacuum vector $|0\rangle$ defines a quasifree state on $\mathcal{A}(\mathcal{M}, g)$. Conversely, given a quasifree state ω we can always find a one particle Hilbert space structure $(\kappa, \mathcal{K})$ where ω is represented by $|0\rangle$ in the corresponding Fock space representation. For details of this construction see the discussion in [Wal94]. We will come back to this issue later when giving the explicit form of $(\kappa, \mathcal{K})$ for a pure quasifree isotropic and homogenous state.

2.3.1. Hadamard States

After having quantised our scalar field on an algebraic level and having introduced the concept of states, there remains the question how to characterise physically admissible states. In Minkowski space this is rather straightforward, since there is a unique vacuum state ω_{Mink} which is invariant under all isometries of the spacetime and carries the lowest possible energy. Moreover, all states in its folium have an interpretation in terms of particles. This situation changes when we consider QFT on general spacetimes. Since spacetime symmetries will then generally be absent, they cannot be used to single out a preferred vacuum state. For the same reason, there is also no unique particle interpretation. Apart from that, we want to define expectation values of observables like $:\Phi^2:$ and $:T_{ab}:$. We know already from Minkowskian theory that, in order to define such objects, we have to subtract an infinite counterterm, which can be achieved by a manipulation called *normal ordering*. On Minkowski, space this manipulation relies on the existence of the unique vacuum state. Therefore, on generic spacetimes a more abstract characterisation for physical states is needed. It turnes out that the *Hadamard condition* for a state ω is a necessary and sufficient requirement in order to define expectation values of field products at the same point. One may reach at this condition by requiring that, locally, ω should "look like" the Minkowski vacuum. We want to make this idea precise for the example of ω_{Mink} of the massless scalar field. Its two–point function in position space reads

$$\omega_{Mink}(\Phi(f)\Phi(g)) = \lim_{\epsilon\downarrow 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \mathrm{d}x^4 \mathrm{d}y^4 \frac{f(x)g(y)}{-(x-y)^2 + 2i\epsilon(x_0 - y_0) + \epsilon^2}. \tag{2.3}$$

The idea is now to generalise this expression to an arbitrary curved spacetime $\mathcal{M}$. To this end, we choose a time function $T : \mathcal{M} \to \mathbb{R}$ and define the function

$$\sigma_\epsilon : (x,y) \mapsto \sigma(x,y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2,$$

where σ denotes the signed[4] squared geodesic distance between x and y. According to (2.3), we then require that the leading singularity of a physically sensible state should be proportional to σ_ϵ^{-1}. Setting

$$\mathcal{G} \doteq \lim_{\epsilon\downarrow 0} \frac{1}{4\pi^2} \left(\frac{\Delta^{1/2}}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{L^2}\right) \right), \tag{2.4}$$

this leads to the local ansatz[5]

$$\mathcal{W}_2^\omega = \mathcal{G} + W^\omega,$$

where $\Delta^{1/2}, V$ and W^ω are supposed to be smooth functions and L is a constant with the dimension of length. The appearance of the logarithmic term in the singular part $\mathcal{G}$ is a consequence of the fact that $\mathcal{W}_2^\omega$ must be a bisolution of the Klein–Gordon operator. Namely, this condition implies

$$P\mathcal{G} \in \mathcal{E}(\mathcal{M}^2) \tag{2.5}$$

$\mathcal{G}$ is called *Hadamard parametrix*, where "parametrix" means that $\mathcal{G}$ defines an equivalence class of distributive solutions of (2.1), the equivalence relation being to differ by a smooth function. Using the property [Fri75]

$$g^{ab}(\nabla_a\sigma)(\nabla_b\sigma) = -4\sigma \tag{2.6}$$

[4] Signed means that $\sigma(x,y)$ is negative if x and y can be connected by a timelike geodesic.

[5] It is only defined on a gedesically convex neighborhood.

and making the ansatz

$$V = \frac{1}{L^2}\sum_{j=0}^{\infty} v_j \left(\frac{\sigma}{L^2}\right)^j ,$$

equation (2.5) leads to the following system of partial differential equations, the so called *Hadamard recursion relations*:

$$2g^{ab}(\nabla_a\sigma)\nabla_b\Delta^{1/2} + (8 + \Box_g\sigma)\Delta^{1/2} = 0 \tag{2.7}$$

$$2g^{ab}(\nabla_a\sigma)\nabla_b v_0 + (4 + \Box_g\sigma)v_0 = -L^2 P\Delta^{1/2} \tag{2.8}$$

$$2(j+1)g^{ab}(\nabla_a\sigma)\nabla_b v_{j+1} + (j+1)(\Box_g\sigma - 4j)v_{j+1} = -L^2 P v_j \tag{2.9}$$

Imposing the initial condition $\Delta^{1/2}(x,x) = 1$ (which follows from comparison with the Minkowski case), this system of equation has uniquely determined solutions [BGP07; Mor99]. However, the representation of V by a power series in σ is only known to exist on analytic spacetimes. That is why one truncates the series for V at some finite k,

$$V^{(k)} \doteq \frac{1}{L^2}\sum_{j=1}^{k} v_j \left(\frac{\sigma}{L^2}\right)^j , \tag{2.10}$$

and defines the distribution $\mathcal{G}_k$ by replacing V by $V^{(k)}$ in (2.4). We are now prepared to define the notion of a Hadamard state on $\mathcal{A}(\mathcal{M}, g)$:

Definition 2.3.5. *A state ω on $\mathcal{A}(\mathcal{M}, g)$ is called **Hadamard** if for any geodesic convex neighborhood $\mathcal{N}$ of any given point $x \in \mathcal{M}$ one can find a sequence $W_k^\omega \in C^k(\mathcal{N}\times\mathcal{N})$ such that its two point distribution can be written as*

$$\mathcal{W}_2^\omega(f,g) = \mathcal{G}_k(f,g) + \int\limits_{\mathcal{N}\times\mathcal{N}} W_k^\omega(x,x')f(x)g(x')\mathrm{d}\mu_g(x)\mathrm{d}\mu_g(x'). \tag{2.11}$$

By means of (2.7) one can always compute $\mathcal{G}_k$ and does not have to worry about convergence questions of the expansion of V. The dependence of $\mathcal{W}_2^\omega$ on the state ω is encoded in the series W_k^ω. In [KW91] it was proven that $\mathcal{G}_k$ does not depend on the choice of the time function T entering in the definition of σ_ϵ, so it does not appear in definition 2.3.5. Apart from this characterisation of Hadamard states in position space, there is an equivalent one due to Radzikowski [Rad96], which is formulated in the language of *microlocal analysis*. It imposes a certain condition on the *wavefront set* of the two–point distribution of ω. We will include it in our exposition, since it will play an important role in the proof of Fewster's energy inequality, which is the original motivation for the construction of SLE's. In order to formulate this microlocal characterisation, we need to introduce some facts about distributions on manifolds and the analysis of their singularities. A standard reference for this topic is [Hör90]. We start with distributions on $\mathbb{R}^n$. On $\mathcal{D}(\mathbb{R}^n)$ we have a natural topology, defined as follows: A sequence f_j is said to converge to 0 in $\mathcal{D}(\mathbb{R}^n)$ if there exists a compact set $K \subset \mathbb{R}^n$ with $\mathrm{supp} f_j \in K$ for all j and if for every multi index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we have

$$\lim_{j\to 0} \sup |D^\alpha f_j| = 0,$$

where
$$D^\alpha \doteq \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The space of distributions on $\mathbb{R}^n$ is then defined as the topological dual of $\mathcal{D}(\mathbb{R}^n)$, that is the continous linear maps $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$, for which we write $\mathcal{D}'(\mathbb{R}^n)$. There is a natural embedding $\mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ by setting
$$T_f(g) \doteq \int_{\mathbb{R}^n} \mathrm{d}^n x f(x) g(x).$$

Such distributions are called smooth. Given $T \in \mathcal{D}'(\mathbb{R}^n)$ one can ask if it is smooth at $p \in \mathbb{R}^n$. For $h \in \mathcal{D}(\mathbb{R}^n)$, define $hT(f) \doteq T(hf)$. We say that T is smooth at p if there exist $h \in \mathcal{D}(\mathbb{R}^n)$ with $h(p) = 1$ and $\phi \in \mathcal{E}(\mathbb{R}^n)$ such that $hT(f) = \int f(x)h(x)\phi(x)\mathrm{d}^n x$ for all $f \in \mathcal{D}(\mathbb{R}^n)$. We denote all points p at which T is smooth by $\{C^\infty T\}$ and define the *singular support* of T by
$$\text{sing supp}(T) \doteq \mathbb{R}^n \setminus \{C^\infty T\},$$
which is a closed set. However, the singular support gives no information about the directions of the singularity at $p \in \text{sing supp}(T)$. This can be achieved by looking at the Fourier transform $\widehat{hT}$ of T, localised around p (i.e. $p \in \text{supp}h$). Since hT has compact support, $\widehat{hT}$ is smooth. If T were smooth at p, then then $\widehat{hT}$ would be of fast decrease. If $p \in \text{sing supp}T$, one can analyse in which directions $\widehat{hT}$ does not decrease rapidly. We now want to make this idea precise. The Fourier transform on $\mathcal{D}(\mathbb{R}^n)$ is defined by
$$\hat{f}(k) \doteq (2\pi)^{-n/2} \int f(x) e^{ikx} \mathrm{d}^n x, \tag{2.12}$$
where kx is here the usual euclidean product in $\mathbb{R}^n$. (2.12) extends to distributions via
$$\hat{T}(f) \doteq T(\hat{f}).$$

Definition 2.3.6. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $(p, k) \in \mathbb{R}^n \times \dot{\mathbb{R}}^n$, where $\dot{\mathbb{R}}^n \doteq \mathbb{R}^n \setminus 0$. The point (p, k) is called* ***regular directed*** *if for some $h \in \mathcal{D}(\mathbb{R}^n)$ with $h(p) = 1$ there exists an open neighborhood $V \subset \dot{\mathbb{R}}^n$ of k such that*
$$\sup_{k' \in V} |\widehat{hT}(\tau k')| \sim O(\tau^{-N})$$
for all $N \in \mathbb{N}$ as $\tau \to \infty$. The set of all regular directed points of T is called $\{\mathrm{regd}T\}$.

Now we can introduce the central notion of the wavefront set of a distribution:

Definition 2.3.7. *The* ***wavefront set*** *of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is defined by*
$$WF(T) \doteq \mathbb{R}^n \times \dot{\mathbb{R}}^n \setminus \{\mathrm{regd}T\}.$$

The following proposition collects some important properties of wavefront sets of distributions in $\mathcal{D}'(\mathbb{R}^n)$, whose proves can be found in [Hör90]:

Proposition 2.3.8.

1. sing supp$T = \{x \in \mathbb{R}^n : (x, k) \in WF(T)\}$

2. *Let P be a partial differential operator with smooth coefficients. Then $WF(PT) \subseteq WF(T)$.*

3. *Let $U, V \subset \mathbb{R}^n$, $T \in \mathcal{D}'(V)$ and $\varphi : U \to V$ a diffeomorphism. The pullback $\varphi_* T$, defined by $\varphi_* T(f) = T(\varphi^* f)$ for all $f \in \mathcal{D}(U)$ fulfils*

$$WF(\varphi_* T) = \varphi_* WF(T) = \{(\varphi^{-1}(x), \varphi_* k) : (x, k) \in WF(T)\}.$$

The third statement says that the wavefront set transforms like covectors under coordinate transformations, which means that the wavefront set of T is a subset of the cotangent bundle. Therefore the concept of wavefront sets can be used in a meaningful way for distributions on a general manifold $\mathcal{M}$, since it is independent of the chosen chart.

As mentioned above, we want to find a criterion on the wavefront set of the two–point distribution of ω in order to be a Hadamard state. We return to ω_{Mink} of the massless Klein–Gordon field. From its position space representation (2.3) we infer that $(x, y) \in \text{sing supp}\mathcal{W}_2 \Leftrightarrow (x-y)^2 = 0$. Its Fourier representation is given by

$$\mathcal{W}_2 = (2\pi)^{-2} \int \mathrm{d}^4 p \delta(p^2) \Theta(p_0) e^{-ipx},$$

where we put $y = 0$. We localise $\mathcal{W}_2$ with $h \in \mathcal{D}(\mathbb{R}^4)$ and take the Fourier transform,

$$\widehat{h\mathcal{W}_2}(p) = \left(\hat{h} * \hat{\mathcal{W}_2}\right)(p) = (2\pi)^{-4} \int \mathrm{d}^4 q \hat{h}(q) \hat{\mathcal{W}}_2(p-q).$$

Since $\hat{\mathcal{W}}_2$ has support on the boundary of the forward light cone V_+ and $\hat{h}$ decreases rapidly in all directions, it follows that the only directions in which $\widehat{h\mathcal{W}_2}(p)$ is not of rapid decrease are given by lightlike future directed covectors p. Repeating the same argument for $\mathcal{W}_2$ when regarded as distribution in two independent variables (x, y) we get the result that for singular directions (k, k') at (x, y) with $(x-y)^2 = 0$ it must hold $k = -k'$. The wavefront set of $\mathcal{W}_2$ of the massless Klein Gordon field on Minkowski space therefore reads

$$WF(\mathcal{W}_2) = \{(x, y, k, -k) \in \mathbb{R}^n \times \dot{\mathbb{R}}^n : (x-y)^2 = 0, k^2 = 0, k||(x-y), k_0 > 0\}.$$

The fact that $k_0 > 0$ reflects the positivity of the energy, also known as the relativistic spectrum condition

$$\text{spec} P_\mu \subset \overline{V_+},$$

that has to be satisfied by the representors of the generators of translations in the GNS representation of the vacuum.

To proceed, we introduce the notation

$$\mathcal{N} \doteq \{(x, k) \in \dot{T}^* \mathcal{M} : g^{ab}(x) k_a k_b = 0\}$$

Furthermore, we write $(x, k) \sim (x', k')$ for $(x, k), (x', k') \in \dot{T}^* \mathcal{M}$ iff there exists a null geodesic γ connecting x with x' such that k and k' are cotangent to γ and k' is the parallel transport of k. We can now state the following proposition [Hör90]:

Proposition 2.3.9. *Let $\Lambda \in \mathcal{D}'(\mathcal{M}^2)$ be a distributive bisolution of P on the globally hyperbolic spacetime $(\mathcal{M}, g)$. Then*

1. $WF(\Lambda) \subset \mathcal{N} \times \mathcal{N}$

2. $(x, k; x', k') \in WF(\Lambda)$ *with* $k, k' \neq 0 \Rightarrow (y, l; y', l') \in WF(\Lambda)$ *for all* $(y, l) \sim (x, k)$ *and* $(y', l') \sim (x', k')$.

If Λ is the two point function of a Hadamard state, we already know that its singular support contains only elements (x, x') which are lightlike related. Furthermore, from the wavefront set of $\mathcal{W}_2$ for ω_{Mink} of the massless Klein–Gordon it follows that the corresponding singular directions are antiparallel, where the first one must be future directed. One might thus expect that it is exactly this last condition on the wavefront set that the two point function of a Hadamard state has to satisfy. Indeed, Radzikowski [Rad96] could prove the following equivalence:

Proposition 2.3.10. *Let ω be a state on $\mathcal{A}(\mathcal{M}, g)$ and $\mathcal{W}_2^\omega$ be its two–point function. Then the following statements are equivalent:*

1. $WF(\mathcal{W}_2^\omega) = \{(x, k; x', -k') \in \dot{T}^*\mathcal{M} \times \dot{T}^*\mathcal{M} : (x, k) \sim (x', k'), k \rhd 0\}$

2. ω *is a Hadamard state in the sense of definition 2.3.5.*

Because of the condition $k \rhd 0$, the microlocal characterisation of Hadamard states is also called *microlocal spectrum condition*, since it generalises the positivity of energy in a local way.

2.3.2. Pure Quasifree Isotropic and Homogenous States on FRW Spacetimes

After having discussed the general aspects of states, we return to the class of spatially flat FRW spacetimes. We would like to characterise the class of isotropic homogenous quasifree and pure states. Since such states are entirely characterised by their two–point distribution $\mathcal{W}_2^\omega$, the symmetry requirement can be formulated as follows:

Definition 2.3.11. *Let $(\mathcal{M}, g)$ be spatially flat FRW spacetime and let $E(3)$ denote its isometry group. A quasifree state ω on $\mathcal{A}(\mathcal{M}, g)$ is homogenous and isotropic (we will also say $E(3)$–invariant) if*

$$\mathcal{W}_2^\omega(f, h) = \mathcal{W}_2^\omega(f \circ g^{-1}, h \circ g^{-1})$$

for all $g \in E(3)$ and for all $f, h \in \mathcal{D}(\mathcal{M})$.

Lüders and Roberts derived the the general form of isotropic homogenous quasifree and pure states. We will state their result for such states on spatially flat FRW spacetimes [LR90, Theorem 2.3.]

Theorem 2.3.12. *Let $(\mathcal{M}, g)$ be a spatially flat FRW spactime. The quasifree pure homogenous and isotropic states ω on $\mathcal{A}(\mathcal{M}, g)$ are given by the two–point function*

$$\mathcal{W}_2^\omega(f, g) = \int_{\mathbb{R}^3} \left(\overline{\widehat{Ef}}(t_0, \mathbf{k}) \ \overline{\widehat{\partial_t Ef}}(t_0, \mathbf{k}) \right) \mathbf{S}(k) \left(\widehat{Eg}(t_0, \mathbf{k}) \ \widehat{\partial_t Eg}(t_0, \mathbf{k}) \right)^t \mathrm{d}^3\mathbf{k},$$

where $k \doteq |\mathbf{k}|$, the spatial Fourier transform is defined as

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} \mathrm{d}^3\mathbf{x}$$

and the matrix $\mathbf{S}(k)$ is given by

$$\mathbf{S}(k) \doteq \begin{pmatrix} a^6(t_0)|q(k)|^2 & a^3(t_0)\overline{q}(k)p(k) \\ a^3(t_0)\overline{p}(k)q(k) & |p(k)|^2 \end{pmatrix}.$$

p and q are bounded measurable functions on $\mathbb{R}^3$ satisfying $\overline{q}p - q\overline{p} = i$ and t_0 is an arbitrary initial time.

Remark 2.3.13. *The causal propagator (for the case $\Sigma = \mathbb{R}^3$) is explicitly given by*

$$(Ef)(t,\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k} \int_I \mathrm{d}t' G_k(t,t') a^3(t') \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}' e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} f(t',\mathbf{x}')$$

where $G_k(t,t') = i\left(\overline{S}_k(t')S_k(t) - \overline{S}_k(t)S_k(t')\right)$ and the mode function $S_k(t)$ is a solution of the time part of the Klein–Gordon equation,

$$\ddot{S}_k(t) + 3\frac{\dot{a}(t)}{a(t)}\dot{S}_k(t) + \left(m^2 + k^2 a^{-2}(t)\right) S_k(t) = 0, \tag{2.13}$$

fulfilling the constraint

$$\overline{S}_k \dot{S}_k - S_k \dot{\overline{S}}_k = i a^{-3}. \tag{2.14}$$

Note that $G_k(t,t')$ is independent of a specific choice of $S_k(t)$ due to (2.14). Now given a pure, quasifree, isotropic and homogenous state, characterised by the functions q and p from the above theorem, we may single out mode functions $T_k(t)$ by imposing the following initial conditions at t_0:

$$T_k(t_0) = -p(k)a^{-3}(t_0), \quad \dot{T}_k(t_0) = q(k). \tag{2.15}$$

In the sense of distributions, $\mathcal{W}_2^\omega(x,x')$ is then given as

$$\mathcal{W}_2^\omega(x,x') = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k} \overline{T}_k(t) T_k(t') e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \tag{2.16}$$

The information about the state is therefore contained in the specification of the modes $T_k(t)$.

Via equation (2.15), the state ω can be seen as being equivalent with the choice of a set of preferred modes (sometimes called "positive frequency modes"). With the help of these modes, or equivalently, their initial conditions, we can define the following one particle Hilbert space structure: For $f \in \mathcal{D}(\mathcal{M})$ set

$$\begin{aligned} \kappa([f]) =& iq(k)a^3(t_0)\widehat{Ef(t_0)}(\mathrm{k}) + ip(k)(\widehat{\partial_t Ef)(t_0})(\mathrm{k}) \\ =& ia^3(t_0)\left(\dot{T}_k(t_0)\widehat{Ef(t_0)}(\mathrm{k}) - T_k(t_0)(\widehat{\partial_t Ef)(t_0})(\mathrm{k})\right). \end{aligned} \tag{2.17}$$

$\kappa([f]) \in L_2(\mathbb{R}^3) = \mathcal{K}$ can be interpreted as one particle wave function in spatial momentum space. This construction gives the explicit representation of $\mathcal{A}(\mathcal{M},g)$ on the symmetric Fock space $\mathcal{F}_s(L_2(\mathbb{R}^3))$, induced by the quasifree state ω of the above theorem.

2.4. The Renormalised Stress–Energy–Tensor

The algebra $\mathcal{A}(\mathcal{M}, g)$ of the Klein Gordon field contains only sums of products of the field at different points. However, we want to be able to calculate expectation values of quantum observables which are the counterparts of classical quantities like $\phi^2(x)$ or the energy–momentum tensor $T_{ab}(x)$, which is also quadratic in the field and its derivatives. In Minkowski space, this is done by *normal ordering* using the unique vacuum state ω_{Mink}. In the corresponding GNS representation, the quantum field is represented as the sum of creation and annihilation operators,

$$\Phi(f) = a^\dagger(\kappa([f])) + a(\kappa([f])).$$

One then defines normal ordered products of fields by putting all creation operators to the left:

$$\begin{aligned}
&: \Phi(f)\Phi(g) : \\
&=: a^\dagger(\kappa([f]))a(\kappa([g])) + a(\kappa([f]))a(\kappa([g])) + a^\dagger(\kappa([f]))a^\dagger(\kappa([g])) + a(\kappa([f]))a^\dagger(\kappa([g])) : \\
&\doteq a^\dagger(\kappa([f]))a(\kappa([g])) + a(\kappa([f]))a(\kappa([g])) + a^\dagger(\kappa([f]))a^\dagger(\kappa([g])) + a^\dagger(\kappa([g]))a(\kappa([f])),
\end{aligned}$$

which is equivalent to

$$: \Phi(f)\Phi(g) := \Phi(f)\Phi(g) - \mathcal{W}_2(f, g). \tag{2.18}$$

The expectation value of $: \Phi^2(x) :$, evaluated in an arbitrary state λ in the folium of ω_{Mink} is then a smooth function of x, since λ and ω_{Mink} are both Hadamard states. One may now try to generalise this prescription to curved spacetimes by taking an arbitrary Hadamard state ω_0 as reference state and replacing $\mathcal{W}_2$ by $\mathcal{W}_2^{\omega_0}$ in formula (2.18). However this will not yield a locally covariant quantum field, as shown in [HW01]. Instead, the authors of [HW01] constructed locally covariant Wick powers by using $\mathcal{G}$ instead of $\mathcal{W}_2^\omega$ of some Hadamard state ω for normal ordering. By imposing further physical requirements like scaling behaviour and commutation relation with the free field, they could show that renormalisation ambiguities already arise for the definition of the Wick polynomials. For instance, two prescriptions for the Wick square are related by

$$: \widetilde{\Phi^2}(x) :=: \Phi^2(x) : + \alpha m^2 + \beta R,$$

i.e. we have a two-parameter renormalisation ambiguity. This is in sharp contrast with the situation in Minkowski space. There such ambiguities arise only on the level of time ordered products, that is, for the interacting field. To summarise, it is possible to construct an enlarged algebra of observables $\mathcal{W}(\mathcal{M}, g)$, which contains in particular the quantum energy–momentum tensor $: T_{ab} :$ and is subject to renormalisation ambiguities already for free fields. However, the admissible states on $\mathcal{W}(\mathcal{M}, g)$ have to be Hadamard.

For our purposes it is sufficient to work with $\mathcal{A}(\mathcal{M}, g)$ and to define expectation values of products of fields at one point via the *Hadamard point–splitting procedure*. According to (2.18), but using $\mathcal{G}$ instead of the two point function of a reference state, for a Hadamard state ω on $\mathcal{A}(\mathcal{M}, g)$ one defines

$$\omega(: \Phi^2(x) :) \doteq \lim_{y \to x} \left(\mathcal{W}_2^\omega(x, y) - \mathcal{G}(x, y) \right).$$

This quantity depends manifestly only on the state, since $\mathcal{G}(x, y)$ is constructed out of the local geometry of $(\mathcal{M}, g)$. Here, one part of the ambiguity mentioned above can already be seen by changing the lengthscale in the definition of $\mathcal{G}$. Now, a similar ansatz can be made in order to define $\omega(: T_{ab} :)$. We first want to state an important uniqueness theorem due to Wald:

Theorem 2.4.1. *Let $\omega(: T_{ab} :) : \mathcal{S}(\mathcal{A}(\mathcal{M}, g)) \to \mathcal{E}(\mathcal{M})$ be a map from the states on $\mathcal{A}(\mathcal{M}, g)$ to the smooth functions on $\mathcal{M}$, which satisfies the following conditions:*

1.

$$\begin{aligned}&\omega_1(: T_{ab} :) - \omega_2(: T_{ab} :)\\&= \lim_{x' \to x} \left(\nabla_a \otimes Y_b^{b'}(x, x') \nabla_{b'} - \frac{1}{2} g_{ab}(x) \left(g_{cd}(x) \nabla^c Y_{e'}^d(x, x') \nabla^{e'} - m^2 \right) \right) F(x, x')\end{aligned}$$

where $F(x, x') \doteq \mathcal{W}_2^{\omega_1} - \mathcal{W}_2^{\omega_2}$ and $Y_{e'}^b(x, x')$ denotes the bitensor which identifies the tangent spaces at x and x' via parallel transport along the unique geodesic linking x and x' (for all point–splitting constructions we can always restrict to some convex geodesic neighborhood $\mathcal{N}$).

2. $\nabla^a \omega(: T_{ab} :) = 0$ *for all* ω.

3. *Let $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ be two globally hyperbolic spacetimes. Let $y \in \mathcal{M}$ and let $\mathcal{O}$ be a globally hyperbolic neighborhood of y, such that $\mathcal{O} \cap \Sigma$ is a Cauchy surface for $\mathcal{O}$, where Σ is a Cauchy surface for $(\mathcal{M}, g)$. Let $i : \mathcal{O} \to \mathcal{O}' \subseteq \mathcal{M}'$ be an isometry, where $i(\mathcal{O} \cap \Sigma)$ is a Cauchy surface of the form $\mathcal{O}' \cap \Sigma'$, with Σ' a Cauchy surface of $\mathcal{M}'$. Let furthermore ω, ω' be states on $\mathcal{A}(\mathcal{M}, g)$ and $\mathcal{A}(\mathcal{M}', g')$, respectively, such that*

$$\omega \restriction_{\mathcal{A}(\mathcal{O}, g)} = \omega' \restriction_{\mathcal{A}(\mathcal{O}', i^* g)} .$$

Then there should hold

$$\omega(: T_{ab}(y) :) = i_* \omega'(: T_{ab}(i(y)) :).$$

Now consider two such maps $\omega(: T_{ab}^{(1)} :)$ and $\omega(: T_{ab}^{(2)} :)$. Then their difference

$$C_{ab} \doteq \omega(: T_{ab}^{(1)} :) - \omega(: T_{ab}^{(2)} :)$$

is independent of ω, $\nabla^a C_{ab} = 0$ and C_{ab} depends only on g_{ab} and its derivatives at x.

The last condition in the above list requires $\omega(: T_{ab} :)$ to be a locally covariant tensor field. The appropriate condition for the observable T_{ab} itself would require it to be a locally covariant quantum field. This was one of the underlying ideas towards the formulation of locally covariant QFT ([HW01; BFV03]). On the level of expectation values, this principle entails that, like for the definition of $\omega(: \Phi^2 :)$, we should invoke a point–splitting subtraction scheme using the Hadamard parametrix $\mathcal{G}$ rather than a reference state. So let us construct such a map $\omega \mapsto \omega(T_{ab}(x))$ fulfilling the above requirements. Clearly, the first one is motivated by the canonical form of the classical expression T_{ab}, which is given by

$$T_{ab} \doteq \frac{2}{\sqrt{|\det(g_{ab})|}} \frac{\delta S_{KG}}{\delta g^{ab}},$$

where S_{KG} denotes the action of the Klein–Gordon field [Wal84]. For the case of minimal coupling ($\xi = 0$) we obtain

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\left(\nabla_c\phi\nabla^c\phi - m^2\phi^2\right). \tag{2.19}$$

Now define the bidifferential operator

$$D_{ab}(x,x') \doteq \nabla_a \otimes Y_b^{b'}(x,x')\nabla_{b'} - \frac{1}{2}g_{ab}(x)\left(g_{cd}(x)\nabla^c Y_{e'}^{d}(x,x')\nabla^{e'} - m^2\right). \tag{2.20}$$

In the following, we will often consider coincidence limits $x \to x'$ of bitensors, for which we use the notation

$$[f(x,x')] \doteq \lim_{x'\to x} f(x,x').$$

Now set

$$\omega\left(:\widetilde{T}_{ab}:\right) \doteq [D_{ab}(\mathcal{W}_2^\omega(x,x') - \mathcal{G}(x,x'))],$$

which obviously fulfils the first requirement. However, it turns out that

$$\nabla^a\omega\left(:\widetilde{T}_{ab}:\right) = -\frac{1}{3}\nabla^a g_{ab}[P_x\mathcal{G}(x,x')],$$

which is in conflict with the second requirement. We thus redefine[6]

$$\omega(:T_{ab}:) \doteq \omega\left(:\widetilde{T}_{ab}:\right) + \frac{1}{3}g_{ab}[P_x\mathcal{G}] + C_{ab}, \tag{2.21}$$

which now fulfils all wanted conditions. It turns out that, upon imposing further reasonable conditions[7] on C_{ab}, its most general form is

$$C_{ab} = Am^4 g_{ab} + Bm^2 G_{ab} + \Gamma I_{ab} + \Delta J_{ab},$$

where the tensors I_{ab} and J_{ab} were introduced in chapter 1. They can be calculated as

$$\begin{aligned} I_{ab} =& 2\nabla_a\nabla_b R + 2RR_{ab} - g_{ab}\left(\frac{1}{2}R^2 + 2\Box_g R\right) \\ J_{ab} =& -\Box_g R_{ab} - \frac{1}{2}g_{ab}(R_{cd}R^{cd} + \Box_g R) + \nabla_b\nabla_a R + 2R^{cd}R_{cadb}. \end{aligned}$$

A, B, Γ and Δ are a priori undetermined dimensionless renormalisation parameters, which have to be fixed by comparison with experiment and/or other physical arguments[8].

[6] An alternative way proposed in [Mor03] consists in adding the term $\frac{1}{3}\phi P\phi$ to the classical energy–momentum tensor, thus altering the bidifferential operator $D_{ab}(x,x')$ by adding the term $\frac{1}{3}g_{ab}P_x$. We will calculate this quantity in the appendix.

[7] It should of course have mass dimension 4 and be analytic in R_{abcd} and m, see [HW05]

[8] An example is the requirement that the above construction reduces to the normal ordering prescription known from Minkowski QFT. This would fix A as a function of the chosen length scale L.

2.4.1. A General Worldline Quantum Inequality

In this subsection we would like to review the general worldline inequality for the minimally coupled Klein–Gordon field on a globally hyperbolic spacetime, which is due to Fewster [Few00]. Apart from being the original motivation for the construction of SLE's by Olbermann, it provides a nice illustration of the power of the microlocal characterisation of Hadamard states. Consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ and an observer, described by a smooth timelike curve $\gamma : \mathbb{R} \supseteq I \to \mathcal{M}$ with unit tangent vector $u^a(t)$ at $\gamma(t)$. The (classical) energy density measured by this observer is given by $\rho(t) = u^a(t)u^b(t)T_{ab}(\gamma(t))$, with T_{ab} given by (2.19). In a tubular neighborhood of γ we may introduce a smooth orthonormal frame, i.e. a collection of smooth vector fields $\{v_\mu^a\}_{\mu=0}^3$ satisfying $g^{ab} = \eta^{\mu\nu} v_\mu^a v_\nu^b$ and $v_0^a(\gamma(t)) = u^a(t)$. The classical expression for $\rho(t)$ in the field configuration ϕ then reads

$$\rho(t) = \frac{1}{2} \sum_{i=0}^{3} \left(v_i^a \nabla_a \phi v_i^b \nabla_b \phi \right) |_{\gamma(t)} + \frac{1}{2} m^2 \phi^2(\gamma(t)),$$

which is the restriction of the point split quantity

$$\rho(t,t') = \frac{1}{2} \left(\sum_{i=0}^{3} v_i^a(\gamma(t)) v_i^{b'}(\gamma(t')) \right) \nabla_a \phi|_{\gamma(t)} \nabla_{b'} \phi|_{\gamma(t')} + \frac{1}{2} m^2 \phi(\gamma(t))\phi(\gamma(t'))$$

to the diagonal $t = t'$. In order to define it as expectation value in the Hadamard state ω, we proceed in the same way like in the definition of the quantised stress–energy tensor. Defining the map $\chi : (t,t') \mapsto (\gamma(t), \gamma(t'))$, we set

$$\omega(\rho)(t,t') = \frac{1}{2} \sum_{i=0}^{3} \chi_* \left(v_i^a \nabla_a \otimes v_i^b \nabla_b \mathcal{W}_2^\omega \right) + \frac{1}{2} m^2 \chi_* \mathcal{W}_2^\omega, \tag{2.22}$$

χ_* denoting the pullback of bidistributions from $\mathcal{M} \times \mathcal{M}$ to $\mathbb{R}^2$. It can be rigorously defined[9] if $N_\chi \cap WF(\mathcal{W}_2^\omega) = \varnothing$, where N_χ is the set of normals associated with χ, i.e.

$$N_\chi = \{(\gamma(t), k; \gamma(t'), k') : k_a u^a(t) = 0 = k'_b u^b(t')\}.$$

The above condition is certainly met in our case since γ is timelike and covectors in $WF(\mathcal{W}_2^\omega)$ are lightlike. The wavefront set of the pullback of $\mathcal{W}_2^\omega$ satisfies

$$WF(\chi_* \mathcal{W}_2^\omega) \subset \{(t, \xi; t', -\xi') : \xi, \xi' > 0\}.$$

The same condition is satisfied by

$$WF\left(\chi_* \sum_{i=0}^{3} \left(v_i^a \nabla_a \otimes v_i^{b'} \nabla_{b'} \mathcal{W}_2^\omega \right) \right),$$

since differential operators with smooth coefficients do not increase the wavefront set. Furthermore, since $\mathcal{W}_2^\omega$ is induced by a state, it is of positive type:

$$\mathcal{W}_2^\omega(\overline{f} \otimes f) \geq 0.$$

The same is then true for $\omega(\rho)$. We can now state Fewsters main result:

[9]For technical details see [Few00] and the literature cited therein.

Theorem 2.4.2. *Let ω and ω_0 be Hadamard states on $\mathcal{A}(\mathcal{M}, g)$ for the minimally coupled Klein–Gordon field and let $\langle : \rho : \rangle_\omega \doteq \omega(\rho) - \omega_0(\rho)$ (see definition(2.22)). Then for all $f \in \mathcal{D}(I)$ the quantum inequality*

$$\int \mathrm{d}t (f(t))^2 \langle : \rho : \rangle_\omega(t,t) \geq -\frac{1}{\pi} \int_0^\infty \mathrm{d}\alpha (\widehat{f \otimes f) \omega_0}(\rho)(-\alpha, \alpha) \tag{2.23}$$

holds true and the right–hand side of (2.23) converges for all such f.

Proof. Since ω and ω_0 are Hadamard states, the difference of their two point functions is a smooth function on $\mathcal{M}^2$ and so is $\langle : \rho : \rangle_\omega$, whose restriction to $t = t'$ is therefore also smooth. Furthermore, $\mathcal{W}_2^\omega$ and $\mathcal{W}_2^{\omega_0}$ share the same antisymmetric part, so $\langle : \rho : \rangle_\omega(t, t')$ is also symmetric. Using the Fourier representation of the delta distribution and setting $f_\alpha \doteq f e^{i\alpha t}$ we obtain

$$\begin{aligned}
\int \mathrm{d}t (f(t))^2 \langle : \rho : \rangle_\omega(t,t) &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}\alpha \langle : \rho : \rangle_\omega (f_{-\alpha} \otimes f_\alpha) \\
&= \frac{1}{\pi} \int_0^\infty \mathrm{d}\alpha \langle : \rho : \rangle_\omega (f_{-\alpha} \otimes f_\alpha) \\
&\geq -\frac{1}{\pi} \int_0^\infty \mathrm{d}\alpha \omega_0(\rho) \rangle_\omega (f_{-\alpha} \otimes f_\alpha) \\
&= -\frac{1}{\pi} \int_0^\infty \mathrm{d}\alpha (\widehat{f \otimes f) \omega_0}(\rho)(-\alpha, \alpha) \\
&> -\infty,
\end{aligned}$$

where the last line follows from the fact that the Fourier transform of $\omega_0(\rho)$ is of fast decay in the direction $(-\alpha, \alpha), \alpha > 0$ on account of the above mentioned form of its wavefront set. □

We would like to add some remarks on (2.23): Due to the construction of the bidistribution $\omega_0(\rho)(t, t')$, the right–hand side of (2.23) depends on the reference state ω_0 as well as on the choice of the vierbein. However, in the case of a FRW–spacetime we can use an isotropic and homogenous state ω_0 and then this dependence will vanish. Furthermore, since the functional

$$\omega \mapsto \int \mathrm{d}t f(t)^2 \langle : \rho : \rangle_\omega(t)$$

is bounded from below on the class of Hadamard states, one can ask if there exists a Hadamard state which minimises this quantity. It turns out that this is true when restricting ω to the class of pure quasifree isotropic and homogenous states on FRW spacetimes. The corresponding state is called state of low energy induced by f. We will return to its construction in section 2.5.

2.4.2. Renormalisation of the Energy Density on FRW Spacetimes

The aim of this subsection is to derive a general expression for the renormalised energy density measured by an isotropic observer in a pure quasifree homogenous and isotropic Hadamard state ω on a FRW spacetime. We know already the general form of the two point distribution for such states, when given by their set of mode functions $T_k(t)$. In order to calculate $\omega(: \rho(t) :)$, where the normal ordering does not refer to some reference state, but rather to the subtraction of the state independent Hadamard parametrix $\mathcal{G}$, we have to know the form of $\mathcal{G}$ on spatially flat FRW spacetimes. This task was rigorously solved in [Sch10], and we will use this strategy. The point–split energy density on FRW spacetimes is obtained by inserting the vierbein

$$v_0^a = \left(\frac{\partial}{\partial_t}\right)^a$$
$$v_i^a = a^{-1}\left(\frac{\partial}{\partial_{x_i}}\right)^a, \; i = 1, .., 3$$

into the expression (2.22). In order for the result to be smooth, we have to replace $\mathcal{W}_2^\omega$ by $\mathcal{W}_2^\omega - \mathcal{G}$. However, we only need to be able to take its coincidence limit $x \to x'$. Therefore, it is sufficient to take the truncated Hadamard parametrix $\mathcal{G}_1$, since the restriction to the diagonal $x = x'$ of a first order bidifferential operator applied to $\mathcal{G} - \mathcal{G}_k$ vanishes for $k \geq 1$. Furthermore, as we have seen in the discussion of the renormalised energy–momentum tensor, we have to "add by hand" the term

$$\frac{1}{3}[P_x \mathcal{G}]$$

in order for $\omega(:\rho:)$ to stem from a covariantly conserved $:T_{ab}:$. Finally, the tt component of the renormalisation freedom, $C_{tt}(A, B, \Gamma, \Delta)$ has to be added. The result then reads

$$\begin{aligned}\omega(:\hat{\rho}(x):) = &\Bigg[\underbrace{\frac{1}{2}\left(\partial_t\partial_{t'} + \frac{1}{a^2}\sum_{i=1}^{3}\partial_i\partial_{i'} + m^2\right)}_{\doteq \mathcal{R}}\left(\mathcal{W}_2^\omega(x,x') - \mathcal{G}_1(x,x')\right)\Bigg] \\ &+ \frac{1}{3}[P_x\mathcal{G}(x,x')] + (\partial_t)^a(\partial_t)^b C_{ab}(x).\end{aligned} \tag{2.24}$$

From the representation (2.16) of $\mathcal{W}_2^\omega$, we infer that also $\mathcal{R}\mathcal{W}_2^\omega$ has the simple form of a mode integral. Furthermore, it can be restricted to a surface of constant (conformal) time τ on account of its Hadamard property. Such restrictions to the "partial diagonal" $\tau = \tau'$ will be denoted by $[.]_\tau$ in the sequel. Assuming that ω is induced by the choice of mode functions $T_k(t)$, the corresponding expression is easily obtained as

$$[\mathcal{R}\mathcal{W}_2^\omega]_t = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k}\left(|\dot{T}_k(t)|^2 + \omega^2(k,t)|T_k(t)|^2\right)e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \tag{2.25}$$

with $\omega^2(k,t) \doteq a^{-2}(t)k^2 + m^2$. Now, $[\mathcal{R}\mathcal{G}_1]_t$ will usually be calculated in position space. However, it turns out that it can also be rewritten as a mode integral, which allows to perform the subtraction appearing in (2.24) under the mode integral before doing the spatial coincidence limit. The result will be a convergent integral in momentum space, which is suited for numerical evaluation. When working in conformal time τ we therefore need expressions for $[\mathcal{G}_1]_\tau, [\partial_\tau\mathcal{G}_1]_\tau, [\partial_{\tau'}\mathcal{G}_1]_\tau$ and $[\partial_\tau\partial_{\tau'}\mathcal{G}_1]_\tau$. Spatial derivatives can then be taken after the temporal coincidence limit, since these operations commute. In order to do the explicit calculation, we can replace $\mathcal{W}_2^\omega(x,x') - \mathcal{G}_1(x,x')$ in (2.24) by $\mathcal{W}_2^{\omega,s}(x,x') - \mathcal{G}_1^s(x,x')$, where the superscript "s" denotes symmetrisation w.r.t. the two arguments. Thus we will only need the symmetric parts of the distributions $\lim_{\epsilon\downarrow 0}\sigma_\epsilon^{-1}$ and $\lim_{\epsilon\downarrow 0}\log(\sigma_\epsilon)$ from which $\mathcal{G}_1^s$ is built. We denote them by

$\left(\frac{1}{\sigma_+}\right)^s$ and $(\log\sigma_+)^s$, respectively. A last remark concerns some statements[10] about symmetric functions and distributions, which will be important later on: On spatially flat FRW spacetimes, the symmetry group of every time slice is the euclidean group $E(3)$ of translations and rotations. Let $\hat{I}$ be the range of conformal time $\tau(t)$ and let f be a function on $D \subseteq (\hat{I}\times\mathbb{R}^3)^2$, open and invariant under $E(3)$ in the following sense:

$$\forall g\in E(3),\ \forall(\tau,\mathbf{x},\tau',\mathbf{x}')\in D:(\tau,g\mathbf{x},\tau',g\mathbf{x}')\in D.$$

If also f is invariant under $E(3)$, i.e. $f(\tau,\mathbf{x},\tau',\mathbf{x}')=f(\tau,g\mathbf{x},\tau',g\mathbf{x}')\ \forall g\in E(3)$, then

$$f(\tau,\mathbf{x},\tau',\mathbf{x}')=\tilde{f}(\tau,\tau',\|\mathbf{x}-\mathbf{x}'\|).$$

A bidistribution on $\mathcal{D}(\mathbb{R}^3)$ which is invariant under $E(3)$, i.e. $T(f,f')=T(f_g,f'_g)\ \forall g\in E(3)$ can be written as $T(f,f')=\tilde{T}(f*f')$, the $*$ denoting convolution. Furthermore, $\tilde{T}\in\mathcal{D}(\mathbb{R}^3)$ is already determined by its action on radially symmetric test functions. We will refer to $\tilde{f}$ and $\tilde{T}$ as the symmetry reduced counterparts of f and T, respectively.

In the following we sketch the important steps in order to calculate (2.24), following the reference [Sch10]. All technical proofs can of course be found there and we will omit them.

Step 1: The Distributions $\left(\frac{1}{\sigma_+}\right)^s$ and $(\log\sigma_+)^s$

In spatially flat FRW spacetimes, σ can be written as

$$\sigma(x,x')=\rho(x,x')q(x,x'), \tag{2.26}$$

where $\rho(x,x')\doteq|\mathbf{x}-\mathbf{x}'|^2-(\tau-\tau')^2$ is the signed "Minkowskian" squared geodesic distance. The relation (2.6) furnishes a differential equation for q which can be used to compute a small distance expansion of q in terms of $r^2\doteq|\mathbf{x}-\mathbf{x}'|^2$ and $\tau-\tau'$ (see appendix A). A statement analogous to (2.26) holds also true for the distributions

$$\left(\frac{1}{\sigma_+}\right)^s \quad\text{and}\quad (\log\sigma_+)^s,$$

which is the content of the following lemmas from [Sch10]. They also furnish explicit expressions for the restrictions of these distributions (and temporal derivatives thereof) to the partial diagonal $\tau'=\tau$:

Lemma 2.4.3. *Let* $\left(\frac{1}{\tilde{\sigma}_+}\right)^s : C_0^\infty(\hat{I}\times\hat{I}\times\mathbb{R}^3)\to\mathbb{C}$ *be given by*

$$\begin{aligned}\left(\frac{1}{\tilde{\sigma}_+}\right)^s(f)\doteq&\\ \lim_{\epsilon\to+0}\int_{\hat{I}}\int_{\hat{I}}\int_{\mathbb{R}^3}\frac{1}{2\tilde{q}(\tau,\tau',||\mathbf{x}||)}&\left(\frac{1}{\mathbf{x}^2-(\tau-\tau')^2+\frac{2i\epsilon(\tau-\tau')}{\tilde{q}(\tau,\tau',||\mathbf{x}||)}+\frac{\epsilon^2}{\tilde{q}(\tau,\tau',||\mathbf{x}||)}}\right.\\ &\left.+\frac{1}{\mathbf{x}^2-(\tau-\tau')^2+\frac{2i\epsilon(\tau'-\tau)}{\tilde{q}(\tau,\tau',||\mathbf{x}||)}+\frac{\epsilon^2}{\tilde{q}(\tau,\tau',||\mathbf{x}||)}}\right)\\ &\times f(\tau,\tau',\mathbf{x})C^2(\tau)C^2(\tau')\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}^3\mathbf{x}.\end{aligned}$$

[10]The proofs can be found in [Sch10] and the literature cited there.

Then $\left(\frac{1}{\tilde{\sigma}_+}\right)^s$ *can be written as*

$$\left(\frac{1}{\tilde{\sigma}_+}\right)^s(f) = \int_{\hat{I}}\int_{\hat{I}} \tilde{\sigma}^{-1}_{\tau-\tau'}\left(\frac{f(\tau,\tau',.)}{\tilde{q}(\tau,\tau',||.||)}\right) C^2(\tau)C^2(\tau')\mathrm{d}\tau\mathrm{d}\tau'$$

where the function $\Delta\tau \mapsto \tilde{\sigma}^{-1}_{\Delta\tau}(h)$ *is (for fixed* $h \in C_0^\infty(\mathbb{R}^3)$*) twice continously differentiable with*

$$\begin{aligned}
\tilde{\sigma}^{-1}_{\Delta\tau}(h) \restriction_{\Delta\tau=0} &= \lim_{\epsilon\to+0}\int_{\mathbb{R}^3}\frac{h(\mathbf{x})}{\mathbf{x}^2+\epsilon^2}\mathrm{d}\mathbf{x} =: \frac{1}{r_+^2}(h)\\
\partial_{\Delta\tau}\left(\tilde{\sigma}^{-1}_{\Delta\tau}(h)\right)\restriction_{\Delta\tau=0} &= 0\\
\partial_{\Delta\tau\Delta\tau}\left(\tilde{\sigma}^{-1}_{\Delta\tau}(h)\right)\restriction_{\Delta\tau=0} &= \lim_{\epsilon\to+0}\int_{\mathbb{R}^3}\frac{\triangle h(\mathbf{x})}{\mathbf{x}^2+\epsilon^2}\mathrm{d}\mathbf{x} =: \frac{2}{r_+^4}(h).
\end{aligned}$$

A similar statement holds for the distribution $[\log(\tilde{\sigma}_+)]^s$:

Lemma 2.4.4. *Let* $(\log\tilde{\sigma}_+)^s$ *be given by the same regularisation and symmetrisation prescription as above. Then*

$$\begin{aligned}
(\log\tilde{\sigma}_+)^s = &\int_{\hat{I}}\int_{\hat{I}}\int_{\mathbb{R}^3}\log(\tilde{q}(\tau,\tau',||\mathbf{x}||))f(\tau,\tau',\mathbf{x})C^2(\tau)C^2(\tau')\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}^3\mathbf{x}\\
&+\int_{\hat{I}}\int_{\hat{I}}\tilde{\mathrm{lo}}_{\tau-\tau'}(f(\tau,\tau',.))C^2(\tau)C^2(\tau')\mathrm{d}\tau\mathrm{d}\tau'
\end{aligned}$$

where the function $\Delta\tau \mapsto \tilde{\mathrm{lo}}_{\Delta\tau}(h)$ *is (for fixed* $h \in C_0^\infty(\mathbb{R}^3)$*) twice continously differentiable with*

$$\begin{aligned}
\tilde{\mathrm{lo}}_{\Delta\tau}(h)\restriction_{\Delta\tau=0} &= \int_{\mathbb{R}^3}\log(\mathbf{x}^2)h(\mathbf{x})\mathrm{d}^3\mathbf{x}\\
\partial_{\Delta\tau}\left(\tilde{\mathrm{lo}}_{\Delta\tau}(h)\right)\restriction_{\Delta\tau=0} &= 0\\
\partial_{\Delta\tau\Delta\tau}\left(\tilde{\mathrm{lo}}_{\Delta\tau}(h)\right)\restriction_{\Delta\tau=0} &= -2\lim_{\epsilon\to+0}\int_{\mathbb{R}^3}\frac{h(\mathbf{x})}{\mathbf{x}^2+\epsilon^2}\mathrm{d}^3\mathbf{x}.
\end{aligned}$$

Step 2: The Hadamard Recursion Relations

The fact that our spatially flat FRW spacetime $(\mathcal{M}, g)$ can be conformally embedded in Minkowski space can be used to bring the Hadamard parametrix in a form similar to the one that it assumes in Minkowski space. By representing $\Box_g$ in the coordinates $(\tau, \mathbf{x})$ and looking at the recursion relations (2.7), it can be shown that the coefficients in the Hadamard parametrix take the following form:

$$\frac{\Delta^{1/2}}{q} = \frac{1}{\sqrt{C(\tau)C(\tau')}}(1+\rho R_\Delta) \tag{2.27}$$

$$V^{(k)} = \frac{1}{\sqrt{C(\tau)C(\tau')}}\left(\sum_{j=0}^{k}\rho^j \overset{o}{v}_j + \rho^{k+1}R_v\right) \tag{2.28}$$

Using again the small distance expansion for q and the first formula of the recursion relations (2.7), one derives a small distance expansion for $\tilde{R}_\Delta(\eta, r)$, which is given in appendix A.

The $\overset{o}{v}_j$ in (2.28) are defined to be the unique bounded solutions to the modified recursion relations

$$2\epsilon_{\mu\nu}\overset{o}{\nabla}_\mu\rho\overset{o}{\nabla}_\nu\overset{o}{v}_0 - 4\overset{o}{v}_0 = -L^2 Q \tag{2.29}$$

$$2(j+1)\epsilon_{\mu\nu}\overset{o}{\nabla}_\mu\rho\overset{o}{\nabla}_\nu\overset{o}{v}_{j+1} - 4(j+1)(j+2)\overset{o}{v}_{j+1} = -L^2\overset{o}{P}_x\overset{o}{v}_j. \tag{2.30}$$

$\epsilon_{\mu\nu}$ denotes the Minkowski metric and the circle over the derivative operators and the v_k indicates that these objects are those defined on Minkowski space[11]. Namely, equations (2.29) and (2.30) are exactly the Hadamard recursion relations for the Minkowski case except for the appearance of the $\tau-$ dependent term

$$Q(\tau) \doteq \left(m^2 - \frac{1}{6}R(\tau)\right)C(\tau). \tag{2.31}$$

One can then show that the difference between the "true" truncated series $V^{(k)}$ and

$$\overset{o}{V}^{(k)} \doteq \frac{1}{\sqrt{C(\tau)C(\tau')}}\sum_{j=0}^{k}\rho^j\overset{o}{v}_j \tag{2.32}$$

vanishes like σ^{k+1} when approaching the lightcone. So when taking the coincidence limits of (derivatives of) $\mathcal{G}_k^s$ later on, this difference can be ignored and it is sufficient for such calculations to make the replacement

$$V^{(k)} \to \overset{o}{V}^{(k)}.$$

Using the method of characteristics, one finds that the $\overset{o}{v}_j$ depend only on τ and τ'. Equations (2.29) and (2.30) thus read

$$(\tau-\tau')\partial_\tau\overset{o}{v}_0 + \overset{o}{v}_0 = \frac{L^2}{4}Q \tag{2.33}$$

$$(j+1)(\tau-\tau')\partial_\tau\overset{o}{v}_{j+1} + (j+1)(j+2)\overset{o}{v}_{j+1} = \frac{L^2}{4}(\partial_\tau^2 + Q)\overset{o}{v}_j. \tag{2.34}$$

One can now compute coincidence limits of $\overset{o}{v}_j$ and their time derivatives w.r.t. τ and τ' by differentiating equations (2.33) and (2.34) sufficiently often and taking the coincidence limit at the end (again, we refer to appendix A). We have now all the ingredients to calculate the coordinate space expressions for $[\tilde{\mathcal{G}}_k^s]_\tau$, $[\partial_\tau\tilde{\mathcal{G}}_k^s]_\tau$ and $[\partial_{\tau\tau'}\tilde{\mathcal{G}}_k^s]_\tau$, which are homogenous and isotropic distributions on $C_0^\infty(\mathbb{R}^3)$. Putting them together, we obtain

$$\begin{aligned}[\widetilde{\mathcal{R}\mathcal{G}_1^s}]_\tau = \frac{1}{4\pi^2}\Bigg\{ &-\frac{2}{C^2 r_+^4} + \left(\frac{C'^2}{8C^4} + \frac{m^2}{2C}\right)\frac{1}{r_+^2} \\ &+ \left(\frac{m^4}{16} - \frac{m^2C'^2}{32C^3} + \frac{9C'^4}{256C^6} - \frac{C'^2C''}{16C^5} - \frac{C''^2}{64C^4} + \frac{C'C'''}{32C^4}\right)(\mathrm{lo}_0 + \log C) \\ &+ \frac{\Box R}{120} + \frac{m^2C''}{8C^2} - \frac{19C''^2}{960C^4} - \frac{11m^2C'^2}{96C^3} - \frac{33C'^4}{640C^6} - \frac{11C'''C'}{480C^4} \\ &+ \frac{53C'^2C''}{480C^5} - \frac{m^4}{8}\Bigg\}.\end{aligned} \tag{2.35}$$

[11]For instance, $\overset{o}{P}_x$ is the Minkowskian wave operator with "time dependent mass": $\overset{o}{P}_x = \partial_\tau^2 - \triangle_{\mathbf{x}} + Q$

Step 3: The Hadamard Parametrix as Mode Integral

The last step consists in rewriting the singular parts of (2.35), i.e. the distributions r_+^{-4}, r_+^{-2} and lo_0 as mode integrals over the momentum space associated with Σ. This is achieved by the following lemma [Sch10, Lemma 5.6.]:

Lemma 2.4.5. *Let $\Omega : \mathbb{R}^+ \to \mathbb{R}$ have asymptotic behaviour*

$$\Omega(p) = \sum_{j=-1}^{k'} \frac{b_j}{p^{2j+1}} + O\left(p^{-2k'-3}\right), \quad p \to \infty.$$

Then for $h \in C_0^\infty(\mathbb{R}^3)$, the distribution

$$W_\Omega \doteq \lim_{\epsilon \to \infty} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-p\epsilon} e^{i\mathbf{p}\mathbf{x}} \Omega(p) \mathrm{d}^3\mathbf{p}$$

is given by

$$W_\Omega(h) = \frac{1}{2\pi^2} \left(-2\frac{b_{-1}}{r_+^4}(h) + \frac{b_0}{r_+^2}(h) + V^{k'-1}\mathrm{lo}_0(h) + \mathcal{R}_{\Omega,L}^{2k'-1}(h) \right),$$

where

$$\begin{aligned}
V^{k'-1}\mathrm{lo}_0(h) &= -4\pi \int_0^\infty \sum_{l=0}^{k'-1} \frac{b_{l+1}}{(2l+1)!} (-r^2)^l \log(r/L) h(r) r^2 \mathrm{d}r \\
\mathcal{R}_{\Omega,L}^{2k'-1}(h) &= 4\pi \int_{\mathbb{R}^+} R_{\Omega,L}^{2k'-1}(r) r^2 h(r) \mathrm{d}r \\
R_{\Omega,L}^{2k'-1}(r) &= \sum_{l=0}^{k'-1} \frac{R_{2l+1}}{(2l+1)!} (-r^2)^l + o(r^{2k'-1}) \\
R_{2l+1} &= \lim_{M \to \infty} \left(\int_0^M p^{2l+1} \left(p\Omega(p) - b_{-1}p^2 - b_0\right) \mathrm{d}p - \sum_{j=1}^{l} \frac{b_{l+1-j}}{2j} M^{2j} - b_{l+1} \log(ML) \right) \\
&\quad + b_{l+1} \left(-\gamma + \sum_{n=1}^{2l+1} \frac{1}{n} \right)
\end{aligned}$$

Remark 2.4.6.

1. *The definition of $V^{k'-1}\mathrm{lo}_0(h)$ and $\mathcal{R}_{\Omega,L}^{2k'-1}(h)$ differs here from [Sch10] by a factor 4π, since we use the definition*

$$\mathrm{lo}_0(h) = \int \mathrm{d}\mathbf{r}^3 h(\mathbf{r}) \log \mathbf{r}^2 = 4\pi \int \mathrm{d}r r^2 h(r) \log r^2.$$

2. *Since later we will apply this lemma with regard to the coincidence limit $r \to 0$ of $[\omega(\rho)]_\tau - [\mathcal{R}\mathcal{G}_1]_\tau$, we only need the contribution of $R^{2k'-1}_{\Omega,L}$ (which is the integral kernel of the distribution $\mathcal{R}^{2k'-1}_{\Omega,L}$) which remains finite for $r \to 0$, i.e. we choose $k' = 1$, since all higher order contributions will vanish for $r \to 0$.*

We are now able to compute the expectation value of the energy density in a general Hadamard state on a spatially flat FRW spacetime. It will be given as a convergent integral in momentum space. The calculation of $[P_x\mathcal{G}_1]$ and the renormalisation freedom C_{tt} will be done in appendix A and B.

2.5. States of Low Energy on FRW Spacetimes

2.5.1. Adiabatic States

Parker was the first author who introduced the notion of *adiabatic states* on FRW spacetimes. He noticed that the instantanous particle picture at t_0 is different from that at t_1, leading to the production of infinitely many particles. Motivated by the idea to minimise this particle production he made the following WKB-type ansatz for the mode functions $T_k(t)$:

$$T_k(t) = (2a^3(t)\Omega(k,t))^{-\frac{1}{2}} e^{-i\int_{t_0}^{t}\Omega(k,t')\mathrm{d}t'} \tag{2.36}$$

with a positive function $\Omega(k,t)$. Inserting this ansatz in the mode equation (2.13) leads to the following differential equation for Ω:

$$\Omega^2 = \omega^2 - \frac{3}{4}\left(\frac{\dot a}{a}\right)^2 - \frac{3}{2}\frac{\ddot a}{a} + \frac{3}{4}\left(\frac{\dot\Omega}{\Omega}\right)^2 - \frac{1}{2}\frac{\ddot\Omega}{\Omega} \doteq F(\Omega),$$

with $\omega \doteq \sqrt{k^2a^{-2} + m^2}$. Next one makes an iteration ansatz

$$\left(\Omega^{(n+1)}\right)^2 = F\left(\Omega^{(n)}\right), \quad \Omega^{(0)} \doteq \omega,$$

hoping that the sequence $\Omega^{(n)}$ converges to the exact solution Ω for $n \to \infty$. However, almost nothing seems to be known about its convergence properties. Parker originally defined the adiabatic state of order n via the approximate modes $T_k^{(n)}(t)$, obtained by plugging $\Omega^{(n)}$ directly into the ansatz (2.36). However, a better method, due to Lüders and Roberts [LR90], consists in prescribing the *initial conditions* of the exact modes $T_k(t)$ at time t_0 via

$$\begin{aligned} T_k(t_0) &= T_k^{(n)}(t_0) = \left(2a^3\Omega^{(n)}\right)^{-\frac{1}{2}}\Big|_{t_0} \\ \dot T_k(t_0) &= \dot T_k^{(n)}(t_0) = \left(2a^3\Omega^{(n)}\right)^{-\frac{1}{2}}\left(-i\Omega^{(n)} - \frac{3}{2}H - \frac{1}{2}\frac{\dot\Omega^{(n)}}{\Omega^{(n)}}\right)\Big|_{t_0}. \end{aligned} \tag{2.37}$$

The state induced by such mode functions $T_k(t)$ is then called adiabatic state of order n. Note that this procedure has to be regarded as a way to fix the asymptotic behaviour of the initial values of the modes $T_k(t)$ on a Cauchy surface Σ for large k, since the positivity of $\left(\Omega^{(n)}\right)^2$ can in general only be ensured for large $k > k_0$. In this general situation one can prescribe arbitrary initial values for $k < k_0$. To summarise, such adiabatic states depend on

1. the initial time t_0

2. the order n

3. the choice of the initial conditions of $T_k(t)$ for small k

Now, the asymptotics of $T_k(t_0)$ and $\dot{T}_k(t_0)$ in k correspond to the regularity properties of the Hadamard point–splitting–regularised two–point function in position space (which is of course a smooth function for Hadamard states). Indeed, this regularity behaviour was used by Junker and Schrohe [JS02] for the more general definition of adiabatic states ω_n of order n on arbitrary spacetimes by requiring the *Sobolev wavefront set* WF^s of its two point function to be contained in the wave front set of a Hadamard two point function for $s < n + 3/2$. The notion of Sobolev wavefront sets $WF^s(u)$ is a refinement of the earlier introduced $C^\infty-$ wavefront sets. It consists, roughly speaking, of requiring directions in the complement of $WF^s(u)$ to be those in which the Fourier transform of u has a decrease faster than an inverse power of maximal order s, instead of any inverse power (for a precise definition see [DH72]). The authors of [JS02] then showed that adiabatic states on FRW spacetimes of infinite order n in the sense of definition (2.37) are Hadamard states.

However, for computations of the energy–momentum tensor, the point–splitting method already works if n is low, allowing for the use of such states for computations in cosmology. The authors of [JS02] also suggested how one could in principle experimentally distinguish two adiabatic states of different order, thus providing a physical interpretation. Quite recently, Olbermann [Olb07b] could improve the concept of adiabatic states by introducing the *states of low energy* on FRW spacetimes, which have the clear physical interpretation of minimising the time- smeared energy density of all isotropic observers, where the smearing is performed over a finite time interval with a smooth test function f of compact support. Furthermore, he could show that they are Hadamard states. Based on this work, Küskü constructed the class of almost equilibrium states [Küs08], which minimise the free energy density w.r.t. f, while Them [The10] tried to generalise the construction of SLE's to Bianchi spacetimes. We will review the construction of SLE's in the following subsection.

2.5.2. Construction of SLE's

This subsection is devoted to a brief review of the construction of SLE's on FRW spacetimes. According to Fewsters quantum inequality (2.23), the quantity

$$\varrho_\omega[f] \doteq \int_I f^2(t)\omega(:\rho(t):)\mathrm{d}t$$

is bounded from below as ω ranges in the class of Hadamard states. I is the range of the proper time of an observer γ, to which the energy density refers and : ... : denotes in this case normal ordering w.r.t. a reference state ω_0. As argued before, in a spatially flat FRW spacetime one can choose a homogenous and isotropic reference state ω_0, in which case the bound won't depend on the chosen vierbein entering in the expression for the point–split energy density (2.22). Restricting to the class of homogenous isotropic pure and quasifree states ω on $\mathcal{A}(\mathcal{M}, g)$ one can ask if there is a state ω which minimises the functional $\omega \mapsto \varrho_\omega[f]$ for fixed $f \in \mathcal{D}(I)$, I being the range of cosmological time. We would call such a state *state of low energy induced by* f^2 and denote it by ω_{f^2}. If we denote the corresponding mode functions by $T_k(t)$, the minimisation requirement means that the quantity

$$\varrho_k \doteq \frac{1}{2} \int f^2(t) \left(|\dot{T}_k(t)|^2 + (a^{-2}(t)k^2 + m^2)|T_k(t)|^2 \right) \mathrm{d}t$$

has to be minimal for all modes $k \in (0, \infty)$ separately. Olbermann calculated the minimising modes $T_k(t)$ by writing them as a *Bogolubov transformation* of arbitrary reference modes $S_k(t)$ fulfilling (2.13) and (2.14):

$$T_k(t) = \lambda(k)S_k(t) + \mu(k)\overline{S}_k(t) \tag{2.38}$$

Since $T_k(t)$ have to fulfil (2.14) as well, the Bogolubv coefficients must satisfy

$$|\lambda(k)|^2 - |\mu(k)|^2 = 1.$$

Furthermore, if $T_k(t)$ minimises ϱ_k then also $e^{i\alpha(k)}T_k(t)$ does so for all $\alpha(k) \in \mathbb{R}$. Thus, $\mu(k)$ can be chosen real without loss of generality. ϱ_k is thus an ordinary function of two variables and can be straightforwardly minimised. We cite the result from [Olb07a, Theorem 3.1.]

Theorem 2.5.1. *Let $(\mathcal{M}, g)$ be a spatially flat FRW spacetime and $\mathcal{A}(\mathcal{M}, g)$ the field algebra of the free minimally coupled Klein–Gordon field on $(\mathcal{M}, g)$, $\gamma : I \to \mathcal{M}$ an isotropic geodesic parametrised by cosmological time t and $f \in C_0^\infty(I)$. In the set of homogenous isotropic pure and quasifree states on $\mathcal{A}(\mathcal{M}, g)$ there exists a unique (up to a phase) state ω_{f^2} for which the smeared energy density functional*

$$\varrho_\omega[f] \doteq \int \mathrm{d}t f^2(t)\omega(:\hat{\rho}(t):)$$

assumes its minimum. This state is given by its two–point function (2.16) and the corresponding modes $T_k(t)$ are given by

$$T_k(t) = \lambda(k)S_k(t) + \mu(k)\overline{S}_k(t),$$

where the reference modes $S_k(t)$ satisfy (2.13) and (2.14) and $\lambda(k)$ and $\mu(k)$ carry the information about the state via

$$\begin{aligned}
\lambda(k) &= e^{i(\pi - \arg c_2(k))} \sqrt{\frac{c_1(k)}{2\sqrt{c_1(k)^2 - |c_2(k)|^2}} + \frac{1}{2}} \\
\mu(k) &= \sqrt{\frac{c_1(k)}{2\sqrt{c_1(k)^2 - |c_2(k)|^2}} - \frac{1}{2}} \\
c_1(k) &= \frac{1}{2} \int \mathrm{d}t f^2(t) \left(|\dot{S}_k(t)|^2 + (k^2 a^{-2} + m^2)|S_k(t)|^2 \right) \\
c_2(k) &= \frac{1}{2} \int \mathrm{d}t f^2(t) \left(\dot{S}_k(t)^2 + (k^2 a^{-2} + m^2)S_k(t)^2 \right).
\end{aligned}$$

We would like to add a few remarks. First of all, this construction also works for the other two possible spatial geometries, which we skipped here on notational grounds. Furthermore, it is easy to see that ω_{f^2} reduces for all f to the Minkowski vacuum if $a = \mathrm{const}$. That is why the SLE's deserve to be seen as generalised vacua. However, ω_{f^2} is not automatically a Hadamard state by construction. This requires an own proof, which was also given by Olbermann [Olb07a, Theorem 4.9.]. A last remark concerns the choice of minimal coupling: The starting point of the construction of SLE's was Fewster's proof of the general wordline inequality. However, for the sake of validity of his result for arbitrary observers and general globally hyperbolic spacetimes, Fewster considered only the case of minimal coupling, and so did Olbermann. However, in case of FRW spacetimes, Fewster's and hence Olbermann's results might be extended to the case of arbitrary coupling ξ of the quantum field to the curvature.

3. Calculation of $\omega(:\rho:)$ in SLE's on de Sitter Spacetimes

In this section we demonstrate that it is possible to obtain explicit results for the energy density in SLE's under certain assumptions. We shall consider the minimally coupled Klein–Gordon field on the cosmological part of de Sitter spacetime, i.e. $\mathcal{M} = \mathbb{R}^4$ and $a(t) = e^{Ht}$. With this assumptions it is possible to solve (2.13) in order to obtain a set of reference modes $S_k(t)$ needed as starting point for the calculation of the SLE. We go over to conformal time τ and define the function $\chi_k(\tau)$ via

$$\chi_k(\tau(t)) \doteq a(t)S_k(t),$$

which then obeys the differential equation

$$\chi_k''(\tau) + \left(k^2 + Q(\tau)\right)\chi_k(\tau) = 0, \tag{3.1}$$

with the "time dependent mass"

$$Q(\tau) \doteq C(\tau)\left(m^2 - \frac{1}{6}R(\tau)\right).$$

The condition (2.14) on $S_k(t)$ then becomes

$$\chi_k'\overline{\chi}_k - \chi_k\ \overline{\chi}_k' = i. \tag{3.2}$$

A solution of (3.1) for our spacetime and arbitrary mass is given by

$$\chi_k(\tau) = \frac{\sqrt{-\tau\pi}}{2}e^{\frac{-i\pi\nu}{2}}H_\nu^{(2)}(-k\tau), \tag{3.3}$$

where $H_\nu^{(2)}$ denotes the Hankel function of the second kind [AS64] and

$$\nu \doteq \sqrt{\frac{9}{4} - 2\left(\frac{m^2}{2H^2}\right)},$$

where ν and its imaginary part (if the expression under the square root becomes negative) are taken to be positive. We observe that there are two cases for which $\chi_k(\tau)$ can be expressed in terms of elementary functions, namely $\nu = 1/2$ and $\nu = 3/2$, which corresponds to $m^2 = 2H^2$ and $m^2 = 0$, respectively.

Now, given a state which is parametrised by a Bogolubov transformation w.r.t. the reference modes $S_k(t)$ according to (2.38), we may express the temporal coincidence limit of the unrenormalised point split energy density (2.25) as:

$$[\mathcal{RW}_2^{\omega,s}]_{\tau(t)} = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\Bigg(\left(1+2\mu(k)^2\right)\left(|\dot{S}_k(t)|^2 + (k^2e^{-2Ht}+m^2)|S_k(t)|^2\right) + 2\mu(k)|\lambda(k)|\mathrm{Re}\left\{e^{i\mathrm{arg}\lambda(k)}\left(\dot{S}_k(t)^2+(k^2e^{-2Ht}+m^2)S_k(t)^2\right)\right\}\Bigg), \tag{3.4}$$

where it is assumed that $\mu(k)$ is chosen to be real. There remains the task to calculate the Bogolubov coefficients $\lambda(k), \mu(k)$ according to Olbermanns theorem, which we shall now tackle. For calculational simplicity, we will use a Gaussian

$$f(t) = \frac{1}{\sqrt{\pi}\epsilon}e^{-\frac{(t-t_0)^2}{\epsilon^2}} \tag{3.5}$$

as smearing function which induces[1] the SLE ω_f, which was already employed in [DV10] for the calculation of particle production. Obviously, f is not of compact support. However, our choice is physically justified since the cosmological time interval is $I=\mathbb{R}$ and the Hadamard property of ω_f is ensured on account of the smoothness and rapid decrease of f.

3.1. Calculation of $\omega(:\rho:)$ for Different Regimes of the Mass

We come now to the calculation of the state ω_f and the corresponding energy density expectation value $\omega_f(:\rho:)$ for the choice (3.5) on de Sitter space. We start with

The Case $m^2 = 2H^2$

Up to multiplication with an irrelevant phase factor, (3.3) reduces to

$$\chi_k(\tau) = \frac{1}{\sqrt{2k}}e^{ik\tau},$$

and thus we get

$$[\mathcal{RW}_2^{\omega,s}]_\tau = \frac{H^4\tau^2}{2(2\pi)^3}\int_{\mathbb{R}^3}\left((1+2\mu_k^2)\left(k\tau^2+\frac{3}{2k}\right) + 2\mu_k|\lambda_k|\mathrm{Re}\left\{e^{i(\mathrm{arg}\lambda_k+2k\tau)}\left(\frac{3}{2k}+i\tau\right)\right\}\right)e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\mathrm{d}^3\mathbf{k} \tag{3.6}$$

The remaining task is the calculation of λ_k and μ_k for the given test function f according to Olbermann's prescription. $c_1(k)$ can be computed exactly, because it involves only standard Gaussian integrals:

$$\begin{aligned} c_1(k) &= \int_{\mathbb{R}}\mathrm{d}t f(t)\left(ke^{-4Ht}+\frac{3H^2}{2k}e^{-2Ht}\right) \\ &= He^{-3Ht_0}\left(ze^{4\alpha^2}+\frac{3}{2z}e^{\alpha^2}\right), \end{aligned}$$

[1] Thus, f defined in (3.5) will be play the role of f^2 from theorem 2.5.1.

where we introduced the new variables

$$z \doteq \frac{k}{Ha(t_0)} \qquad\qquad \alpha \doteq \epsilon H. \tag{3.7}$$

c_2 is given by the expression

$$\begin{aligned} c_2 =& \frac{1}{\sqrt{\pi}\epsilon}\int_{\mathbb{R}} \mathrm{d}t f(t)\left(e^{2ik\tau(t)}\left(\frac{3H^2}{2k}e^{-2Ht} - iHe^{-3Ht}\right)\right) \\ =& \frac{3H^2}{2k\sqrt{\pi}\epsilon}\int_{-\infty}^{\infty}\mathrm{d}t \exp\left(-\frac{(t-t_0)^2}{\epsilon^2} - 2Ht + 2ik\tau(t)\right) \\ &- \frac{iH}{\sqrt{\pi}\epsilon}\int_{-\infty}^{\infty}\mathrm{d}t \exp\left(-\frac{(t-t_0)^2}{\epsilon^2} - 3Ht + 2ik\tau(t)\right). \end{aligned}$$

In order to evaluate this integral, we have to make an approximation: Under the assumption $\alpha \ll 1$ we may perform a Taylor expansion of $\tau(t)$ to first order around t_0 and obtain standard integrals. Clearly, the original integral falls off faster than any inverse power of k and so does the approximated integral. Thus the error we made has the same decay behaviour. This observation ensures the existence of the mode integral in the calculation of the renormalised energy density. We obtain

$$c_2 \approx He^{-\alpha^2 z^2}e^{-3Ht_0}e^{-2iz}\left(\frac{3}{2z}e^{\alpha^2 - 2i\alpha^2 z} - ie^{9\alpha^2/4 - 3i\alpha^2 z}\right).$$

We notice that c_2 decays $\propto e^{-\alpha^2 z^2}$. Thus, for z small enough such that c_2 is important we may use the simplifications $e^{\alpha^2 z} \approx 1$ and $e^{\alpha^2} \approx 1$, since $\alpha \ll 1$ by assumption. This leads finally to

$$c_1 \approx He^{-3Ht_0}\left(z + \frac{3}{2z}\right) \tag{3.8}$$

$$c_2 \approx He^{-3Ht_0}e^{-\alpha^2 z^2}\sqrt{1 + 9/(4z^2)}\exp\left(-i\left(\arctan\frac{2z}{3} + 2z\right)\right). \tag{3.9}$$

For our choice of m and the scale factor, we can easily determine the to-be subtracted singularity from (2.35). It reads

$$[\widetilde{\mathcal{RG}_1^s}]_\tau = \frac{H^4}{2\pi^2}\left(-\frac{\tau^4}{r_+^4} + \frac{3\tau^2}{4r_+^2} + \frac{23}{480}\right).$$

Applying lemma 2.4.5 to the function $\Omega(k) = \frac{H^4}{2}\left(k\tau^4 + \frac{3\tau^2}{2k}\right)$ yields[2]

$$[\widetilde{\mathcal{RG}_1^s}]_\tau = \frac{H^4}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}e^{i\mathbf{kr}}\left(k\tau^4 + \frac{3\tau^2}{2k}\right) + \frac{1}{4\pi^2}\frac{23H^4}{240}.$$

The remaining ingredients we need are (see (A.6 and (B.1))

$$\begin{aligned} [P_x\mathcal{G}_1] &= \frac{1}{4\pi^2}\frac{H^4}{20} \\ C_{tt} &= H^4(4A - 6B), \end{aligned}$$

[2] Up to terms which vanish for $r \to 0$.

and $\omega(:\hat{\rho}(t):)$ follows according to (2.24). As renormalisation condition we require[3] that

$$\lim_{t\to\infty}\omega(:\hat{\rho}(t):)=0, \tag{3.10}$$

which implies

$$4A-6B=\frac{19}{240}\frac{1}{4\pi^2}. \tag{3.11}$$

Introducing the auxiliary function

$$z\mapsto u(z)\doteq\frac{c_1}{2\sqrt{c_1^2-|c_2|^2}}=\frac{z^2+3/2}{2\sqrt{(z^2+3/2)^2-e^{-2\alpha^2z^2}\left(9/4+z^2\right)}},$$

the energy density for the SLE induced by the Gaussian with parameters (ϵ,t_0) now reads

$$\begin{aligned}\omega(:\hat{\rho}:)=&\frac{H^4}{4\pi^2}e^{-2H(t-t_0)}\int\limits_{\mathbb{R}_+}\mathrm{d}zz\Bigg(2(u(z)-1/2)\left(z^2e^{-2H(t-t_0)}+\frac{3}{2}\right)\\&-\frac{\sqrt{u(z)^2-1/4}}{\sqrt{1+4z^2/9}}\Bigg(3\cos\left(2z(1-e^{-H(t-t_0)})\right)\left(\frac{4z^2e^{-H(t-t_0)}}{9}+1\right)\\&+2z\sin\left(2z(1-e^{-H(t-t_0)})\right)\left(e^{-H(t-t_0)}-1\right)\Bigg)\Bigg).\end{aligned} \tag{3.12}$$

This formula can now be evaluated numerically. Figures 3.1-3.3 show the behaviour of $\omega_f(:\rho(t):)$ for various choices of the Gaussian test function f. The expectation value of the energy density is given in units of H^4 and all numerical values of time variables refer to the unit H^{-1}. The plots show very nicely how the form of the test function affects the energy density curve of the corresponding SLE. It is remarkable that a small smearing time leads to a very high energetic excitation of the state in contrast to broader smearings (figure 3.2). This observation shows why the concept of instantanous vacua (i.e. states which are required to minimise energy density at one sharp instant of time) is not a good one. Namely, such states are not sufficiently regular in order to allow for the definition of the expectation value of the energy density via the Hadamard point–splitting procedure. The divergence of the energy density of SLE's as $\epsilon\to 0$ is a hint for this. Furthermore, as we can already infer from formula (3.12), the energy density depends on $t-t_0$ instead of t and t_0 seperately. The reason for this behaviour seems to be that the energy density for an SLE is only nontrivial if curvature is present (recall that on Minkowski space, the SLE's reduce to the Minkowski vacuum whose energy density is zero). The curvature in de Sitter space in turn does not depend on time. The shifting symmetry of the energy density curve leads to the question whether the sequence of SLE's, induced by test functions whose support gets shifted to $-\infty$, converges to the Bunch Davies state as limit state. We will treat this problem in greater generality in chapter 4.

[3]This is an arbitrary requirement, however this reveals information about the choice of renormalisation which corresponds to the normal order prescription used for renormalising the energy density of the Bunch–Davies state.

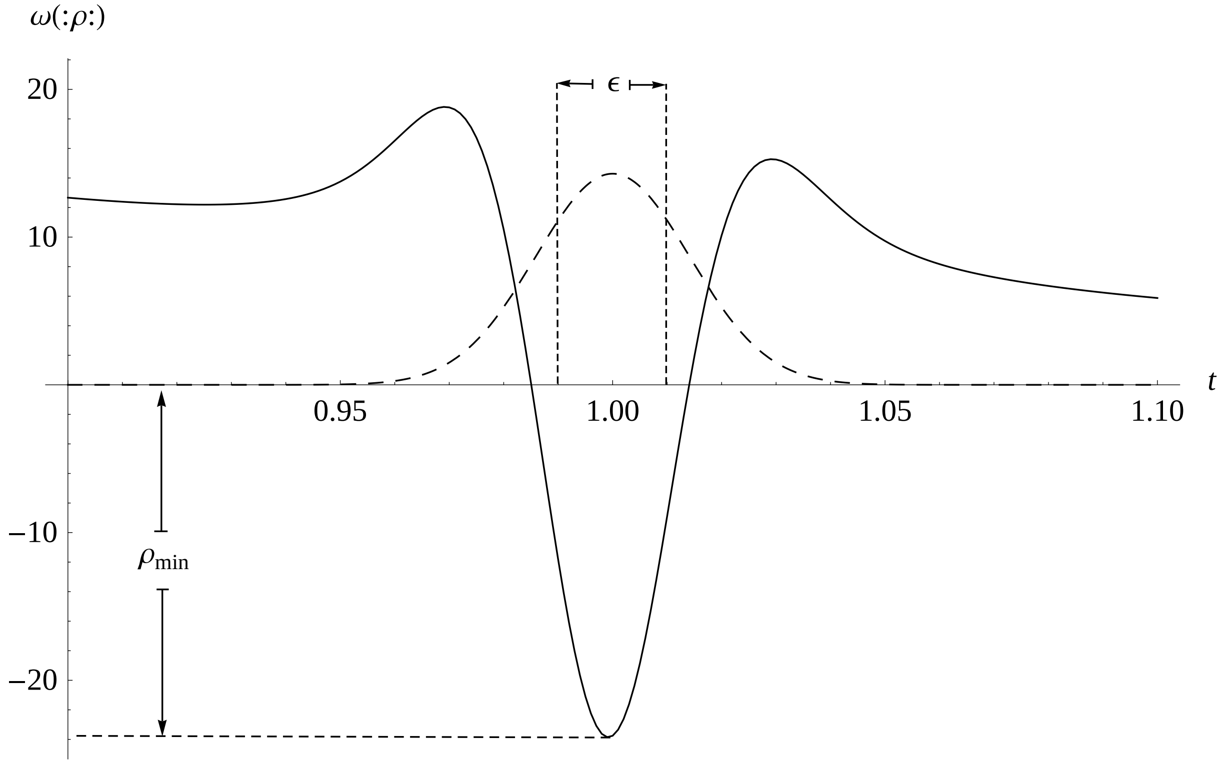

Figure 3.1.: Renormalised energy density (solid line) for $m^2 = 2H^2$, $\epsilon = 0.02$ and $t_0 = 1$, plotted against cosmological time t. For comparison we showed the corresponding (but rescaled) test function f (dashed line). It is clearly seen that the test function "stamps" its form on the energy density curve. Furthermore, there are characteristic "bumps" in the energy density at the times where the test function has significantly fallen off. In the plot we indicated the typical quantities ρ_{min} and ϵ.

The Case $m^2 = 0$

In order to supplement our result of the previous subsection, we will repeat the corresponding calculations for the case $m^2 = 0$. The reference modes are readily calculated:

$$S_k(t) = \frac{1}{\sqrt{2k^3}}(H + ie^{-Ht}k)e^{-i\exp(-Ht)k/H}.$$

It follows

$$\begin{aligned}|\dot{S}_k(t)|^2 + k^2e^{-2Ht}|S_k(t)|^2 &= ke^{-4Ht} + \frac{H^2}{2k}e^{-2Ht}\\ \dot{S}_k(t)^2 + k^2e^{-2Ht}S_k(t)^2 &= e^{-2i\exp(-Ht)k/H}e^{-2Ht}\left(\frac{H^2}{2k} + iHe^{-Ht}\right).\end{aligned}$$

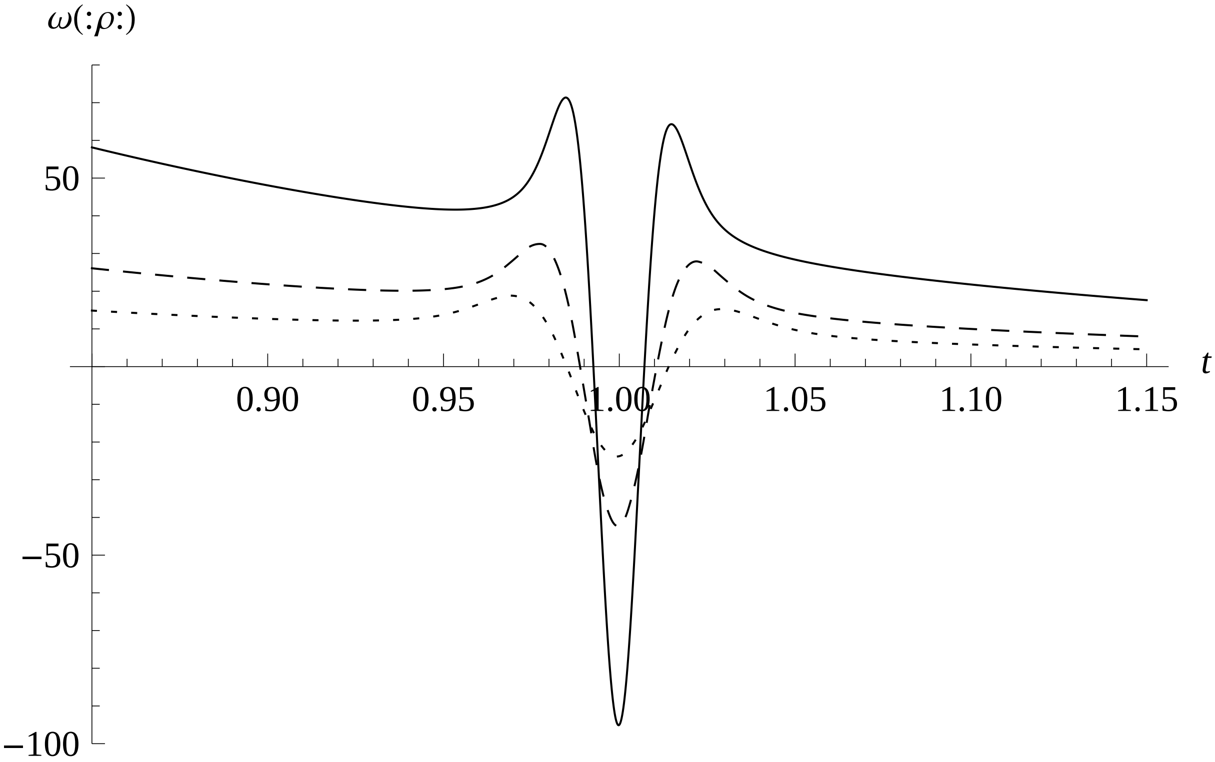

Figure 3.2.: Renormalised energy density against cosmological time for $m^2 = 2H^2$, $t_0 = 1$ and different smearing widths: $\epsilon = 0.01$ (solid), $\epsilon = 0.015$ (dashed) and $\epsilon = 0.02$ (dotted). The broader the smearing, the lower is the global excitation of the state. Correspondingly, ρ_{min} can reach very large values if the smearing is performed over a small time.

Thus, the same integrals like in the case $m^2 = 2H^2$ have to be evaluated. Using the same approximations and variables like in the previous subsection we obtain

$$\begin{aligned} c_1 &= He^{-3Ht_0}\left(\frac{\exp(\alpha^2)}{2z} + z\exp(4\alpha^2)\right) \\ &\approx He^{-3Ht_0}\left(\frac{1}{2z} + z\right) \\ c_2 &= He^{-3Ht_0}e^{-\alpha^2 z^2}e^{-2iz}\left(\frac{\exp(\alpha^2 - 2iz\alpha^2)}{2z} + i\exp(9\alpha^2/4 - 3i\alpha^2 z)\right) \\ &\approx He^{-3Ht_0}e^{-\alpha^2 z^2}e^{-2iz}\sqrt{1 + 1/(4z^2)}e^{i\arctan(2z)}, \end{aligned}$$

We introduce again the function

$$u : z \mapsto \frac{c_1}{2\sqrt{c_1^2 - |c_2|^2}}.$$

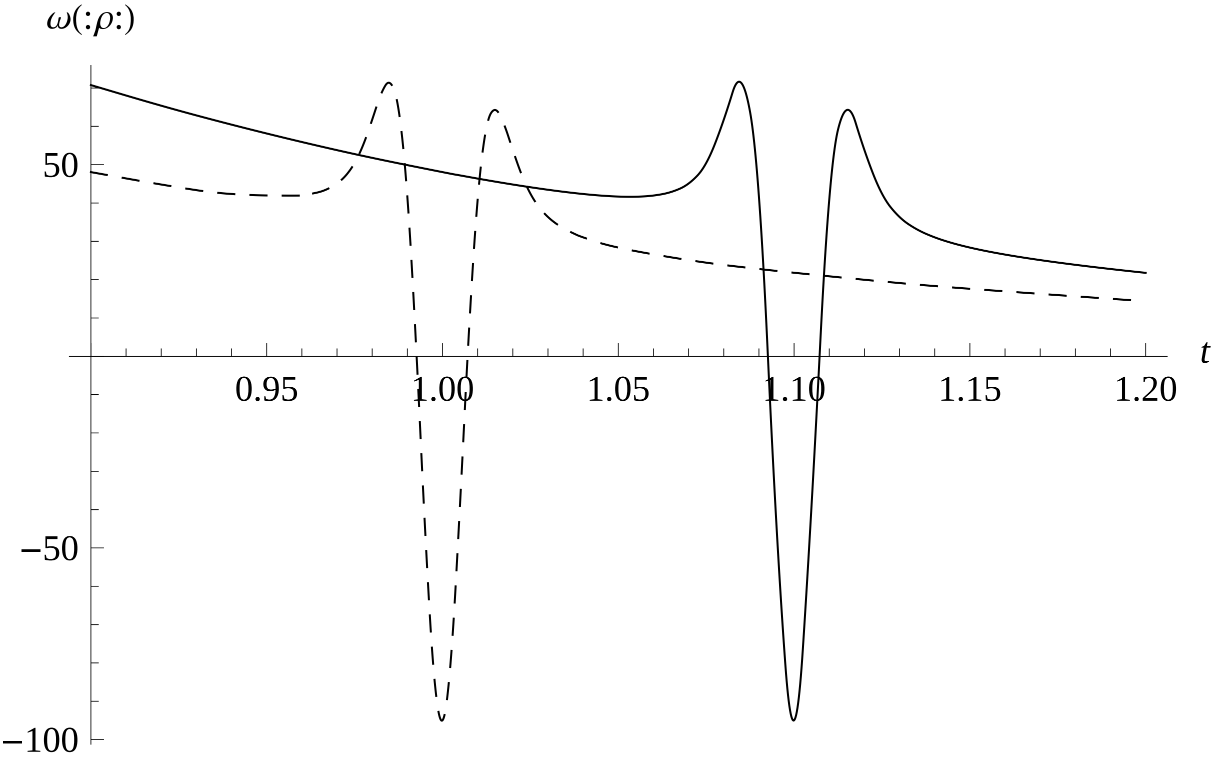

Figure 3.3.: Renormalised energy density against cosmological time for $m^2 = 2H^2$, $\epsilon = 0.01$ and different preparation times: $t_0 = 1$ (dashed), $t_0 = 1.1$ (solid). A later preparation time results in a shift to the right while the form of the curve is unaffected (this can be seen immediately from (3.12)). Thus, if the state is prepared at early times w.r.t. a fixed test function, then it will show a lower excitation in a fixed time span than a later prepared one.

Note that for $m = 0$ we do not have any renormalisation freedom at our disposal. Hence, the energy density follows as

$$\begin{aligned}\omega(:\hat{\rho}:) =& \frac{H^4}{4\pi^2} e^{-2(t-t_0)} \int\limits_{\mathbb{R}_+} \mathrm{d}z z \Bigg(2(u_0 - 1/2) \left(z^2 e^{-2(t-t_0)} + \frac{1}{2} \right) \\ & - \frac{\sqrt{u^2 - 1/4}}{\sqrt{1+4z^2}} \bigg(\cos\left(2z(1-e^{-H(t-t_0)})\right) \left(4z^2 e^{-H(t-t_0)} + 1\right) \\ & + 2z \sin\left(2z(1-e^{-H(t-t_0)})\right) \left(1 - e^{-H(t-t_0)}\right) \bigg) \Bigg) + \frac{H^4}{4\pi^2} \frac{1}{240}. \end{aligned} \tag{3.13}$$

Apart from the different asymptotic behaviour for $t \to \infty$, there is no significant difference to the case $m^2 = 2H^2$. The difference of the integrands of (3.12) and (3.13) is only notable for small z. However, for small α the main contributions to the integral come from the larger values of z.

The Case $m^2 \gg H^2$

After having investigated the small mass regime, we would like to obtain the energy density also for large masses. However, the solutions of the mode equations then involve Bessel functions of large imaginary order, which are hard to evaluate numerically (especially when the mode integral has to be performed). In order to calculate the SLE's for this regime, we will work with the following approximation: Writing the exact reference modes $S_k(t) = a^{-1}(t)\chi_k(\tau(t))$ as before, $\chi_k(\tau)$ must obey the differential equation (3.1). Define $\Omega(\tau)^2 \doteq k^2 + Q(\tau)$. A special solution of (3.1) can always be written as

$$\chi_k(\tau) = \frac{1}{\sqrt{2\Omega}} e^{i\int_{\tau_0}^{\tau} \Omega(\tau')\mathrm{d}\tau'}(1+\delta),$$

where δ is an error term whose modulus can be estimated from above [Olv74]. Next we define the approximated reference modes

$$P_k(t) \doteq \frac{1}{\sqrt{2\Omega}a} e^{i\int_{\tau_0}^{\tau} \Omega(\tau')\mathrm{d}\tau'},$$

where the lower bound of integration τ_0 is defined to correspond to the localisation time t_0 of the test function f in cosmological time, i.e. $\tau_0 \doteq \tau(t_0)$. The idea is to use the $P_k(t)$ instead of the exact reference modes $S_k(t)$ for the calculation of c_1 and c_2. The $S_k(t)$ are defined by having the same initial conditions like $P_k(t)$ at t_0. It is now straightforward to calculate the following expressions:

$$\begin{aligned}|\dot{P}_k(t)|^2 + (k^2a^{-2}(t)+m^2)|P_k(t)|^2 &= \frac{\Omega}{a^4} + \frac{3H^2}{2a^2\Omega} + \frac{nH^4}{2\Omega^3} + \frac{n^2a^2H^6}{8\Omega^5} \\ \dot{P}_k(t)^2 + (k^2a^{-2}(t)+m^2)P_k(t)^2 &= \frac{1}{2}e^{2i\int_{\tau_0}^{\tau}\Omega(\tau')\mathrm{d}\tau'}\left(\frac{n^2a^2H^6}{4\Omega^5} + \frac{3H^2}{\Omega a^2} + \frac{nH^4}{\Omega^3} - \frac{2iH}{a^3} - \frac{inH^3}{\Omega^2 a}\right),\end{aligned}$$

where we have defined

$$n \doteq \frac{m^2}{H^2} - 2.$$

In order to evaluate the integrals c_1 and c_2 it is necessary to make the following approximations: We replace the non-oscillating terms (e.g. Ωa^{-4}) by their zeroth–order Taylor series in cosmological time t around t_0. This is justified since the correction coming from the linear term is supressed due to its asymmetry in $t - t_0$ (recall that f is a Gaussian in $t - t_0$). Furthermore we linearise the term

$$\int_{\tau_0}^{\tau} \mathrm{d}\tau'\Omega(\tau') = \int_{t_0}^{t} \mathrm{d}t'\Omega(\tau(t'))a^{-1}(t') \approx \sqrt{z^2+1}\sqrt{n}H(t-t_0).$$

It follows

$$
\begin{aligned}
c_1 &= \frac{H}{a(t_0)^3}\left(\sqrt{n}\sqrt{z^2+1} + \frac{3}{2\sqrt{n}\sqrt{z^2+1}} + \frac{1}{2\sqrt{n}\sqrt{z^2+1}^3} + \frac{1}{8\sqrt{n}\sqrt{z^2+1}^5}\right)\\
&\approx \frac{H\sqrt{n}\sqrt{z^2+1}}{a(t_0)^3}\\
c_2 &= \frac{H}{2a(t_0)^3}e^{-(z^2+1)n\alpha^2}\left(\frac{1}{4\sqrt{n}\sqrt{z^2+1}^5} + \frac{1}{\sqrt{n}\sqrt{z^2+1}^3} + \frac{3}{\sqrt{n}\sqrt{z^2+1}} - i(2+\frac{1}{(z^2+1)})\right)\\
&\approx \frac{-iH}{2a(t_0)^3}e^{-\alpha^2 n(z^2+1)}\left(2+\frac{1}{z^2+1}\right),
\end{aligned}
\tag{3.14}
$$

where this time

$$
z \doteq \frac{k}{\sqrt{n}He^{Ht_0}}
$$

and α as already defined in (3.7). We neglected contributions which become small for large masses, i.e. large n. Like in the previous subsections, we were assuming that $\alpha \ll 1$ in order to make the simplifications in the integrals meaningful. This also justifies the following approximation done for c_2: Consider one of its contributions,

$$
\int \mathrm{d}t f(t) e^{2i\int_{\tau_0}^{\tau}\Omega(\tau')\mathrm{d}\tau'}\frac{3H^2}{2\Omega a^2}.
$$

We claim that the linear term in the Taylor expansion of $\frac{3H^2}{\Omega a^2}$ around t_0, given by

$$
-\frac{3H^2e^{-3Ht_0}(3+2z^2)}{\sqrt{n}\sqrt{z^2+1}^3}(t-t_0),
$$

is negligible for the result of this contribution for c_2. We have

$$
\frac{1}{\sqrt{\pi}\epsilon}\int e^{-(t-t_0)^2/\epsilon^2}e^{2i\sqrt{z^2+1}\sqrt{n}H(t-t_0)}(t-t_0) = i\epsilon^2 H\sqrt{n}\sqrt{z^2+1}e^{-\epsilon^2(1+z^2)H^2n}
$$

Thus we would get an additional contribution

$$
-3i\alpha^2\left(2+\frac{1}{z^2+1}\right)
$$

in the bracket of c_2 in formula (3.14)). But since $\alpha \ll 1$, it is strongly suppressed. A similar discussion applies to the other terms in c_2. Starting from result (3.14) we can now calculate the Bogolubov coefficients λ, μ according to Olbermann's theorem. They can be expressed in a simple manner, using the fact that

$$
\frac{|c_2|}{c_1} = \frac{e^{-\alpha^2 n(z^2+1)}}{2\sqrt{n}\sqrt{z^2+1}}\left(2+\frac{1}{z^2+1}\right)
$$

is small for all values of z and thus

$$\begin{aligned}
\mu &= \sqrt{\frac{1}{2\sqrt{1-|c_2|^2/c_1^2}} - \frac{1}{2}} \approx \frac{e^{-\alpha^2 n(z^2+1)}}{4\sqrt{n}\sqrt{z^2+1}}\left(2+\frac{1}{z^2+1}\right) \\
|\lambda| &= \sqrt{\frac{1}{2\sqrt{1-|c_2|^2/c_1^2}} + \frac{1}{2}} \approx 1 + \frac{e^{-2\alpha^2 n(z^2+1)}}{32n(z^2+1)}\left(2+\frac{1}{z^2+1}\right)^2 \\
\arg\lambda &= \pi - \arg c_2 \approx \frac{3}{2}\pi.
\end{aligned} \tag{3.15}$$

In order to get the time dependence of the energy density for the SLE, we would have to insert in (3.4) the exact mode solutions $S_k(t)$ whose initial values at t_0 coincide with those of the approximate modes $P_k(t)$. However, since they are difficult to treat numerically, we make a second approximation and use once more the approximate modes $P_k(t)$ instead of the $S_k(t)$. That is,

$$\begin{aligned}
[\mathcal{R}\mathcal{W}_2^s]_t \approx & \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{k}\Big((\mu^2+1/2)(|\dot{P}_k(t)|^2 + (k^2a^{-2}(t)+m^2)|P_k(t)|^2) \\
& + \mu|\lambda|\mathrm{Re}\left\{e^{i\arg\lambda}(\dot{P}_k(t)^2 + (k^2a^{-2}(t)+m^2)P_k(t)^2)\right\}\Big)e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}
\end{aligned} \tag{3.16}$$

We have

$$\begin{aligned}
\mathrm{Re}\left\{e^{i\arg\lambda}(...)\right\} = & -\cos\left(2\int_{\tau_0}^{\tau}\mathrm{d}\tau'\Omega(\tau')\right)\left(\frac{H}{a^3}+\frac{nH^3}{2\Omega^2 a}\right) \\
& + \sin\left(2\int_{\eta_0}^{\tau}\mathrm{d}\tau'\Omega(\tau')\right)\left(\frac{n^2a^2H^6}{8\Omega^5}+\frac{3H^2}{2\Omega a^2}+\frac{nH^4}{2\Omega^3}\right)
\end{aligned}$$

Obviously, the contribution $\propto \sin(...)$ is supressed by the factor $n^{-1/2}$ and we will neglect it.

We turn now to the calculation of the counterterms using the Hadamard subtraction scheme. We have

$$\begin{aligned}
[\widetilde{\mathcal{R}\mathcal{G}_1^s}]_\tau &= \frac{H^4}{4\pi^2}\Bigg(-\frac{2\tau^4}{r_+^4} + \frac{3+n}{2}\frac{\tau^2}{r_+^2} + \frac{(2+n)n}{16}\left(\mathrm{lo}_0 + \log\frac{1}{H^2\tau^2}\right) + \frac{23-50n-30n^2}{240}\Bigg) \\
[P_x\mathcal{G}] &= \frac{H^4}{4\pi^2}\left(\frac{1}{20}-\frac{3n^2}{8}\right).
\end{aligned}$$

On the other hand, an asymptotic expansion for large k yields

$$|\dot{P}_k(t)|^2 + (k^2a^{-2}(t)+m^2)|P_k(t)|^2 = H^4\tau^4 k + \frac{H^4\tau^2(n+3)}{2k} - \frac{H^4(n^2+2n)}{8k^3} + O(k^{-5}).$$

Applying lemma 2.4.5 to the function $\Omega(k) \doteq |\dot{P}_k(t)|^2 + (k^2a^{-2}(t)+m^2)|P_k(t)|^2$ we get

$$\begin{aligned}
& \lim_{\epsilon\to 0}\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3\mathbf{k}\left(|\dot{P}_k|^2 + (k^2a^{-2}(t)+m^2)|P_k|^2\right)e^{i\mathbf{k}\mathbf{r}}e^{-\epsilon k} \\
&= [\widetilde{\mathcal{R}\mathcal{G}_1^s}]_\tau + \frac{H^4}{4\pi^2}\Bigg(\frac{5n^2+4n}{32} - \frac{23}{240} + \frac{n^2+2n}{8}(\ln\sqrt{n} - \ln 2 - 1 - \gamma)\Bigg),
\end{aligned}$$

up to terms which vanish for $r \to 0$. One may explicitly convince oneself that a similar formula holds for arbitrary scale factors, i.e. not only "by accident" in de Sitter case. This means that the approximate modes $P_k(t)$ can be used for a renormalisation prescription for the calculation of the energy momentum tensor, since they produce the "right divergencies" which cancel those from the unrenormalised mode expression of $\omega(T_{ab})$ for the Hadamard state ω in question. This goes also under the name *adiabatic renormalisation* (for concrete calculational examples see e.g. [BD82]). Schlemmer [Sch10] discussed the relation to the rigorous Hadamard subtraction scheme in more detail.

For the renormalised expectation value of the energy density it follows

$$\begin{aligned}
&\omega(:\rho:) = \\
&\frac{nH^4 e^{-3H(t-t_0)}}{4\pi^2} \int_0^\infty \mathrm{d}z z^2 \Bigg(\frac{e^{-2\alpha^2 n(z^2+1)}}{8(z^2+1)} \left(2 + \frac{1}{z^2+1}\right)^2 \sqrt{z^2 e^{-2H(t-t_0)}+1} \\
&- \cos(2\sqrt{n} f(z,t-t_0)) \frac{e^{-\alpha^2 n(z^2+1)}}{4\sqrt{z^2+1}} \left(2 + \frac{1}{z^2+1}\right) \left(2 + \frac{1}{(z^2 e^{-2H(t-t_0)}+1)}\right) \Bigg) \\
&+ \frac{H^4}{4\pi^2} \left(\frac{n^2+4n}{32} - \frac{19}{240} + \frac{n^2+2n}{8} (\ln\sqrt{n} - \ln 2 - 1 - \gamma) \right) \\
&+ H^4 \left((n+2)^2 \alpha - 3(n+2)\beta \right),
\end{aligned} \tag{3.17}$$

where

$$f(z,t-t_0) = \sqrt{z^2 e^{-2H(t-t_0)}+1} - \sqrt{z^2+1} + \log \frac{e^{-H(t-t_0)}(1+\sqrt{z^2+1})}{1+\sqrt{z^2 e^{-2H(t-t_0)}+1}}.$$

Note that we neglected terms of order $O(n^0)$, i.e. the above formula is supposed to be meaningful for large n. Figures 3.4 and 3.5 show the energy density (in units of H^4) against cosmological time t (in units of H^{-1}) for the regime $m^2 \gg H^2$, where $t_0 = 0$ and the renormalisation constants for each curve were chosen such that

$$\lim_{t\to\infty} \omega(:\rho:) = 0.$$

However, if the renormalisation constants A and B are held fixed, the energy density curve will get shifted by an n-dependent term with leading contribution

$$\frac{H^4(n^2+2n)\log n}{64\pi^2}.$$

Plot (3.4) nicely illustrates that bigger masses "damp" the fluctuations of $\omega_f(:\rho:)$. At the same time they lead to characteristic oscillations of the energy density, whose frequency depends on m. Plot (3.5) shows that, for fixed mass, the smearing width of the test function f has the same effect which we already found for the low mass regime. It is also seen that the oscillation frequency is not influenced by the smearing width.

3.2. Comparison with Fewsters Bound

Since we chose an example of a spacetime where many results can be obtained analytically, it is instructive to examine Fewster's general worldline QEI (2.23) explicitly for our situation. We

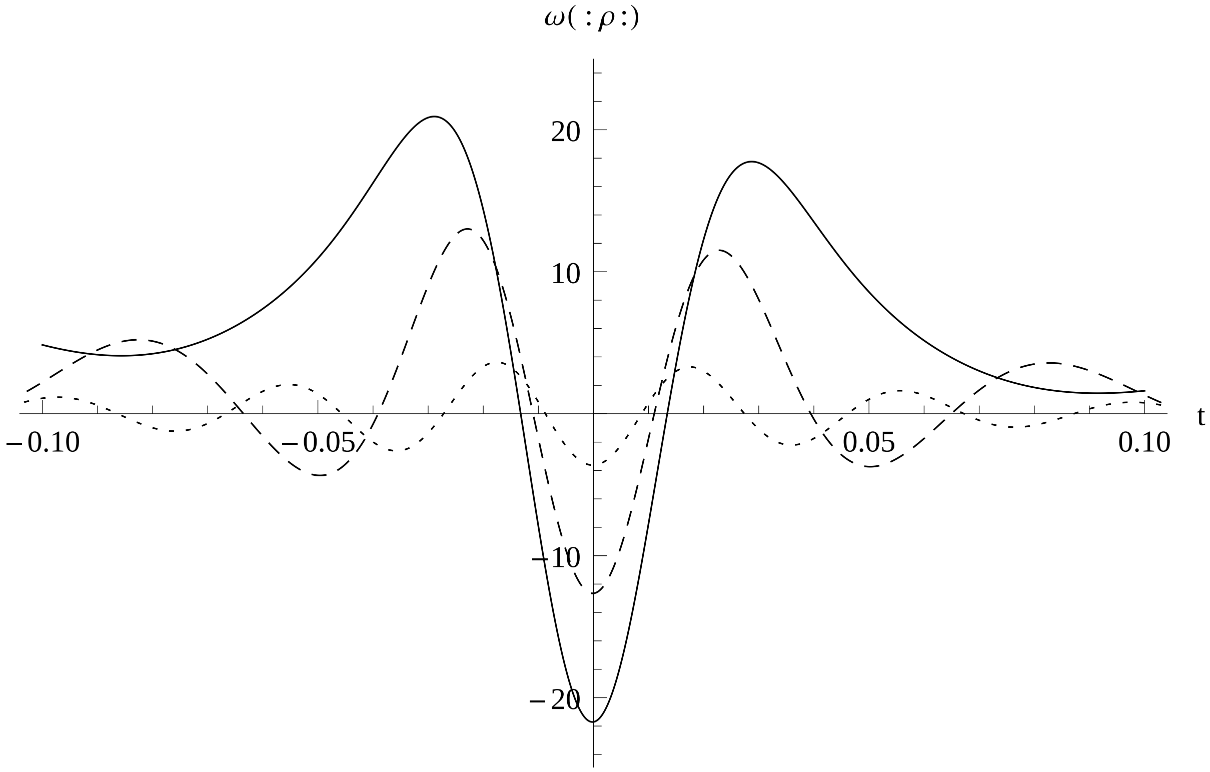

Figure 3.4.: Renormalised energy density for SLE's against cosmological time for $t_0 = 0$, $\alpha = 0.02$ and different values of n: $n = 500$ (solid),$n = 2000$ (dashed) and $n = 5000$ (dotted).

shall only consider the case $m^2 = 2H^2$. The expression for $\omega(:\rho:)$ we found above corresponds to a normal ordering prescription w.r.t. the Bunch–Davies state, which thus plays the role of ω_0 in (2.23). Its unrenormalised point–split energy density reads in conformal time

$$\omega_0(\rho(\tau,\tau')) = \frac{H^2}{8\pi^2}\int_0^\infty \mathrm{d}k k\tau\tau' e^{-ik(\tau-\tau')}\Big(2k^2\tau\tau' + 3 - ik(\tau-\tau')\Big).$$

In order to compute the bound in Fewsters general worldline QEI, we have to evaluate the expression

$$B_f \doteq \int_0^\infty \frac{\mathrm{d}\gamma}{\pi}\int \mathrm{d}t\mathrm{d}t' e^{-i\gamma(t-t')}\sqrt{f(t)}\sqrt{f(t')}\omega_0(\rho(t,t')) \tag{3.18}$$

for the Gaussian

$$f(t) = \frac{1}{\sqrt{\pi}\epsilon}e^{\frac{H^2(t-t_0)^2}{\alpha^2}},$$

with $\alpha \doteq H\epsilon$. Assuming again $\alpha \ll 1$ we may linearise $\tau(t)$ around t_0 and carry out both time integrations in (3.18). The intermediate result reads

$$B_f = \frac{H^4}{4\pi^{5/2}}\int_0^\infty \mathrm{d}\gamma'\int_0^\infty \mathrm{d}k' e^{-(k'+\gamma')^2}\left(\frac{2k'^3}{\alpha^4}e^{4\alpha^2} + 3\frac{k'}{\alpha^2}e^{\alpha^2} - 2\sin(\alpha(k'+\gamma'))e^{5\alpha^2/2}\frac{k'^2}{\alpha^3}\right),$$

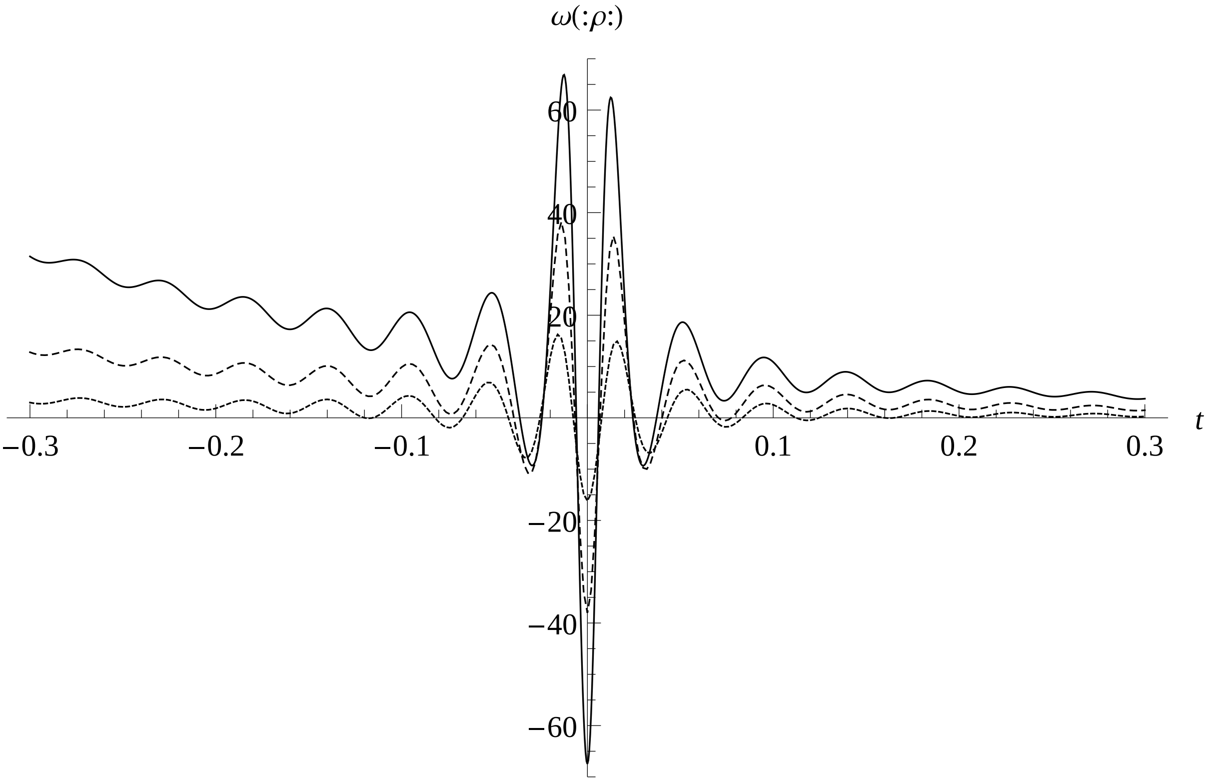

Figure 3.5.: Renormalised energy density for SLE's against cosmological time for $t_0 = 0$, $n = 5000$ and different values of α: $\alpha = 0.01$ (solid),$\alpha = 0.012$ (dashed) and $\alpha = 0.015$ (dotted).

where we performed the substitutions $k' = \epsilon e^{-Ht_0} k$ and $\gamma' = \epsilon\gamma$. This integral can be explicitly calculated and yields

$$B_f = \frac{H^4}{32\pi^2}\left(\frac{3e^{4\alpha^2}}{2\alpha^4} + \frac{3e^{\alpha^2}}{\alpha^2} + \frac{e^{9\alpha^2}(\alpha^2 H^2 - 6)}{3\alpha^2}\right). \tag{3.19}$$

Now we calculate the l.h.s. of (2.23) for the SLE induced by f, i.e. we calculate

$$Q_f \doteq \int \mathrm{d}t f(t)\omega_f(:\rho(t):). \tag{3.20}$$

Starting from the simplified formula (3.12) for $\omega_f(:\rho(t):)$, we interchange the t- und z- integration and obtain

$$Q_f \approx \frac{H^4}{4\pi^2}\int_0^\infty \mathrm{d}z z\left(2\left(u-\frac{1}{2}\right)\left(z^2+\frac{3}{2}\right) - 3\sqrt{u^2-\frac{1}{4}}e^{-\alpha^2 z^2}\sqrt{1+\frac{4}{9}z^2}\right), \tag{3.21}$$

where we used $e^{-H(t-t_0)} \approx 1 - H(t-t_0)$ in order to perform the t-integration and the approximations

$$e^{\alpha^2} \approx 1, \qquad z\alpha^2 \ll 1,$$

from which the last one is satisfied since the integrand in (3.21) only contributes for $z\alpha^2 \ll 1$ on account of the decay behaviour of the function $u(z)$. We are interested in the behaviour of Q_f for small α. The integrand of (3.21) may be bounded independently of α for $z \in (0, z_0)$ and thus the contribution coming from integrating from $z = 0$ to z_0 can be estimated by a finite constant. We choose $z_0 = 2$ and perform the following expansions for $z \geq 2$:

$$u \approx \frac{1}{2} + e^{-2\alpha^2 z^2} \frac{9/4 + z^2}{4(3/2 + z^2)^2}, \qquad \sqrt{u^2 - 1/4} \approx e^{-\alpha^2 z^2} \frac{\sqrt{9/4 + z^2}}{2(3/2 + z^2)}$$

Note that this appoximation is good for all α. For $z \geq 2$ the integrand is now given by

$$-z \frac{9/4 + z^2}{2(3/2 + z^2)} e^{-2\alpha^2 z^2}.$$

For small α the exact behaviour for small z becomes less important and it is justified to write

$$Q_f \approx -\frac{H^4}{4\pi^2} \int_0^\infty \mathrm{d}z \frac{z}{2} e^{-2\alpha^2 z^2} = -\frac{H^4}{32\pi^2 \alpha^2}. \tag{3.22}$$

Comparing this expression with (3.19) we see that $Q_f \gg -B_\alpha$ for small α. That is, the f–smeared energy density in the SLE ω_f does not attain its lower bound. Moreover, it scales differently with the smearing width. This was to be expected a priori, since Fewster obtained a bound valid for all Hadamard states, while the class of SLE's contains only homogenous and isotropic Hadamard states and is thus much smaller. Physically speaking, Fewsters bound says that there might exist states (in particular anisotropic and inhomogenous ones) whose time-smeared energy density for certain isotropic observers becomes much lower than that of SLE's belonging to this smearing function. The different scaling in α of Q_α and B_f for decreasing α is due to the fact that the SLE has to minimise the energy density *globally*, that is in all space.

3.3. SLE's as Reference Ground States and the Regime of Backreaction

Our results have shown that SLE's become more vacuum–like when we choose a larger rather than a shorter smearing width α for the test function f, which is clearly visible in figure 3.2. A question which arises naturally in this context is for which order of magnitude of α the backreaction to the spacetime is no longer negligible. We will answer this question in a quantitative way in this subsection.

Before doing that, let us discuss a numerical example for SLE's, illustrating their properties as reference ground states for the particle interpretation. Starting from the observation that the present epoch of our universe can be well described by a spatially flat de Sitter space with Hubble parameter

$$H_0 \approx 2 \times 10^{-42}\mathrm{GeV}$$

and corresponding critical energy density

$$\rho_C = \frac{3H_0^2}{8\pi G} \approx 7 \times 10^{-47}(\mathrm{GeV})^4$$

(we took these numbers from Appendix A.2 of [KT90], with $h \approx 0.9$), we may take as a reference state for a scalar QFT an SLE, induced by a test function with localisation time of "today" and a certain smearing width ϵ in cosmological time, which we take to be $\epsilon = 0.02H_0^{-1}$. On cosmological scales, this is a rather short time. The corresponding measure of the temporal fluctuations of the energy density in this state, ρ_{min} in figure 3.1, can be read off this plot to be

$$\rho_{min} \approx 23H_0^4 \approx 370 \times 10^{-168}\mathrm{GeV}^4.$$

This is 120 orders of magnitude smaller than ρ_C, and thus negligible for local backreaction effects. Moreover, one can argue in a quantitative manner in which sense the particle picture built on the SLE's is stable under the variation of the test function f: Consider the SLE's ω_f and ω_g, induced by smearing functions $f, g \in \mathcal{D}(\mathbb{R})$, respectively. Consider furthermore a test function $h \in \mathcal{D}(\mathcal{M})$. By using the GNS–representation of the SLE's obtained by the one particle Hilbert space structure (2.17), we can define the one–particle states

$$\begin{aligned}\Psi[h, f] &\doteq \pi_f(\Phi(h))\Omega_f = a^\dagger(k_f([h]))\Omega_f \\ \Psi[h, g] &\doteq \pi_g(\Phi(h))\Omega_g = a^\dagger(k_g([h]))\Omega_g,\end{aligned}$$

where Ω_f and Ω_g are the vacuum vector states representing ω_f and ω_g. The one particle Hilbert space is in both cases $L^2(\mathbb{R}^3)$ and the elements are momentum space wave functions. Let $\lambda_{f/g}$ and $\mu_{f/g}$ be the Bogolubov coefficients describing $\omega_{f/g}$ w.r.t. the fixed reference modes $S_k(t)$. Now, according to (2.17) the respective momentum–space wave functions read:

$$\begin{aligned}k_f([h])(\mathbf{k}) = ia^3(t_0)\Big(&\left(\lambda_f \dot{S}_k(t_0) + \mu_f \dot{\overline{S}}_k(t_0)\right) \widehat{Eh(t_0)}(\mathbf{k}) \\ &- \left(\lambda_f S_k(t_0) + \mu_f \overline{S}_k(t_0)\right) (\widehat{\partial_t Eh)(t_0})(\mathbf{k})\Big) \\ k_g([h])(\mathbf{k}) = ia^3(t_0)\Big(&\left(\lambda_g \dot{S}_k(t_0) + \mu_g \dot{\overline{S}}_k(t_0)\right) \widehat{Eh(t_0)}(\mathbf{k}) \\ &- \left(\lambda_g S_k(t_0) + \mu_g \overline{S}_k(t_0)\right) (\widehat{\partial_t Eh)(t_0})(\mathbf{k})\Big)\end{aligned}$$

Figures 3.6 and 3.7 show some numerical examples for the difference of the particle pictures built on different choices of f and g on de Sitter space for the case $m^2 = 2H^2$. The momenta k are given in units of H. The plots illustrate that the particle interpretation for low momenta will be different in the states of low energy ω_f and ω_g. This regime of low momenta is determined by the smallest smearing width of f and g on account of the decay properties of $\mu(k)$, calculated in section 3.1.

Now we would like to answer another question, namely which regime of smearing times is physically meaningful if backreaction is neglected. The semiclassical version of the Friedmann equation (1.13) may be rewritten as

$$\omega_f(:\rho:) = \frac{3H^2}{8\pi G}. \tag{3.23}$$

It is useful to adapt the renormalisation constants such that

$$\lim_{t\to\infty} \omega_f(:\rho:) = \frac{3H^2}{8\pi G},$$

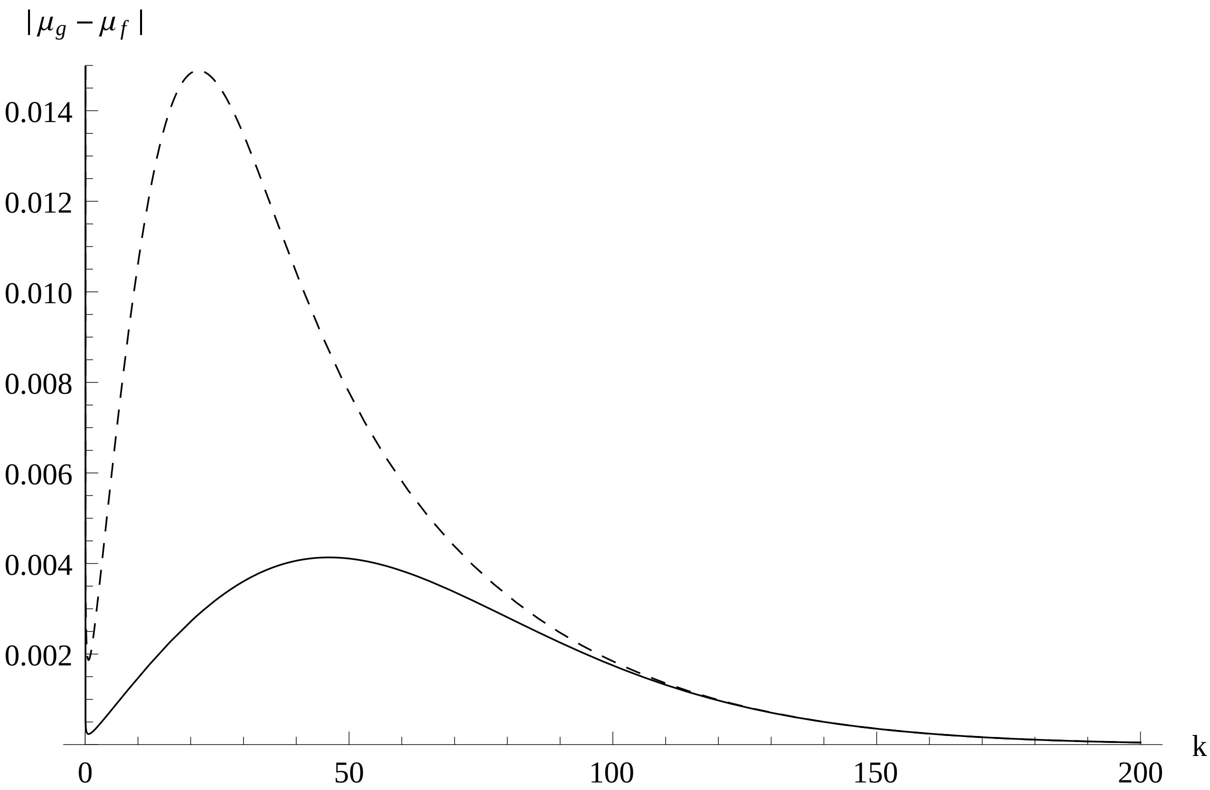

Figure 3.6.: Difference of the Bogolubov coefficients $\mu(k)$, describing the SLE's $\omega_{f/g}$ w.r.t. the Bunch–Davies modes. The test function g is a Gaussian located at $t_g = 0$ and having smearing width $\epsilon_g = 0.01H^{-1}$. The corresponding parameters for f are $\epsilon_f = 0.05H^{-1}$ (dashed) and $\epsilon_f = 0.02H^{-1}$ (solid), whereas the localisation time is for both curves $t_f = t_g$.

which means that (3.23) is fulfilled asymptotically in the future[4]. Now we require that locally (i.e. around the localisation time t_0) there should hold $\omega_f(:\rho:) \approx \frac{3H^2}{8\pi G}$, which leads to the condition

$$\rho_{min} \ll \frac{3H^2}{8\pi G} \tag{3.24}$$

(see figure 3.1 for the definition of ρ_{min}). We already know that small smearing times make ρ_{min} increase. Regarding first the case $m^2 = 2H^2$, we use the same arguments like in the derivation of (3.22) out of (3.21) and obtain

$$\rho_{min} \approx \left| \frac{H^4}{4\pi^2} \int_0^\infty \mathrm{d}z z \frac{z^2 + 9/4}{z^2 + 3/2} e^{-\alpha^2 z^2} (e^{-\alpha^2 z^2}/2 - 1) \right|,$$

which becomes for small α

$$\rho_{min} \approx \left| \frac{H^4}{4\pi^2} \int_0^\infty \mathrm{d}z z e^{-\alpha^2 z^2} (e^{-\alpha^2 z^2}/2 - 1) \right| = \frac{3H^4}{32\alpha^2\pi^2}$$

[4]Note that this is not possible for the case $m^2 = 0$.

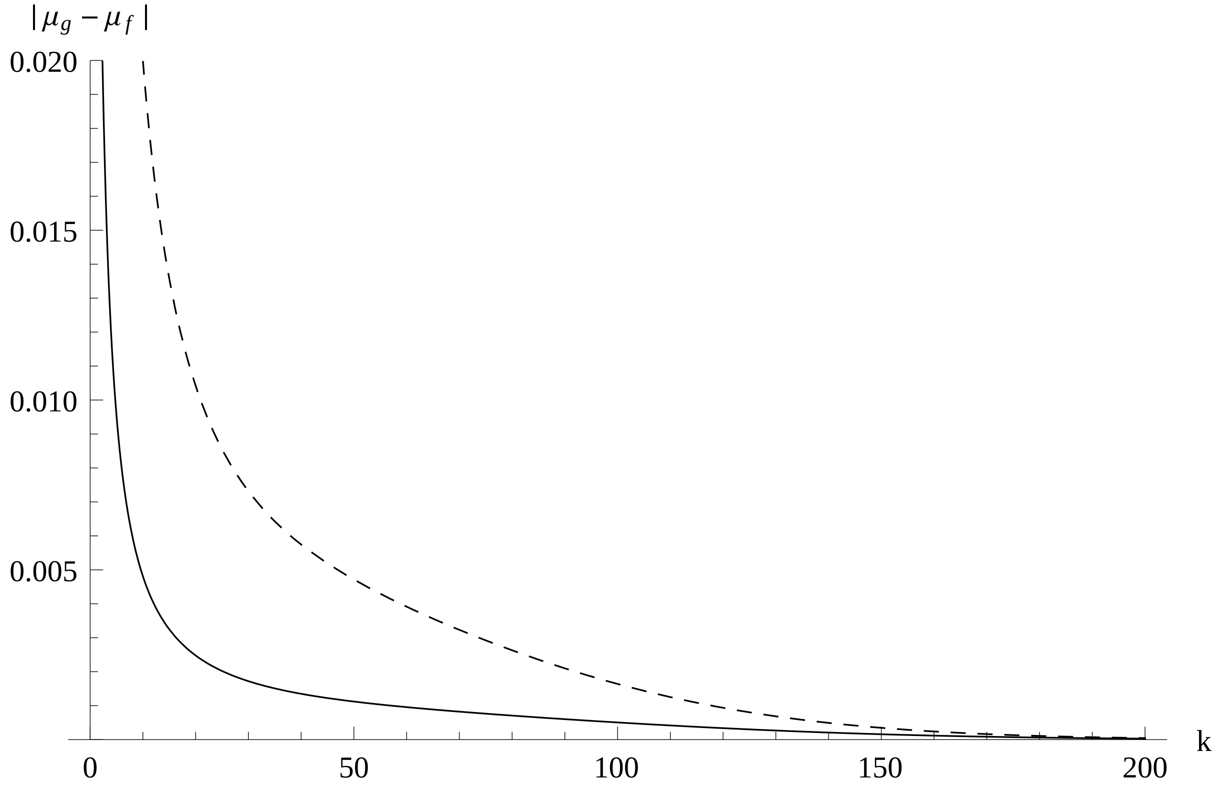

Figure 3.7.: Difference of the Bogolubov coefficients $\mu(k)$, describing the SLE's $\omega_{f/g}$ w.r.t. the Bunch–Davies modes. This time both Gaussians f and g have the same smearing widths $\epsilon_g = \epsilon_g = 0.01H^{-1}$. The dashed curve was obtained for $t_g = 0$ and $t_f = -0.5H^{-1}$; the solid curve shows $t_g = 0$ and $t_f = -0.1H^{-1}$.

Condition (3.24) is then equivalent to

$$\epsilon \gg \sqrt{\frac{G}{4\pi}} = \frac{t_P}{2\sqrt{\pi}}. \tag{3.25}$$

That is, only when the smearing time ϵ becomes as small as the the Planck time, the energy density curve of the SLE belonging to the test function with smearing width ϵ will reach the value of the critical energy density in the vicinity of the localisation time t_0. It is only in this regime where backreaction effects will become important[5]. Since ρ_{min} scales with α^{-2}, this result is independent of the value of the Hubble constant H. It is also independent of the localisation time t_0 (see figure 3.3). In order to exclude the possibility that this result hinges on the very special choice of mass $m^2 = 2H^2$, we repeat this calculation for the remaining two mass regimes. We start with $m^2 = 0$:
We have

$$u \approx \frac{1}{2} + \frac{z^2 + 1/4}{4(z^2 + 1/2)^2} e^{-2\alpha^2 z^2} \qquad \sqrt{u^2 - 1/2} \approx \frac{\sqrt{z^2 + 1/4}}{2z^2 + 1/2} e^{-\alpha^2 z^2}.$$

[5]Of course we know that the energy density diverges for $t \to -\infty$. Condition (3.24) should rather hold for $t > T$, T being some fixed time.

Inserting this into formula (3.13) for $t = t_0$ gives

$$\begin{aligned}\rho_{min} &\approx \frac{H^4}{4\pi^2}\int \mathrm{d}z z\frac{4z^2+1}{4z^2+2}\left(e^{-\alpha^2 z^2} - \frac{1}{2}e^{-2\alpha^2 z^2}\right)\\ &\approx \frac{H^4}{4\pi^2}\int \mathrm{d}z z\left(e^{-\alpha^2 z^2} - \frac{1}{2}e^{-2\alpha^2 z^2}\right)\\ &= \frac{3H^4}{32\alpha^2\pi^2},\end{aligned}$$

which is the same result like in the case $m^2 = 2H^2$. Now we look at the case $m^2 \gg H^2$. From formula (3.17) we get

$$\begin{aligned}\rho_{min} \approx &\frac{H^4 n}{4\pi^2}\int_0^\infty \mathrm{d}z z^2\left(2+\frac{1}{z^2+1}\right)^2\frac{e^{-\alpha^2 n(z^2+1)}}{4\sqrt{z^2+1}}\left(\frac{e^{-\alpha^2 n(z^2+1)}}{2}-1\right)\\ =&\frac{H^4 e^{-2n\alpha^2}}{128\pi^{3/2}\alpha^2}\Big((3+n\alpha^2)\mathrm{U}(3/2,0,2n\alpha^2) - 2e^{n\alpha^2}(6+n\alpha^2)\mathrm{U}(3/2,0,n\alpha^2)\\ &- 8n\alpha^2\left(2e^{n\alpha^2}\mathrm{U}(3/2,1,n\alpha^2) - \mathrm{U}(3/2,1,2n\alpha^2)\right)\Big),\end{aligned}$$

where $\mathrm{U}(a,b,z)$ is the confluent hypergeometric function [AS64]. We want to discuss the regime for n and α for which (3.24) holds, i.e. under which conditions backreaction is negligible. In order to do so, we define

$$x \doteq n\alpha^2 \approx m^2\epsilon^2 = \left(\frac{m}{m_P}\right)^2\left(\frac{\epsilon}{t_P}\right)^2$$

and

$$\begin{aligned}s(x) \doteq \Big|&\frac{e^{-2x}}{48\sqrt{\pi}}\Big((3+x)\mathrm{U}(3/2,0,2x) - 2e^x(6+x)\mathrm{U}(3/2,0,x)\\ &- 8x\left(2e^x\mathrm{U}(3/2,1,x) - \mathrm{U}(3/2,1,2x)\right)\Big)\Big|\end{aligned} \tag{3.26}$$

Condition (3.24) then becomes

$$\left(\frac{\epsilon}{t_P}\right)^2 \gg s(x) \tag{3.27}$$

Figure 3.8 shows a plot of the function $s(x)$. Since s is bounded on $[0,\infty)$, (3.27) is fulfilled for all possible choices of the mass (provided that $m^2 \gg 2H^2$, since this is the assumption for our calculations), if $\epsilon \gg t_P$. This result assures that our findings for the special case $m^2 = 2H^2$ remain valid for other (and more physical) choices of the mass.

3.4. Comparison with Adiabatic States

In the previous section we obtained the energy density in SLE's on de Sitter space. In particular, we discussed how it depends on the smearing width and localisation time of our Gaussian test function f. We would like to compare the results for the case $m^2 = 2H^2$ with an adiabatic

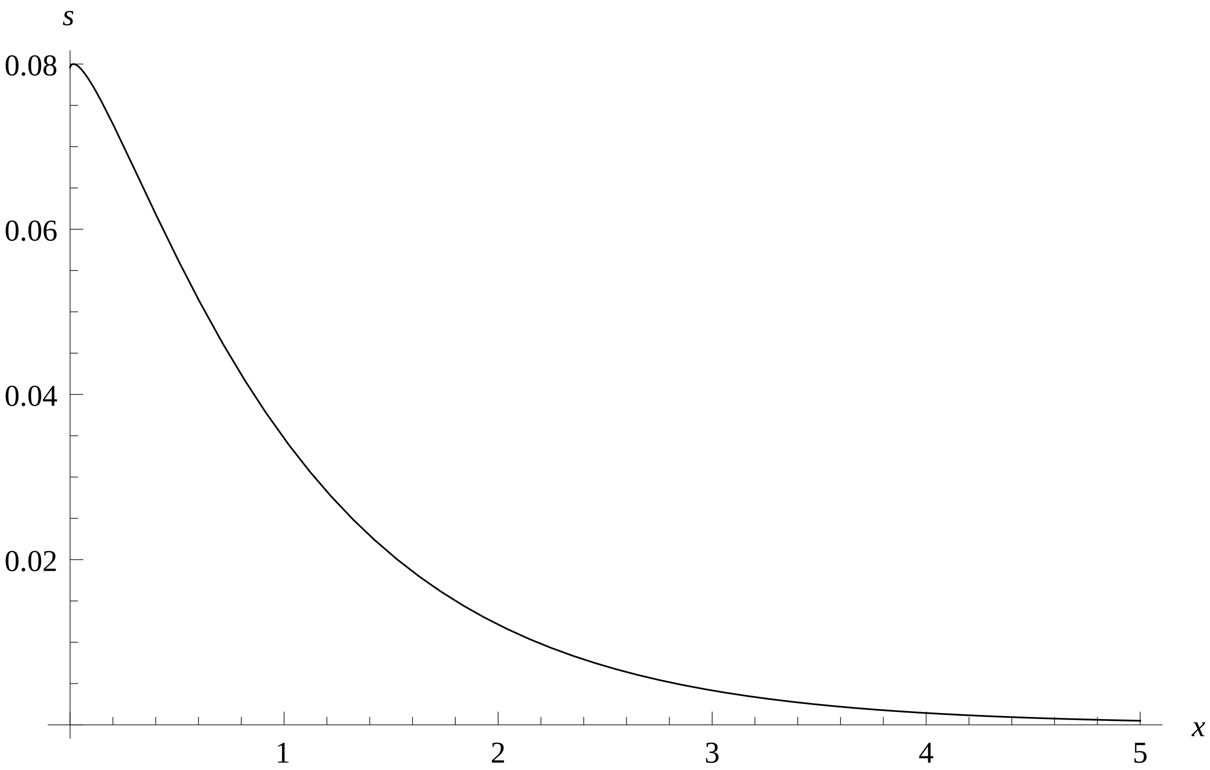

Figure 3.8.: The function $s(x)$, defined in (3.26).

state of order 1, which is regular enough to admit the definition of the energy density via the Hadamard point–splitting method. We write it by means of a Bogolubov transformation w.r.t. our basic modes χ_k:

$$T_k(t) = a^{-1}(t)\left(\alpha_k \chi_k(\tau(t)) + \beta_k \overline{\chi}_k(\tau(t))\right).$$

The Bogolubov coefficients are fixed by the prescription (2.37) for $n = 1$. Setting $\tau_0 = -H^{-1}e^{-Ht_0}$, $z = k|\tau_0|, \Omega = H^{-1}\Omega^{(1)}$, $\Omega_0 = \Omega(t_0)$ and $(\dot{\Omega})_0 = \dot{\Omega}(t_0)$ we obtain

$$\begin{aligned} \alpha_k e^{-iz} + \beta_k e^{iz} &= \sqrt{\frac{z}{\Omega_0}} \\ \alpha_k e^{-iz}(-1+iz) - \beta_k e^{iz}(1+iz) &= \sqrt{\frac{z}{\Omega_0}}\left(-i\Omega_0 - \frac{3}{2} - \frac{(\dot{\Omega})_0}{2H\Omega_0}\right) \end{aligned} \tag{3.28}$$

with the solution

$$\begin{aligned} \alpha_k &= \frac{e^{iz}}{2\sqrt{\Omega_0 z}}\left(z + \frac{i}{2} + \frac{i}{2}\frac{(\dot{\Omega})_0}{H\Omega_0} - \frac{\Omega_0}{H}\right) \\ \beta_k &= \frac{e^{-iz}}{2\sqrt{\Omega_0 z}}\left(z - \frac{i}{2} - \frac{i}{2}\frac{(\dot{\Omega})_0}{H\Omega_0} + \frac{\Omega_0}{H}\right). \end{aligned} \tag{3.29}$$

We calculate

$$\begin{aligned}\Omega_0^2 &= z^2 - \frac{1}{4} - \frac{z^2}{z^2+2} + \frac{5}{4}\frac{z^4}{(z^2+2)^2} \\ \frac{(\dot{\Omega})_0}{H\Omega_0} &= \frac{-2z^2 + \frac{2z^2}{z^2+2} - 7\frac{z^4}{(z^2+2)^2} + 5\frac{z^6}{(z^2+2)^3}}{2\left(z^2 - \frac{1}{4} - \frac{z^2}{z^2+2} + \frac{5}{4}\frac{z^4}{(z^2+2)^2}\right)}.\end{aligned}$$

Ω_0^2 becomes negative for $z < z^*$, with $z^* \approx 0.62$. We therefore have to prescribe α_k and β_k on $(0, k(z^*))$ by arbitrary functions subject to the constraint $|\alpha_k|^2 - |\beta_k|^2 = 1$. The renormalised energy density in the adiabatic state according to the formula in the subsection treating the case $m^2 = 2H^2$ (with the same choice of renormalisation constants) is then given by

$$\begin{aligned}\omega(:\rho:) = \frac{H^4}{4\pi^2}\int_0^\infty \mathrm{d}z z^2 \Bigg(&(|\alpha_k^2| + |\beta_k|^2 - 1)\left(e^{-4H(t-t_0)} + \frac{3}{2}e^{-2H(t-t_0)}\right) \\ &+ 2\mathrm{Re}\left(\alpha_k\overline{\beta_k}e^{-2iz\exp(-H(t-t_0))}\left(\frac{3}{2z}e^{-2H(t-t_0)} - ie^{-3H(t-t_0)}\right)\right)\Bigg).\end{aligned}$$

For the sake of simplicity let us now choose $\beta_k = 0$ on $(0, k(z^*))$ according to the above remark. Introducing the functions

$$\begin{aligned}A(z) &\doteq \frac{1}{2\Omega_0 z}\left(z^2 - \Omega_0^2 - \frac{1}{4}\left(1 + \frac{(\dot{\Omega})_0}{H\Omega_0}\right)^2\right) \\ B(z) &\doteq \frac{1}{2\Omega_0 z}\left(1 + \frac{(\dot{\Omega})_0}{H\Omega_0}\right) \\ C(z) &\doteq \frac{1}{2\Omega_0 z}\left((z - \Omega_0)^2 + \frac{1}{4}\left(1 + \frac{(\dot{\Omega})_0}{H\Omega_0}\right)^2\right)\end{aligned}$$

and setting $\tilde{t} \doteq (t - t_0)$ we get

$$\begin{aligned}\omega(:\rho:) = \frac{H^4}{4\pi^2}\int_{z^*}^\infty \mathrm{d}z z^2 \Bigg(&C(z)\left(e^{-4H\tilde{t}} + \frac{3}{2}e^{-2H\tilde{t}}\right) \\ &+ \cos(2z(1 - \exp(-H\tilde{t})))\left(\frac{3A(z)}{2z}e^{-2H\tilde{t}} + zB(z)e^{-3H\tilde{t}}\right) \\ &+ \sin(2z(1 - \exp(-H\tilde{t})))\left(A(z)e^{-3H\tilde{t}} - \frac{3B(z)}{2}e^{-2H\tilde{t}}\right)\Bigg)\end{aligned}$$

Figure 3.9 shows the time dependence of $\omega(:\rho:)$ for this adiabatic state ($\omega(:\rho:)$ and t are again given in units of H^4 and H^{-1}, respectively.). A comparison with our results for SLE's (figure 3.2) shows that this state is energetically less excited and corresponds rather to an SLE with large smearing width. Note however that this calculation is restricted to de Sitter space. For arbitrary scale factors this correspondence might fail.

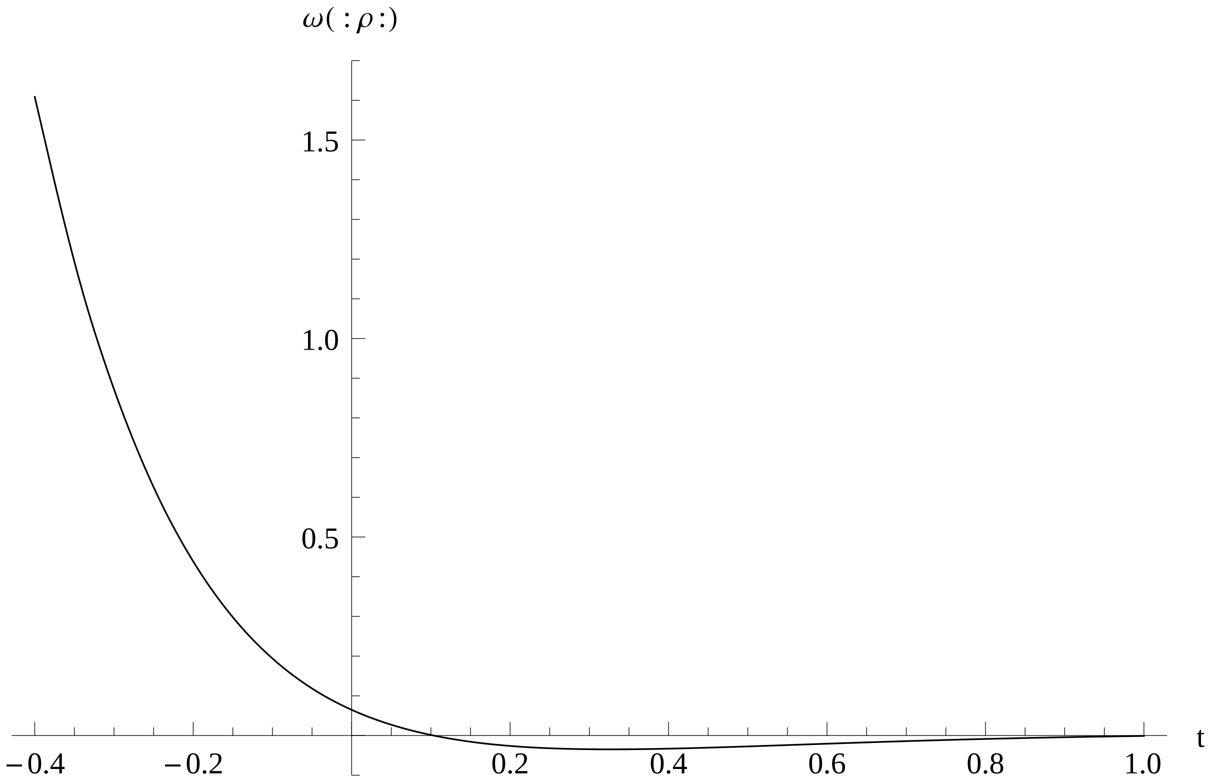

Figure 3.9.: Renormalised energy density for the Klein Gordon field on de Sitter space with $m^2 = 2H^2$ in an adiabatic state of order $n = 1$ and reference time $t_0 = 0$, plotted against cosmological time.

4. Relation of SLE's to Distinguished States on Asymptotic de Sitter Spacetimes

The explicit calculation of the renormalised energy density of SLE's on spatially flat de Sitter spacetimes provided some insight in the dependence of $\omega(:\rho:)$ on the choice of the test function f. Since we chose a Gaussian in order to obtain analytic expressions, we had two parameters at our disposal, namely the localisation time t_0 and the smearing width ϵ. However, it seems that these two numbers are enough to capture the most important features of $\omega(:\rho:)$. In particular, a small smearing width will lead to large local energy density fluctuations. Furthermore we conjecture that the particular form of f (for given smearing width) will affect the form of the energy density curve only *locally*, i.e. near the support of f. We saw that a shift of f (i.e. the choice of another localisation time t_0) results in a corresponding shift of $\omega(:\rho:)$. In other words, on a fixed time interval, the sequence of functions $\omega_{t_0}(:\rho(t):)$ (where ω_{t_0} denotes the SLE induced by a Gaussian with fixed smearing width ϵ and localisation time t_0) tends for $t_0 \to -\infty$ uniformly to 0, which is nothing but the expectation value of the energy density in the Bunch Davies state. A similar statement seems to hold true for the sequence $\omega_\epsilon(:\rho(t):)$ (where t_0 is now held fixed) if $\epsilon \to \infty$. One could furthermore ask if such observations are true for arbitrary observables of the free field theory. This would be the case if we could establish a convergence statement for the two point distribution, since from this object all observables can be calculated. The aim of this section is to establish such a convergence property for SLE's for the case that $t_0 \to -\infty$. We will see that this result holds in fact on a larger class of spacetimes, namely the asymptotic de Sitter spacetimes introduced in section 1.3. As we will explain later in the sequel, they allow for the definition of a *preferred state* $\lambda_\mathcal{M}$ and we will show that the sequence of SLE's will converge to $\lambda_\mathcal{M}$ also in this more general case. We will establish this result for minimal coupling but arbitrary choices of the mass[1] and the form of the compactly supported test function f. This relation of the SLE's on an asymptotic de Sitter space to the corresponding preferred state $\lambda_\mathcal{M}$ allows to interpret $\lambda_\mathcal{M}$ as being a state of low energy for every test function in the infinite past. This is analogous to the situation in Minkowski space, where the distinguished Minkowski vacuum state is a SLE for every test function.

4.1. Convergence of SLE's to Distinguished States: de Sitter Space

In this subsection we will prove the anticipated convergence statement for SLE's on de Sitter spacetime, where the distinguished limiting state $\lambda_\mathcal{M}$ is the well known Bunch Davies state. The more general case of asymptotic de Sitter space will be treated in section 4.2. The object which has to be analysed for this purpose is the two point distribution of the SLE ω_f induced by the arbitrary strictly positive test function $f \in \mathcal{D}(\mathbb{R})$. Without loss of generality, we assume that

[1] However we require $m > 0$ since the distinguished states do not exist for the massless theory.

$\mathrm{supp} f = [-L, L]$. Let us furthermore denote by f_{t_0} the corresponding shifted function, defined by $f_{t_0}(t) \doteq f(t - t_0)$. For a fixed choice of f, we denote the SLE induced by f_{t_0} by ω_{t_0} and its two point distribution by $\mathcal{W}_2^{t_0}$. We want to show that the sequence of two point distributions $\mathcal{W}_2^{t_0}$ converges to the two point distribution $\mathcal{W}_2^{\lambda_\mathcal{M}}$ of the Bunch-Davies state $\lambda_\mathcal{M}$ for $t_0 \to -\infty$ in the natural topology of bidistributions on $\mathcal{M}$. That is, for each pair $g, h \in \mathcal{D}(\mathcal{M})$ we aim to show that

$$\lim_{t_0 \to -\infty} \mathcal{W}_2^{t_0}(g, h) = \mathcal{W}_2^{\lambda_\mathcal{M}}(g, h). \tag{4.1}$$

The distinguished state $\lambda_\mathcal{M}$ for the Klein–Gordon field on spatially flat de Sitter space is defined as follows: Consider the full de Sitter spacetime $(\mathcal{M}_{dS}, g_{dS})$, defined in section 1.3. For the massive scalar field there exists a unique Hadamard state λ which is invariant under the full de Sitter group $O(1,4)$ see e.g. [SS76; All85]. $\lambda_\mathcal{M}$ is then nothing but the restriction of λ to the domain of definition of the cosmological chart. The explicit representation of $\lambda_\mathcal{M}$ in terms of the mode functions reads[2]

$$T_k(t) = \frac{\sqrt{\pi}}{2} e^{-3t/2} e^{i\pi\nu/2} \overline{H_\nu^{(2)}(ke^{-t})}, \tag{4.2}$$

where $H_\nu^{(2)}$ is the Hankel function of the second kind and can be expressed by a linear combination of the Bessel functions $J_{\pm\nu}$ [AS64]. The order ν is given by

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}},$$

if we assume minimal coupling, $\xi = 0$. Furthermore we take both $\mathrm{Re}\nu$ and $\mathrm{Im}\nu$ to be positive. We use the modes (4.2) as reference modes for the construction of ω_{t_0} (i.e. $T_k(t)$ plays the role of $S_k(t)$ in theorem 2.5.1, with which the Bogolubov coefficients $\lambda(k)$ and $\mu(k)$ are calculated). The two point distribution of ω_{t_0} then reads

$$\mathcal{W}_2^{t_0}(x, y) = \mathcal{W}_2^{\lambda_\mathcal{M}}(x, y) + \mathcal{R}^{t_0}(x, y), \tag{4.3}$$

where the bidistribution $\mathcal{R}^{t_0}$ is given by

$$\mathcal{R}^{t_0}(x, y) \doteq \frac{2}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k} \Big(\mu(k)^2 \mathrm{Re}\left(\overline{T}_k(t) T_k(t') \right) + \mu(k) \mathrm{Re}\left(\lambda(k) T_k(t) T_k(t') \right) \Big) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \tag{4.4}$$

$\lambda(k)$ and $\mu(k)$ are of course functionals of f_{t_0}, which we supressed in the notation. In order to arrive at (4.1), we have to show that $\mathcal{R}^{t_0}(x, y)$, applied to any pair of test functions in $\mathcal{D}(\mathcal{M})$, converges to zero as $t_0 \to -\infty$. We remind the reader that the representation of the distribution (4.4) as a mode integral is defined as follows. First the integration with the test functions $g, h \in \mathcal{D}(\mathcal{M})$ in the variables x and y is performed (note that in our standard coordinates we have $\mathrm{d}\mu_g = \mathrm{d}^4 x a(t)^3$) and then the k-integration. For the investigation of the limit $t_0 \to -\infty$ we will have to distinguish between the possible values of ν. In the following, we will use various properties of Hankel- and Besselfunctions, which can be found in [AS64].

[2] We occasionally set $H = 1$.

The Case $\nu \in i\mathbb{R} \setminus \{0\}$

As a warm–up, we treat first the case of imaginary ν. There holds

$$\begin{aligned}
\mathrm{Re}T_k(t) &= e^{-3t_0/2}\frac{\sqrt{\pi}e^{-\pi|\nu|/2}(1-e^{-\pi|\nu|})}{2\sinh\pi|\nu|}\mathrm{Re}J_\nu(ze^{-(t-t_0)})e^{-3(t-t_0)/2} \\
\mathrm{Im}T_k(t) &= -e^{-3t_0/2}\frac{\sqrt{\pi}e^{-\pi|\nu|/2}(1+e^{-\pi|\nu|})}{2\sinh\pi|\nu|}\mathrm{Im}J_\nu(ze^{-(t-t_0)})e^{-3(t-t_0)/2},
\end{aligned} \tag{4.5}$$

and we infer that, on account of the properties of Bessel functions, the mode functions $T_k(t)$ are bounded for $(k,t) \in (0,\infty) \times I$, I being an arbitrary compact interval. We estimate the two contributions of (4.4), applied to the test functions $g, h \in \mathcal{D}(\mathcal{M})$, separately. For the first one we obtain, after integration over the spatial variables,

$$\begin{aligned}
&|\int \mathrm{d}^3\mathbf{k}\mu(k)^2 \int \mathrm{d}t\mathrm{d}t' a(t)^3 a(t')^3 \mathrm{Re}\left(\overline{T}_k(t)T_k(t')\right)\hat{g}(t,\mathbf{k})\hat{h}(t',-\mathbf{k})| \\
&\leq \int_0^\infty \mathrm{d}k k^2\mu(k)^2 \int \mathrm{d}t\mathrm{d}t' a(t)^3 a(t')^3 |T_k(t)||T_k(t')| 4\pi \underbrace{|\max_{(\vartheta,\phi)\in S^2}|\hat{g}(t,\mathbf{k})|}_{=:\hat{g}(t,k)} \underbrace{|\max_{(\vartheta,\phi)\in S^2}|\hat{h}(t',-\mathbf{k})|}_{=:\hat{h}(t',k)}.
\end{aligned}$$

where the hat denotes Fourier transformation in position space and in the second line θ and ϕ are the angular variables of $\mathbf{k}$. Since $\hat{g}$ and $\hat{h}$ are of compact support in time, fall off faster than any power for $k \to \infty$ and are bounded at $k = 0$, we may estimate the integrations w.r.t. t and t' by a k-independent constant. The same arguments apply for the remaining contribution,

$$\int \mathrm{d}^3\mathbf{k}\mu(k) \int \mathrm{d}t\mathrm{d}t' a(t)^3 a(t')^3 \mathrm{Re}\left(\lambda(k)T_k(t)T_k(t')\right)\hat{g}(t,\mathbf{k})\hat{h}(t',-\mathbf{k})|,$$

so that we are left with the discussion of the integrals

$$M \doteq \int_0^\infty \mathrm{d}k k^2\mu^2 \qquad \text{and} \qquad N \doteq \int_0^\infty \mathrm{d}k k^2\mu|\lambda|. \tag{4.6}$$

Now λ and μ are given by thm. 2.5.1 for the shifted test function $f(t-t_0)$. The explicit form (4.2) of $T_k(t)$ allows to infer that the corresponding integrals c_1 and c_2 can each be written as a product of e^{-3t_0} and a function of the variable $z \doteq ke^{-t_0}$, thus μ and λ are functions of z alone, and a change of variables gives

$$M = e^{3t_0}\int_0^\infty \mathrm{d}z z^2\mu^2 \qquad \text{and} \qquad N = e^{3t_0}\int_0^\infty \mathrm{d}z z^2\mu|\lambda|. \tag{4.7}$$

We need now to show the existence of the z-integrals, from which the desired convergence of the SLE ω_{t_0} to $\lambda_{\mathcal{M}}$ for $t_0 \to -\infty$ then follows trivially due to the prefactor e^{3t_0}. We thus need some information about $\mu|\lambda|$ and μ^2 as functions of z. To this avail, it is useful to introduce the auxiliary function

$$u : z \mapsto \frac{c_1}{\sqrt{c_1^2 - |c_2|^2}}. \tag{4.8}$$

We then have $2\mu^2 = (u-1)$ and $2\mu|\lambda| = \sqrt{u^2-1}$. Using the power series representation for the Bessel function,

$$J_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^k}{k!\Gamma(\nu+k+1)}, \tag{4.9}$$

one can evaluate c_1 and c_2 near $z=0$ and finds that (4.8) remains bounded for $z \to 0$. Next we want to show that u is regular for each finite z. To this avail we rewrite

$$c_1^2 - |c_2|^2 = 4\Bigg(\int \mathrm{d}t f_{t_0} \left((\mathrm{Re}\dot{T})^2 + \omega^2 (\mathrm{Re}T)^2\right) \int \mathrm{d}t f_{t_0} \left((\mathrm{Im}\dot{T})^2 + \omega^2 (\mathrm{Im}T)^2\right) \\ - \left(\int \mathrm{d}t f_{t_0} \left(\mathrm{Re}\dot{T}\mathrm{Im}\dot{T} + \omega^2 \mathrm{Re}T\mathrm{Im}T\right)\right)^2 \Bigg),$$

where we suppressed the k and t dependence of $T_k(t)$. We introduce the notation

$$\int \mathrm{d}t f_{t_0} h(t) g(t) \doteq \langle h, g\rangle_{t_0}, \ \int \mathrm{d}t f_{t_0} h(t)^2 \doteq ||h||_{t_0}^2 \tag{4.10}$$

for real h, g and remark that it satisfies the properties of a nondegenerate inner product in the vector space of real measurable functions on $\mathrm{supp} f_{t_0} = [-L+t_0, L+t_0]$. We obtain

$$c_1^2 - |c_2|^2 = \\ 4\Bigg(||\mathrm{Re}\dot{T}||_{t_0}^2 ||\mathrm{Im}\dot{T}||_{t_0}^2 - \langle \mathrm{Re}\dot{T}, \mathrm{Im}\dot{T}\rangle_{t_0}^2 + ||\omega\mathrm{Re}T||_{t_0}^2 ||\omega\mathrm{Im}T||_{t_0}^2 - \langle \omega\mathrm{Re}T, \omega\mathrm{Im}T\rangle_{t_0}^2 \\ + ||\mathrm{Re}\dot{T}||_{t_0}^2 ||\omega\mathrm{Im}T||_{t_0}^2 + ||\mathrm{Im}\dot{T}||_{t_0}^2 ||\omega\mathrm{Re}\dot{T}||_{t_0}^2 - 2\langle \mathrm{Re}\dot{T}, \mathrm{Im}\dot{T}\rangle_{t_0} \langle \omega\mathrm{Re}T, \omega\mathrm{Im}T\rangle_{t_0} \Bigg)$$

Applying the Cauchy-Schwarz inequality twice we conclude that

$$c_1^2 - |c_2|^2 \geq m^4 \left(||\mathrm{Re}T||_{t_0}^2 ||\mathrm{Im}T||_{t_0}^2 - \langle \mathrm{Re}T, \mathrm{Im}T\rangle_{t_0}^2 \right). \tag{4.11}$$

In the definition of $\langle .,. \rangle_{t_0}$ the range of the integration variable t is the interval $[t_0 - L, t_0 + L] = \mathrm{supp} f_{t_0}$. Assume now that $z > 0$. Since $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent (pointwise in x on $(0,\infty)$) for $\nu \notin \mathbb{N}$ and $J_{-\nu} = \overline{J}_\nu$ for imaginary ν, $\mathrm{Re}T_k(t)$ and $\mathrm{Im}T_k(t)$ are also pointwise linearly independent for $t \in [t_0 - L, t_0 + L]$ and each fixed $z > 0$. That is, $\forall z > 0$ and $\forall \lambda \in \mathbb{R}\ \exists t_{(\lambda,z)} \in \mathrm{supp} f_{t_0}$ mit $\mathrm{Re}T_k(t_{(\lambda,z)}) \neq \lambda \mathrm{Im}T_k(t_{(\lambda,z)})$. Since $T_k(t)$ is continous in t, the last inequality holds also on some neighborhood of $t_{(\lambda,z)} \in \mathrm{supp} f_{t_0}$. Choosing $\lambda^*(z) = \int \mathrm{d}t f_{t_0} \mathrm{Re}T_k(t) \mathrm{Im}T_k(t) / \int \mathrm{d}t f_{t_0} (\mathrm{Im}T_k(t))^2$ this yields

$$0 < \epsilon(z) \leq \int \mathrm{d}t f_{t_0}(t) \left(\mathrm{Re}T_k(t) - \lambda^*(z)\mathrm{Im}T_k(t)\right)^2 \\ = ||\mathrm{Re}T||_{t_0}^2 ||\mathrm{Im}T||_{t_0}^2 - \langle \mathrm{Re}T, \mathrm{Im}T\rangle_{t_0}^2, \tag{4.12}$$

which proves that the r.h.s. of (4.11), multiplied by the factor e^{3t_0} is strictly positive for every finite z. Together with the finiteness of $e^{3t_0} c_1$ for finite z this implies regularity of the function u in the regime of finite z. The last step consists in discussing the decay properties of $\mu|\lambda|$ and μ^2 for $z \to \infty$. $H_\nu^{(2)}(x)$ posesses the representation [AS64]

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} (P(\nu,x) - iQ(\nu,x)) e^{-ix} e^{i(\nu/2+1/4)\pi} \tag{4.13}$$

where P and Q can be asymptotically expanded in inverse powers of x. Setting $\tilde{t} = t - t_0$, the integral c_2 assumes the form

$$\begin{aligned} c_2 &= e^{-3t_0} \int \mathrm{d}\tilde{t} f(\tilde{t}) e^{-3\tilde{t}} \sum_{j=0}^{\infty} \frac{a_j(\nu) e^{j\tilde{t}}}{z^j} e^{-2iz\exp(-\tilde{t})} \\ &= -e^{-3t_0} \left(\left(\int \mathrm{d}\tau f(\tilde{t}(\tau)) \tau^2 \sum_{j=0}^{n} \frac{a_j(\nu)}{(z\tau)^j} e^{-2iz\tau} + O\left(z^{-(n+1)}\right) \right) \right) \end{aligned} \tag{4.14}$$

The integral of the sum over j in the second line of (4.14) is decaying faster than any inverse power of z since it is the sum of Fourier transforms of a smooth functions of compact support in τ. The decay properties in z of the error term can be chosen to be good enough for our purposes via a suitable choice of the truncation n. Since we already established $|c_1| \neq |c_2|$ for finite z, we may write $u(z)$ as the following Taylor series:

$$\frac{c_1}{\sqrt{c_1^2 - |c_2|^2}} = \frac{1}{\sqrt{1 - |c_2|^2/c_1^2}} = 1 + \frac{|c_2|^2}{2c_1^2} + ...$$

It is easy to show that the leading asymptotics of c_1 for $z \to \infty$ is $\propto z$. Hence, it follows that μ^2 and $\mu|\lambda|$ decay sufficiently rapid such that the z-integrals in (4.7) exist and thus M and N converge to zero for $t_0 \to -\infty$.

The Case $0 \leq \nu < 3/2$

In the following calculations we will, without loss of generality, use the mode functions $T_k(t)$, given by (4.2), *without* the phase factor $e^{i\nu\pi/2}$. We thus have

$$\mathrm{Re}T_k(t) = \frac{\sqrt{\pi}}{2} e^{-\frac{3}{2}Ht} J_\nu(-kH^{-1}e^{-Ht}), \ \mathrm{Im}T_k(t) = \frac{\sqrt{\pi}}{2} e^{-\frac{3}{2}Ht} Y_\nu(-kH^{-1}e^{-Ht})$$

It turns out that both the functions $k \mapsto T_k(t)$ and $k \mapsto u(ke^{-t_0})$ diverge for $k \to 0$, which makes it necessary to estimate (4.4) more carefully. Namely, introducing the notation

$$I(\alpha) \doteq \int_{-L}^{L} \mathrm{d}t f(t) e^{\alpha Ht}$$

and using again the series representation of Bessel functions for small arguments, it follows for $\nu > 0$ and small z

$$\begin{aligned} c_1^2 - |c_2|^2 \geq & m^4 \left(||\mathrm{Re}T||_{t_0}^2 ||\mathrm{Im}T||_{t_0}^2 - \langle \mathrm{Re}T, \mathrm{Im}T \rangle_{t_0}^2 \right) \\ = & m^4 e^{-6Ht_0} \left(\frac{I(2\nu - 3)I(-2\nu - 3) - I(-3)^2}{16 \sin^2(\pi\nu)\Gamma(1+\nu)^2\Gamma(1-\nu)^2} + O(z^{2\nu}) \right) \end{aligned}$$

and

$$c_1 = e^{-3Ht_0} \left(\frac{(m^2 + (\nu - 3/2)^2)\Gamma(\nu)^2)2^{2\nu}\pi}{4\sin^2(\pi\nu)\Gamma(\nu+1)^2 z^{2\nu}} I(2\nu - 3) + O(1) \right)$$

so that

$$u(z) = O(z^{-2\nu}).$$

The values $\nu = 0$ and 1 require a special discussion, since then $J_{\pm\nu}$ are not linearly independent any more[3]. For instance, in the case $\nu = 0$ we have

$$u(z) = O\left((\ln z)^2\right).$$

For $\nu \in (0, 3/2)$ we may bound $T_k(t)$ as follows:

$$|T_k(t)| \leq D\frac{\sqrt{-\pi\tau}}{2a}\left((k\tau)^{-\nu} + \frac{1}{\sqrt{k\tau}}\right). \tag{4.15}$$

Thus, for $\nu < 1/2$, (4.15) combined with the above established small $z-$ behaviour of $u(z)$ are sufficient in order to show the existence of the z-integral for small z in $\mathcal{R}^{t_0}$. This can be proven in the same manner like for the case of imaginary ν, since the measure factor z^2 is sufficient to avoid "infrared problems". However, for $\nu \in (1/2, 3/2)$ we need a refined investigation of the integral kernel in (4.4). We have

$$\begin{aligned}
&\mu^2\mathrm{Re}\left(\overline{T}_k(t)T_k(t')\right) + \mu\mathrm{Re}\left(\lambda T_k(t)T_k(t')\right) = \\
&\frac{\pi e^{-\frac{3}{2}(t+t')}}{4}\Big(Y_\nu(w)Y_\nu(w')\left(\mu^2 + \mu|\lambda|\cos(\arg c_2)\right) + J_\nu(w)J_\nu(w')\left(\mu^2 - \mu|\lambda|\cos(\arg c_2)\right) \\
&-\mu|\lambda|\sin(\arg c_2)\left(J_\nu(w)Y_\nu(w') + J_\nu(w')Y_\nu(w)\right)\Big)
\end{aligned} \tag{4.16}$$

where we have set $w = ke^{-t} = ze^{-(t-t_0)}$ and $w' = ke^{-t} = ze^{-(t'-t_0)}$. The first summand on the r.h.s. of (4.16) is potentially the most singular one for $z \to 0$ In this regime of ν, the real part of c_2 dominates the imaginary part, as we will now show. Namely, we have for $\nu > 1/2$ and small z:

$$\begin{aligned}
\mathrm{Re}c_2 &= e^{-3t_0}\left(-\frac{(m^2 + (\nu - 3/2)^2)\pi 2^{2\nu}}{8\sin^2(\pi\nu)\Gamma(1-\nu)^2 z^{2\nu}}I(2\nu - 3) + O(1)\right) \\
\mathrm{Im}c_2 &= e^{-3t_0}\left(-\frac{(m^2 + (\nu - 3/2)^2)\pi}{4\sin(\pi\nu)\Gamma(1-\nu)\Gamma(1+\nu)}I(-3) + O(z^2)\right)
\end{aligned}$$

It follows

$$\begin{aligned}
\arg c_2 =& \pi + \arctan\left(\frac{z^{2\nu}I(-3)\sin(\pi\nu)\Gamma(1-\nu)}{I(2\nu-3)\Gamma(1+\nu)2^{2\nu-1}} + O(z^{2\nu+2})\right) \\
=& \pi + \frac{z^{2\nu}I(-3)\sin(\pi\nu)\Gamma(1-\nu)}{I(2\nu-3)\Gamma(1+\nu)2^{2\nu-1}} + O(z^{2\nu+2})
\end{aligned}$$

and thus

$$\cos(\arg c_2) = -1 + \frac{z^{4\nu}}{2}\left(\frac{I(-3)\sin(\pi\nu)\Gamma(1-\nu)}{I(2\nu-3)\Gamma(1+\nu)2^{2\nu-1}}\right)^2 + O(z^{4\nu+2})$$

We are now in the position to investigate the limit $t_0 \to -\infty$ of the distribution $\mathcal{R}^{t_0}$, applied to the testfunctions g and h. We will demonstrate this only for the most singular part of its integral

[3]This is merely a notational obstacle. We will leave out the explicit treatment of these cases, which works completely analogous.

kernel (4.16): Note that, since $u(z) \geq 1$, there holds $|\mu^2 - \mu|\lambda|| \leq 1$. Thus, $|\mu^2 + \mu|\lambda|\cos(\arg c_2)| = |\mu^2 - \mu|\lambda| + O(z^{2\nu})| \leq C$ for some $z_0, C > 0$ and $z \leq z_0$. Furthermore we know that there exist $w^*, D > 0$ such that $|Y_\nu(w)| \leq D w^{-\nu}$ for all $w < w^*$. In the regime $z < z_0$ we have $w = ke^{-t} = ze^{t_0}e^{-t} \leq z_0 e^{t_0} e^{-t}$, which can be chosen smaller than w^* for all $t \in \mathrm{supp}_t \hat{h}$ by requiring $t_0 < t_0^*$ (and similarly for w'). It follows

$$
\begin{aligned}
&\left| \int_0^\infty \mathrm{d}k k^2 (\mu^2 + \mu|\lambda| \cos(\arg c_2)) \int \mathrm{d}t\mathrm{d}t' a^{3/2}(t) a^{3/2}(t') Y_\nu(ke^{-t}) Y_\nu(ke^{-t'}) \hat{g}(k,t) \hat{h}(k,t') \right| \\
&= e^{3t_0} \left| \int_0^\infty \mathrm{d}z z^2 (\mu^2 + \mu|\lambda| \cos(\arg c_2)) \int \mathrm{d}t\mathrm{d}t' a^{3/2}(t) a^{3/2}(t') Y_\nu(ze^{-(t-t_0)}) Y_\nu(ze^{-(t'-t_0)}) \hat{g}\hat{h} \right| \\
&\leq ECD^2 e^{-(2\nu-3)t_0} \int_0^{z_0} \mathrm{d}z z^{-2\nu+2} + e^{3t_0} \left| \int_{z_0}^\infty \mathrm{d}z ... \right|
\end{aligned}
$$

The constant E stems from estimating the integration over t and t' . The first integral in the above estimate clearly exists and converges to zero for $t_0 \to -\infty$ if $\nu < 3/2$. In order to prove existence and convergence of the second summand we use the estimate $|Y_\nu(w)| \leq Aw^{-1/2} + Bw^{-\nu}$ for $w \in (0, \infty)$, $|\mu^2 + \mu|\lambda|\cos(\arg c_2)| \leq \mu^2 + \mu|\lambda|$ and the fact that μ^2 and $\mu|\lambda|$ decay faster than any inverse power of z (see (4.14) and the subsequent discussion, which is valid also for $\nu \in (0, 3/2)$). Similar arguments can be repeated for the remaining (even less singular) parts of the kernel (4.16) and for the cases $\nu \in \{0, 1\}$. This shows that the distribution $\mathcal{R}^{t_0}$, defined by (4.4), converges to zero for $t_0 \to -\infty$ and we have thus proven the following result:

Theorem 4.1.1. *Let f be an arbitrary real positive test function of compact support on the real line and let ω_{t_0} denote the SLE of the minimally coupled Klein-Gordon field of mass $m^2 > 0$, induced by the shifted function $f_{t_0} = f(t - t_0)$, on the part of de Sitter spacetime which is isometric to a FRW universe $(\mathcal{M}, g)$ with flat spatial sections. Let futhermore $\lambda_\mathcal{M}$ be the restriction of the $O(1,4)$-invariant ground state λ on full de Sitter spacetime to $(\mathcal{M}, g)$. Then*

$$
\lim_{t_0 \to -\infty} \mathcal{W}_2^{t_0} \to \mathcal{W}_2^{\lambda_\mathcal{M}} \tag{4.17}
$$

in the natural topology of $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$.

4.2. Convergence of SLE's to Distinguished States: Asymptotic de Sitter Space

We want now to extend our convergence result to a more general class of FRW spacetimes, namely the asymptotic de Sitter spacetimes, introduced in subsection 1.3. The asymptotic de Sitter spacetimes belong to the class of spacetimes with a cosmological past horizon. Under certain conditions, such spacetimes allow for the definition of a distinguished quantum state $\lambda_\mathcal{M}$ for a scalar field theory [DMP09a; DMP09b]. We will briefly sketch how $\lambda_\mathcal{M}$ arises out of the presence of such an horizon $\mathcal{J}^-$: The scalar QFT defined on our expanding universe $(\mathcal{M}, g)$ – also called the bulk – can be related to a QFT defined on $\mathcal{J}^-$ – the boundary. The latter is obtained as follows: The special geometric structure of $\mathcal{J}^-$ (see definition 1.3.1) allows for the definition of a symplectic space $(\mathcal{S}(\mathcal{J}^-), \sigma_{\mathcal{J}^-})$ which is invariant under the group of diffeomorphisms $SG_{\mathcal{J}^-}$ of $\mathcal{J}^-$. One can now construct the corresponding canonical CCR-Weyl algebra $\mathcal{W}(\mathcal{J}^-)$, giving rise to a full quantum field theory in its own right. $SG_{\mathcal{J}^-}$ has a representation on $\mathcal{W}(\mathcal{J}^-)$ via $*-$ automorphisms and there exists a unique pure quasifree state λ on $\mathcal{W}(\mathcal{J}^-)$ which is invariant under these $*-$automorphisms. This construction as it stands is of course physically meaningless if it were not possible to relate it to the QFT in the bulk, characterised by the Weyl algebra $\mathcal{W}(\mathcal{M})$ belonging to the symplectic space $(\mathcal{S}(\mathcal{M}), \sigma_{\mathcal{M}})$ of solutions of the Klein Gordon equation with compactly supported Cauchy data. However, under certain conditions it is possible to identify a subspace of $\mathcal{S}(\mathcal{J}^-)$ with $\mathcal{S}(\mathcal{M})$ by means of an injective symplectomorphism $\mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathcal{J}^-)$. Namely, since $\mathcal{M}$ can be extended to a bigger spacetime $\hat{\mathcal{M}}$, for every solution ϕ of the Klein Gordon equation with compactly supported Cauchy data in $\mathcal{M}$ there exists a solution ϕ' of the Klein Gordon equation in $\hat{\mathcal{M}}$ with the same Cauchy data, giving rise to the linear map

$$\Gamma\phi \doteq \phi' \restriction_{\mathcal{J}^-} . \tag{4.18}$$

In [DMP09a] it was proven that, for the case of an asymptotic de Sitter spacetime $(\mathcal{M}, g)$, there holds $\Gamma\phi \in \mathcal{S}(\mathcal{J}^-)$ and $\sigma_{\mathcal{J}^-}(\Gamma\phi, \Gamma\psi) = \sigma_{\mathcal{M}}(\phi, \psi)$ (i.e. Γ provides an injective symplectomorphism), under the following conditions[4]: With the definition

$$\nu \doteq \sqrt{\frac{9}{4} - \left(\frac{m^2}{H^2} + 12\xi\right)}, \tag{4.19}$$

there should hold either

$$a = -\frac{1}{H\tau} + O(\tau^{-2}),\ \frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{1}{H\tau^2} + O(\tau^{-3}),\ \frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} = -\frac{2}{H\tau^3} + O(\tau^{-4}) \tag{4.20}$$

if $\mathrm{Re}\nu < 1/2$ or

$$a = -\frac{1}{H\tau} + O(\tau^{-4}),\ \frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{1}{H\tau^2} + O(\tau^{-3}),\ \frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} = -\frac{2}{H\tau^3} + O(\tau^{-6}) \tag{4.21}$$

if $\mathrm{Re}\nu \in (1/2, 3/2)$. These conditions[5] are necessary in order to perturbatively construct and control the behaviour of the mode functions associated to the wave equation with parameters m and ξ (see below). Note that in the following we will work again with minimal coupling, i.e. we will set $\xi = 0$. If such an injective symplectomorphism $(\mathcal{S}(\mathcal{M}), \sigma_{\mathcal{M}}) \to (\mathcal{S}(\mathcal{J}^-), \sigma_{\mathcal{J}^-})$ exists, it follows that the bulk algebra $\mathcal{W}(\mathcal{M})$ is a subalgebra of $\mathcal{W}(\mathcal{J}^-)$. This in turn defines the distinguished state $\lambda_{\mathcal{M}}$ on $\mathcal{W}(\mathcal{M})$ by[6]

$$\lambda_{\mathcal{M}}(W_{\mathcal{M}}(\phi)) \doteq \lambda(W_{\mathcal{J}^-}(\Gamma\phi)) \quad \forall \phi \in \mathcal{S}(\mathcal{M}).$$

While this construction of $\lambda_{\mathcal{M}}$ works so far for the general class (i.e. not necessarily FRW) of spacetimes obeying definition 1.3.1 (provided that the above injective symplectomorphism exists), it is not clear if in general $\lambda_{\mathcal{M}}$ will be Hadamard. However, as shown in [DMP09b], this is true for the class of asymptotic de Sitter spacetimes. The authors characterised the

[4]Which are obviosly stricter than 1.16

[5]They differ here from the expressions in [DMP09b]. We suppose that the authors made an error there, since from their form one does not get the claimed asymptotic behaviour of the perturbation potential (4.23).

[6]For $\phi \in S(\mathcal{M})$, $W_{\mathcal{M}}(\phi) \in \mathcal{W}(\mathcal{M})$ are the generators of the Weyl algebra $\mathcal{W}(\mathcal{M})$; they correspond formally to the exponentiated fields $e^{i\Phi(f)}$ with $\Phi(f) \in \mathcal{A}(\mathcal{M}, g)$ and $Ef = \phi$.

distinguished state explicitly in terms of the mode functions. Starting from the mode equation (2.13) and introducing $\rho_k(\tau) = a(t(\tau))S_k(t(\tau))$, we obtain the differential equation[7]

$$\rho_k''(\tau) + \underbrace{\left(k^2 + (H\tau)^{-2}(m^2 - 2H^2) + V(\tau)\right)}_{\doteq\Omega^2(k,\eta)} \rho_k(\tau) = 0. \tag{4.22}$$

for $\rho_k(\tau)$, where $V(\tau)$ is a perturbation potential coming from deviation of the scale factor from exact de Sitter form. Splitting the scale factor according to $a = a_{dS} + a_p$, where $a_{dS} \doteq -(H\tau)^{-1}$, V is given by

$$V = \left(a_p^2 - 2a_p(H\tau)^{-1}\right) m^2 + \frac{2\tau^{-2}a_p - a_p''}{a_p - (H\tau^{-1})}, \tag{4.23}$$

which has for $\tau \to -\infty$ the asymptotic behaviour $O(\tau^{-3})$ or $O(\tau^{-5})$, when imposing assumption (4.20) or (4.21), respectively. The analogon of constraint (2.14) is

$$\overline{\rho'}_k \rho_k - \overline{\rho}_k \rho_k' = i. \tag{4.24}$$

For the exact de Sitter case (i.e. $V = 0$), we denote the special set of mode functions which solve (4.22) and characterise the Bunch Davies state again by $\chi_k(\tau)$. They are given by

$$\chi_k(\tau) = \frac{\sqrt{-\tau\pi}}{2} e^{\frac{i\pi\nu}{2}} \overline{H_\nu^{(2)}}(-k\tau). \tag{4.25}$$

These modes are now used for a perturbative construction of the modes $\rho_k(\tau)$:

$$\begin{aligned} \rho_k(\tau) &= \chi_k(\tau) \\ &+ \sum_{n=1}^{\infty} (-1)^n \int_{-\infty}^{\tau} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 ... \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n S_k(\tau, t_1) S_k(t_1, t_2) ... S_k(t_{n-1}, t_n) \times \\ &\qquad V(t_1)V(t_2)...V(t_n)\chi_k(t_n), \end{aligned} \tag{4.26}$$

where $S_k(t_1, t_2) \doteq -i\left(\overline{\chi}_k(t_1)\chi_k(t_2) - \overline{\chi}_k(t_2)\chi_k(t_1)\right)$ is the unique retarded fundamental solution of the unperturbed problem. The convergence of the series can be proven if the perturbation potential V has the appropriate asymptotic behaviour, which in turn follows from the asymptotic behaviour of a_p. For $\mathrm{Re}\nu < 1/2$ one has to require (4.20), whereas for $\nu \in (1/2, 3/2)$ it is necessary to have (4.21). The so constructed modes behave then asymptotically like the Bunch Davies modes for $\tau \to -\infty$ by construction. It is shown in [DMP09a; DMP09b] that they deliver the distinguished state $\lambda_{\mathcal{M}}$.

In order to extend our convergence result of theorem 4.1.1 for this more general class of spacetimes, we need some technical preparation. We start with the translation of the asymptotic requirement

$$a(t(\tau)) = -\frac{1}{H\tau} + O(\tau^{-2}), \quad \tau \to -\infty$$

into cosmological time, which then reads

$$a(t) = e^{Ht} + O(e^{2Ht}), \quad t \to -\infty.$$

[7]We work from now on again with minimal coupling, $\xi = 0$.

From this we can deduce

$$a^{-n}(t) = e^{-nHt} + O(e^{(-n+1)Ht}), \quad t \to -\infty$$

and

$$\tau(t) = -\frac{1}{H}e^{-Ht} + O(t), \quad t \to -\infty.$$

These relations which will be needed in the following where we often switch between cosmological and conformal time. We now collect some properties of the modes $\rho_k(\tau)$. We begin with a lemma proven in [DMP09a] which establishes the convergence of ansatz (4.26) for the case $\mathrm{Re}\nu < 1/2$:

Lemma 4.2.1. *Let $\rho_k(\tau)$ be the exact solution of (4.22), obtained by the perturbation series (4.26) w.r.t. the modes χ_k, given by (4.25), where $\mathrm{Re}\nu < 1/2$ and $V(\tau) = O(\tau^{-3})$ (which is the case if (4.20) holds). Then for their difference $R_k(\tau) \doteq \rho_k(\tau) - \chi_k(\tau)$ there holds for some $T < -1$ and $\tau \in (-\infty, T)$:*

1. $|R_k(\tau)| \leq \left(k^{Re\nu} + k^{-Re\nu}\right) C_\nu S_{\nu,T}$

2. $\left|\frac{\partial R_k(\tau)}{\partial \tau}\right| \leq 2\left(k^{Re\nu} + k^{-Re\nu}\right)(1+k) C_\nu S_{\nu,T}$

where

$$S_{\nu,T} = (-T)^{1/2-Re\nu}\left(\exp\left(C(-T)^{2Re\nu-1}\right) - 1\right) \tag{4.27}$$

and C, C_ν are fixed positive constants.

While the content of the preceding lemma will suffice for proving convergence of the SLE's to $\lambda_{\mathcal{M}}$ for imaginary ν, this is not the case for $\nu \in (0, 3/2)$. More precisely, for $\nu \in (0, 1/2)$ we shall require that $V = O(\tau^{-4})$ and for $\nu \in (1/2, 3/2)$ we will need $V = O(\tau^{-7})$. The reason for this stricter assumption is that we need to control the small k behaviour of $\mathrm{Im}\rho_k$ and $\mathrm{Re}\rho_k$ seperately. The next lemma will treat the case $\nu \in (0, 1/2)$:

Lemma 4.2.2. *Let $\nu \in (0, 1/2)$ and $V = O(\tau^{-4})$. Let furthermore $\chi_k(\tau)$ be given by (4.25),* ***without*** *the phase[8] $e^{i\pi\nu/2}$. Then for $R_k(\tau) \doteq \rho_k(\tau) - \chi_k(\tau)$ there holds for some $T < -1$, $\tau \in (-\infty, T)$ and $k < 1$:*

1. $|\mathrm{Re}R_k(\tau)| \leq k^\nu C_\nu U_{\nu,T}$

2. $|\mathrm{Im}R_k(\tau)| \leq k^{-\nu} C'_\nu U_{\nu,T}$,

where C_ν, C'_ν are positive constants depending on ν and

$$U_{\nu,T} = |T|^{\nu+1/2} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{C|T|^{2\nu-2}}{2-2\nu}\right)^n$$

for some positive constant C. For $\nu < 1/2$ there clearly holds $U_{\nu,T} \to 0$ as $|T| \to \infty$.

[8]Note that a multiplication of the modes with a phase does not change the corresponding state.

Proof. With the changed definition of χ_k we have for $\nu \in (0, 1/2)$

$$\mathrm{Re}\chi_k = \frac{\sqrt{-\tau\pi}}{2} J_\nu(-k\tau), \ \mathrm{Im}\chi_k = \frac{\sqrt{-\tau\pi}}{2} Y_\nu(-k\tau).$$

Since the retarded fundamental solution S_k used to construct the perturbation series (4.26) is real it follows that $\mathrm{Re}\rho_k$ is obtained by inserting $\mathrm{Re}\chi_k$ in (4.26) and the same is true for the imaginary part. From standard properties of Bessel functions we obtain the following estimates if $|\tau| > 1$ and $k < 1$:

$$|\mathrm{Re}\chi_k| \leq C_\nu(-\tau)^{1/2+\nu}k^\nu, \ |\mathrm{Im}\chi_k| \leq C'_\nu(-\tau)^{1/2+\nu}k^{-\nu}.$$

Using the estimate $|S_k(t_1, t_2)| \leq D|t_1 t_2|^{\nu+1/2}$ in [DMP09a] and the asymptotic behaviour $O(\tau^{-4})$ of the perturbation potential V, we can do a similar estimate like in [DMP09a] of the perturbation series appearing in (4.26), but this time seperately for $\mathrm{Im}\rho_k$ and $\mathrm{Re}\rho_k$. This yields the expression for $U_{\nu,T}$ in the claim of the lemma. □

The next lemma treats the behaviour of $\rho_k(\tau)$ for small k and the case $\nu \in (1/2, 3/2)$:

Lemma 4.2.3. *Let $\nu \in (1/2, 3/2)$ and $V(\tau) = O(\tau^{-7})$. Let $\rho_k(\tau)$ be the exact solution of (4.22), given by the series (4.26), where $\chi_k(\tau)$ is given by (4.25), again without the phase $e^{i\pi\nu/2}$. Suppose furthermore that $|\tau| > 1$ and $k < 1$ and define $\alpha \doteq \sup(\nu, 2 - \nu)$. Then there holds:*

1. $|\mathrm{Re}R_k(\tau)| \leq k^\nu C_\nu W_{\nu,\tau}$

2. $|\mathrm{Im}R_k(\tau)| \leq k^{-\nu}\tilde{C}_\nu Z_{\nu,\tau}$

with C_ν, C'_ν as in the preceding lemma and

$$W_{\nu,\tau} = |\tau|^{\nu+1/2} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{C_1|\tau|^{2\alpha-5}}{5 - 2\alpha + \nu + 1/2} \right)^n$$
$$Z_{\nu,\tau} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{C_2|\tau|^{2\alpha-5}}{5 - 2\alpha} \right)^n.$$

where C_1, C_2 are positive constants and again $W_{\nu,\tau}, W_{\nu,\tau} \to 0$ for $|\tau| \to \infty$.

Proof. This time we used the estimate $|S_k(t_1, t_2)| \leq E|t_1|^{2\alpha+1}$ for some constant E and $|t_1| > |t_2| > 1$ and $k < 1$ (see [DMP09b]), together with the improved τ uniform estimate $|\mathrm{Im}\chi_k(\tau)| \leq \tilde{C}_\nu k^{-\nu}$ for $k < 1$. Using this, similar estimates like in the proof of the preceding lemma yield then the expression for $W_{\nu,\tau}$ and $Z_{\nu,\tau}$. □

We need furthermore a statement for a certain asymptotic behaviour of $\rho_k(\tau)$:

Lemma 4.2.4. *Consider the perturbed differential equation (4.22) for arbitrary $m > 0$, where the asymptotic behaviour of $V(\tau)$ is at least $O(\tau^{-3})$. Assume in addition that $V'(\tau) = O(\tau^{-3})$.*

Then, the special solution $\rho_k(\tau)$ and its first derivative (defined by (4.26)) can be represented as follows:

$$\rho_k(\tau) = \frac{e^{-i\pi/4}}{\sqrt{2k}} e^{-ik\tau} \left(1 + i \left(\frac{m^2 - 2H^2}{2k\tau} - \frac{1}{2k}\int_{-\infty}^{\tau} V(\tau')\mathrm{d}\tau'\right) + R_1(k,\tau)\right) \tag{4.28}$$

$$\rho_k'(\tau) = \frac{e^{-i\pi/4}}{\sqrt{2}} e^{-ik\tau}\sqrt{k}\left(-i + \frac{n}{2k\tau} - \frac{1}{2k}\int_{-\infty}^{\tau} V + R_2(k,\tau)\right), \tag{4.29}$$

where $|R_i(k,\tau)| = O(x^{-2})$ for $x \to \infty$, with $x \doteq k|\tau|$, provided that $k > 0$ and $|\tau|$ is bigger than some appropriate constant.

Proof. Theorem 2.2 in [Olv74] tells us that a special solution of (4.22) can be written as

$$\rho_k(\tau) = \frac{e^{-i\pi/4} e^{i\phi(k,a)}}{\sqrt{2\Omega(k,\tau)}} e^{-i\int_a^\tau \mathrm{d}\tau' \Omega(k,\tau')} \left(1 + \delta(k,\tau)\right). \tag{4.30}$$

$\phi(k,a)$ is a phase factor (defined below) which serves to match $\rho_k(\tau)$ to $\chi_k(\tau)$ for $\tau \to -\infty$. For the error term there holds

$$|\delta(k,\eta)| \le e^{\mathcal{V}_{-\infty,\tau}(F)} - 1, \quad |\delta'(k,\tau)| \le \Omega(k,\tau)\left(e^{\mathcal{V}_{-\infty,\tau}(F)} - 1\right), \tag{4.31}$$

where $\mathcal{V}_{-\infty,\tau}(F)$ is the variation of the error control function

$$F(\tau) = \Omega(k,\tau)^{-1/2} \partial_\tau^2 \Omega(k,\tau)^{-1/2}$$

and can be written as[9]

$$\mathcal{V}_{-\infty,\tau}(F) = \int_{-\infty}^{\tau} \mathrm{d}\tau' |\Omega^{-1/2}(k,\tau')\partial_{\tau'}^2 \Omega(k,\tau')^{-1/2}|$$

We have

$$\Omega^{-1/2}(k,\tau')\partial_{\tau'}^2\Omega(k,\tau')^{-1/2} = \frac{5}{16}\frac{\left(\frac{-2(m^2-2/H^2)}{H^3\tau^3} + V'(\tau)\right)^2}{\Omega^5} - \frac{1}{4}\frac{\frac{6(m^2-2/H^2)}{H^2\tau^4} + V''(\tau)}{\Omega^3}$$

With the additional assumption on $V'(\tau)$ we obtain $\mathcal{V}_{-\infty,\tau} = k^{-3}O(\tau^{-3})$. That is, $\exists \tau_0 > 0, C$ such that $\mathcal{V} \le Ck^{-3}|\tau|^{-3}$ if $|\tau| \ge \tau_0$. For $\mathcal{V} \to 0$ there holds

$$e^{\mathcal{V}} - 1 = O(\mathcal{V}) \tag{4.32}$$

Thus, $\exists D(\mathcal{V}_0), \mathcal{V}_0$ such that $e^{\mathcal{V}} - 1 \le D(\mathcal{V}_0)\mathcal{V}$ for $\mathcal{V} \le \mathcal{V}_0$. Put now the following condition on $k|\tau|$, by requiring $\mathcal{V}_0 \ge Ck^{-3}|\tau|^{-3} \Leftrightarrow k|\tau| \ge \sqrt[3]{C/\mathcal{V}_0}$. Thus, for $|\tau| \ge |\tau_0|$ and $x \doteq k|\tau| \ge \sqrt[3]{C/\mathcal{V}_0}$ we have $e^{\mathcal{V}} - 1 \le DCx^{-3}$. Next we investigate the term

$$\Omega^{-1/2} = k^{-1/2}(1+y)^{-1/4}, \qquad y(k,\tau) \doteq \frac{m^2 - 2H^2}{H^2\tau^2k^2} + \frac{V(\tau)}{k^2} = k^{-2}O(\tau^{-2}).$$

[9] We refer the reader to [Olv74] for the proofs.

For $k > 0$ we obtain

$$\Omega^{-1/2} = \frac{1}{\sqrt{k}} \left(1 - \frac{n}{4k^2(H\tau)^2} - \frac{V(\tau)}{4k^2} + g(y) \right), \qquad g(y) = O(y^2).$$

Using the same argument as before we have

$$|g(y)| \leq Ex^{-4}$$

for some constant E and the condition that both $|\tau|$ and $x = k|\tau|$ are sufficiently big. The last task is to investigate the integral

$$\int_a^\tau \Omega(k, \tau')\mathrm{d}\tau'.$$

Again, we may write

$$\Omega(k, \tau') = k \left(1 + \frac{m^2 - 2H^2}{2H^2\tau'^2k^2} + \frac{V(\tau')}{2k^2} + h(y(k, \tau')) \right).$$

with

$$h(y) = O(y^2) \text{ as } y \to 0.$$

Thus,

$$\begin{aligned}\int_a^\tau \mathrm{d}\tau'\Omega(k, \tau') =& k(\tau - a) - \frac{m^2 - 2H^2}{2H^2k} \left(\tau^{-1} - a^{-1}\right) \\ &+ \frac{1}{2k} \left(\int_{-\infty}^\tau V(\tau')\mathrm{d}\tau' - \int_{-\infty}^a V(\tau')\mathrm{d}\tau' \right) \\ &+ k \left(\int_{-\infty}^\tau h(y(k, \tau'))\mathrm{d}\tau' - \int_{-\infty}^a h(y(k, \tau'))\mathrm{d}\tau' \right).\end{aligned}$$

Repeating the above reasoning, we have

$$|k \int_{-\infty}^\tau h(y(k, \tau'))\mathrm{d}\tau'| \leq E(k|\tau|)^{-3}$$

for $|\tau|, k|\tau|$ big enough. Define now

$$\phi(k, a) \doteq -ka + \frac{m^2 - 2H^2}{2H^2ka} - \frac{1}{2k} \int_{-\infty}^a V(\tau')\mathrm{d}\tau' - \frac{1}{2k} \int_{-\infty}^a h(y(k, \tau'))\mathrm{d}\tau'.$$

Then it follows

$$\begin{aligned}e^{i\phi(k,a)} e^{-i\int_a^\tau \Omega(k,\tau')\mathrm{d}\tau'} =& e^{-ik\tau} e^{i\left(\frac{m^2-2H^2}{2H^2k\tau} - \frac{1}{2k}\int_{-\infty}^\tau V(\tau')\mathrm{d}\tau' - \int_{-\infty}^\tau h(y(k,\tau'))\mathrm{d}\tau' \right)} \\ =& e^{-ik\tau} \left(1 + i \left(\frac{m^2 - 2H^2}{2H^2k\tau} - \frac{1}{2k} \int_{-\infty}^\tau V(\tau')\mathrm{d}\tau' \right) + u(k, \tau) \right),\end{aligned}$$

where $|u(k, \tau)| \leq F(k|\tau|)^{-2}$ for some constant F if $|\tau|$ and $|k\tau|$ are sufficiently big. Inserting all the asymptotic expansions we made so far in formula (4.30) as well as in its first derivative (where the estimate for $\delta'(k, \tau)$ in (4.31) has to be used in addition) yields the claim of our lemma. □

In the following we will analyse the distribution $\mathcal{R}^{t_0}$, representing the difference between the two point distribution of the SLE ω_{t_0} and the distinguished state $\lambda_{\mathcal{M}}$ of our asymptotic de Sitter spacetime. It is defined by (4.4), where $T_k(t)$ are now of course the modes describing the state $\lambda_{\mathcal{M}}$. The Bogolubov coefficients λ and μ describing the SLE ω_{t_0} are now functions of $z = kH^{-1}e^{-Ht_0}$ *and* t_0; z will again be used as integration variable. We now proceed like in the case of de Sitter spacetimes: In order to show that $\lim_{t_0 \to -\infty} \mathcal{R}^{t_0}(g,h) = 0$ for arbitrary $g, h \in \mathcal{D}(\mathcal{M})$, we transform it in a $z-$ integral and show its existence by investigating the integrand for $z \to 0$, finite z and $z \to \infty$. Recall that we imposed additional[10] restrictions on the geometry of the asymptotic de Sitter spacetime (by requiring V to have appropriate asymptotic behaviour in τ), on which our convergence investigations hold. Namely, for $\nu \in (0, 1/2)$ we require $V = O(\tau^{-4})$ and for $\nu \in (1/2, 3/2)$ there should hold $V = O(\tau^{-7})$, whereas for imaginary ν it is sufficient to have $V = O(\tau^{-3})$.

Lemma 4.2.5. *Let $t_0 < t^*$, with t^* sufficiently small. If ν is purely imaginary, then the functions $\mu^2(z,t_0)$ and $|\lambda(z,t_0)|\mu(z,t_0)$ are bounded for $z \to 0$ and have no singularities for finite values of z. If $\nu \in (0, 1/2)$, then the functions $\mu^2(z,t_0)$ and $|\lambda(z,t_0)|\mu(z,t_0)$ are $O(z^{-2\nu})$ for $z \to 0$ and have no singularities for finite z either.*

Proof. We start with the case of imaginary ν. Writing $\rho_k = \chi_k + R_k$ and modifying for the moment the notation $\langle .,. \rangle_{t_0}$ of the previous subsection by setting

$$\langle u, w \rangle_{t_0} \doteq \int \mathrm{d}t f_{t_0} a^{-2} uw$$

we have

$$\begin{aligned}
&c_1^2 - |c_2|^2 \\
&\geq m^4 \left(\int \mathrm{d}t f_{t_0} a^{-2} (\mathrm{Re}\rho_k)^2 \int \mathrm{d}t f a^{-2} (\mathrm{Im}\rho_k)^2 - \left(\int \mathrm{d}t f_{t_0} a^{-2} \mathrm{Re}\rho_k \mathrm{Im}\rho_k \right)^2 \right) \\
&= m^4 \left(||\mathrm{Re}\chi_k||_{t_0}^2 ||\mathrm{Im}\chi_k||_{t_0}^2 - \langle \mathrm{Re}\chi_k, \mathrm{Im}\chi_k \rangle_{t_0}^2 \right) + m^4 W,
\end{aligned} \tag{4.33}$$

where W is some linear combination of the expressions

$$\begin{aligned}
&\langle \mathrm{Re}\chi_k, \mathrm{Re}R_k \rangle_{t_0} \langle \mathrm{Im}\chi_k, \mathrm{Im}R_k \rangle_{t_0}, \ \langle \mathrm{Re}R_k, \mathrm{Re}R_k \rangle_{t_0} \langle \mathrm{Im}\chi_k, \mathrm{Im}R_k \rangle_{t_0}, \\
&\langle \mathrm{Re}\chi_k, \mathrm{Im}R_k \rangle_{t_0} \langle \mathrm{Im}\chi_k, \mathrm{Re}R_k \rangle_{t_0}, ...
\end{aligned} \tag{4.34}$$

We calculate

$$\begin{aligned}
&m^4 \left(||\mathrm{Re}\chi_k||_{t_0}^2 ||\mathrm{Im}\chi_k||_{t_0}^2 - \langle \mathrm{Re}\chi_k, \mathrm{Im}\chi_k \rangle_{t_0}^2 \right) \\
&= \frac{\pi^2 m^4}{16 \sinh^2(|\nu|\pi)} \left(\left(\int \mathrm{d}t f_{t_0}(t) \frac{|\tau(t)|}{a^2(t)} |J_\nu(-k\tau(t))|^2 \right)^2 - \left| \int \mathrm{d}t f_{t_0}(t) \frac{|\tau(t)|}{a^2(t)} \left(J_\nu(-k\tau(t)) \right)^2 \right|^2 \right)
\end{aligned}$$

We use the convergent power series representation for $J_\nu(x) J_{-\nu}(x)$ ([AS64]) and obtain

$$|J_\nu(x)|^2 = J_\nu(x) J_{-\nu}(x) = \frac{\sinh(\pi|\nu|)}{|\nu|\pi} + O(x^2),$$

[10] Additional to the restrictions of [DMP09a; DMP09b] which were necessary to prove the existence of $\lambda_{\mathcal{M}}$.

with

$$x = -k\tau = z\left(e^{-H\tilde{t}} + e^{Ht_0}O(\tilde{t}+t_0)\right), \quad \tilde{t} \doteq t - t_0.$$

Thus,

$$\int \mathrm{d}t f_{t_0} \frac{|\tau(t)|}{a^2(t)} |J_\nu(-k\tau(t))|^2 =$$

$$\frac{e^{-3Ht_0}\sinh(\pi|\nu|)}{\pi|\nu|}\left(\int \mathrm{d}\tilde{t} f(\tilde{t})\frac{e^{-3H\tilde{t}}}{H} + O(e^{Ht_0}t_0)\right) + e^{-3Ht_0}O(z^2)\left(\int \mathrm{d}\tilde{t} f(\tilde{t})\frac{e^{-5H\tilde{t}}}{H^3} + O(e^{Ht_0}t_0)\right)$$

for $ke^{-Ht_0} = z \to 0$ and $t_0 \to -\infty$. Now we use the power series for $J_\nu^2(x)$:

$$J_\nu^2(x) = \left(\frac{x}{2}\right)^{2i|\nu|}\left(\frac{1}{\Gamma(1+i|\nu|)^2} + O(x^2)\right).$$

Furthermore, a Taylor expansion of the logarithm for small α yields

$$\ln(x+\alpha) = \ln x + \alpha/x - \alpha^2/x^2 + ...$$

(for $x > 0$ and $|\alpha| < x$). Thus

$$x^{2i|\nu|} = (z)^{2i|\nu|}\, e^{-2i|\nu|H\tilde{t}}\left(1 + O(e^{Ht_0}t_0)\right)$$

and

$$\begin{aligned}\int \mathrm{d}t f_{t_0} \frac{|\tau(t)|}{a^2(t)} J_\nu(-k\tau)^2 =& \frac{e^{-3Ht_0}}{\Gamma(\nu+1)^2}\left(\frac{z}{2}\right)^{2i|\nu|}\left(\int \mathrm{d}\tilde{t} f(\tilde{t})\frac{e^{-3H\tilde{t}}}{H}e^{-2i|\nu|H\tilde{t}} + O(e^{Ht_0}t_0)\right)\\ &+ e^{-3Ht_0}O(z^2)\left(\frac{z}{2}\right)^{2i|\nu|}\left(\int \mathrm{d}\tilde{t} f(\tilde{t})\frac{e^{-5H\tilde{t}}}{H^3}e^{-2i|\nu|H\tilde{t}} + O(e^{Ht_0}t_0)\right),\end{aligned}$$

again for $z \to 0$ and $t_0 \to -\infty$. It follows

$$\begin{aligned}&m^4\left(||\mathrm{Re}\chi_k||_{t_0}^2 ||\mathrm{Im}\chi_k||_{t_0}^2 - \langle \mathrm{Re}\chi_k, \mathrm{Im}\chi_k\rangle_{t_0}^2\right)\\ =&\frac{m^4 e^{-6Ht_0}}{16|\nu|^2H^2}\underbrace{\left(\left(\int \mathrm{d}\tilde{t} f(\tilde{t})e^{-3H\tilde{t}}\right)^2 - \left|\int \mathrm{d}\tilde{t} f(\tilde{t})e^{-3H\tilde{t}}e^{-2i|\nu|H\tilde{t}}\right|^2\right)}_{\doteq c_{f,\nu}}\\ &+ e^{-6Ht_0}\left(O(z^2) + O(e^{Ht_0}t_0)\right)\end{aligned}$$

where $c_{f,\nu}$ is strictly positive. We look now at the terms contained in W, listed in (4.34), in which the difference function $R_k = \rho_k - \chi_k$ enters. Since the integration range is confined to the support of f_{t_0}, we may choose T (which appears in formula (4.27) of lemma 4.2.1) as a function of t_0, namely by choosing $T_{t_0} = \tau(t_0 + L)$. Thus $t \in \mathrm{supp} f_{t_0} \Rightarrow |\tau(t)| \geq |T_{t_0}|$. Using the expression for $S_{\nu,T}$ in the estimate of $|R_k|$ in lemma 4.2.1 this implies that

$$|R_k(\tau(t))| \leq Ce^{\frac{Ht_0}{2}}$$

if only t_0 small enough. This entails that the strongest divergence for $t_0 \to -\infty$ in (4.33) coming from W is of order $e^{-\frac{11Ht_0}{2}}$. Thus,

$$e^{6Ht_0}(c_1^2 - c_2^2) \geq \frac{m^4 c_{f,\nu}}{16|\nu|^2 H^2} + O(z^2) + O\left(e^{\frac{Ht_0}{2}}\right) \tag{4.35}$$

An estimate of c_1 from above is easily established, by using again lemma (4.2.1) and the above considerations. The result is that $e^{3Ht_0}c_1$ is bounded for $z \to 0$ uniformly in t_0 (which has to be sufficiently small). We conclude that there exists $z^* > 0$ and some t^* such that

$$u(z, t_0) = \frac{c_1}{\sqrt{c_1^2 - |c_2|^2}} \tag{4.36}$$

and thus $|\lambda|\mu$ and μ^2 are bounded from above independently of t_0 for all $t_0 < t^*$ and $z < z^*$. Now assume that z is strictly positive and finite. We look again at the term

$$\left(\int \mathrm{d}t f_{t_0}(t) \frac{|\tau(t)|}{a^2(t)} |J_\nu(-k\tau)|^2\right)^2 - \left|\int \mathrm{d}t f_{t_0}(t) \frac{|\tau(t)|}{a^2(t)} J_\nu(-k\tau)^2\right|^2. \tag{4.37}$$

We may write the argument of J_ν as

$$-k\tau = ze^{-H\tilde{t}} + zO(e^{Ht_0}t_0)$$

for $t_0 \to -\infty$. A Taylor expansion then yields

$$J_\nu(-k\tau) = J_\nu(ze^{-H\tilde{t}}) + O(e^{Ht_0}t_0),$$

valid for fixed z. We obtain

$$\begin{aligned}
&(4.37) \\
=&\frac{e^{-6Ht_0}}{H^2}\left(\left(\int \mathrm{d}\tilde{t} f(\tilde{t}) e^{-3H\tilde{t}} \left|J_\nu\left(ze^{-H\tilde{t}}\right)\right|^2\right)^2 - \left|\int \mathrm{d}\tilde{t} f(\tilde{t}) e^{-3H\tilde{t}} J_\nu\left(ze^{-H\tilde{t}}\right)^2\right|^2\right) \\
&+ e^{-6Ht_0} O(e^{Ht_0}t_0)
\end{aligned}$$

Since the real and imaginary parts of J_ν are pointwise linearly independent and continous on the interval $[ze^{-L}, ze^{L}]$, we conclude (by using the same argument like in the previous chapter) that the above expression is a strictly positive number for every finite strictly positive z and sufficiently small t_0. Returning to W in (4.33), we already saw that its contributions diverge for $t_0 \to \infty$ at most like $e^{-\frac{11Ht_0}{2}}$. Since $e^{3Ht_0}c_1$ is regular for all values of z, we conclude that $c_1/\sqrt{c_1^2 - |c_2|^2}$ does not have any singularities in the transition region between small and large z.

Now consider $\nu \in (0, 1/2)$. We redefine the modes T_k according to lemma 4.2.2. Choose t_0 small enough such that $\tau(L + t_0) \leq -\frac{C}{H}e^{-H(L+t_0)} \doteq T(t_0) \leq T < -1$ for some positive constant C. Then $\tau(t) \leq T(t_0)$ for $t \in \mathrm{supp} f_{t_0}$. The starting point is again formula (4.33). Thanks to the separate estimates for imaginary and real part of R_k in lemma 4.2.2, it is easy to see that W contains no divergencies in k. Using again the power series expansion of J_ν for small arguments, the estimates of lemma 4.2.2 and the fact that $U_{\nu,T} \leq U_{\nu,T(t_0)} = O(e^{3Ht_0(1/2-\nu)})$ it is straightforward to show that[11]

$$\begin{aligned}
&||\mathrm{Re}T_k||_{t_0}^2 ||\mathrm{Im}T_k||_{t_0}^2 - \langle \mathrm{Re}T_k, \mathrm{Im}T_k\rangle_{t_0}^2 \\
\geq &\frac{\pi^2 e^{-6Ht_0}\left(\int \mathrm{d}t f(t) e^{-(3+2\nu)Ht} \int \mathrm{d}t f(t) e^{-(3-2\nu)Ht} - \left(\int \mathrm{d}t f(t) e^{-3Ht}\right)^2\right)}{(4\Gamma(\nu+1)\Gamma(1-\nu)H)^2} \\
&+ e^{-6Ht_0}\left(O(z^2) + O\left(e^{\frac{3}{2}(1-2\nu)Ht_0}\right)\right),
\end{aligned}$$

which is manifestly strictly positive for all sufficiently small z and t_0. Similarly it can be shown that

$$c_1 = e^{-3Ht_0} O(z^{-2\nu})$$

and we conclude that $u(z, t_0)$ and thus μ^2 and $|\lambda|\mu$ diverge at most like $z^{-2\nu}$ for $z \to 0$. In order to show the absence of poles for finite strictly positive z, exactly the same arguments can be used like in the case of imaginary ν. □

Lemma 4.2.6. *Let $\nu \in (1/2, 3/2)$ and $t_0 < t^*$ for some t^* small enough. Then the function*

$$z \mapsto z^2 \left(\mu^2 \mathrm{Re}\left(\overline{T}_{k(z)}(t) T_{k(z)}(t')\right) + \mu \mathrm{Re}\left(\lambda T_{k(z)}(t) T_{k(z)}(t')\right)\right)$$

is integrable for $z \to 0$ and $\mu^2, \mu|\lambda|$ are regular for finite z.

Proof. We start the proof by analysing the behaviour of $\mathrm{Re}c_2$ and $\mathrm{Im}c_2$ for small z. Note that we use again the "rephased" modes according to lemma 4.2.3. Consider the following contribution to c_2,

$$I_1 \doteq \int \mathrm{d}t f_{t_0}(t) m^2 T_k(t)^2. \tag{4.38}$$

Using lemma 4.2.3 we first show that for small $|k\tau| = ze^{-H\tilde{t}}(1 + O(e^{Ht_0}t_0))$, the real part of T_k^2, given by

$$\mathrm{Re}T_k^2 = (\mathrm{Re}T_k)^2 - (\mathrm{Im}T_k)^2,$$

dominates its imaginary part, which reads

$$\mathrm{Im}T_k^2 = 2\mathrm{Re}T_k \mathrm{Im}T_k.$$

When writing asymptotic forms in z and t_0 for $z \to 0$ and $t_0 \to -\infty$ in the following we used the fact that, for the discussion of c_1 and c_2, we need properties of $T_k(t)$ only on the compact interval $t \in [-L + t_0, L + t_0]$ or $\tilde{t} \doteq t - t_0 \in [-L, L]$. Thus,

$$\begin{aligned}
(\mathrm{Re}T_k)^2 =& a^{-2}(\mathrm{Re}\chi_k(\tau))^2 + a^{-2}\left(2\mathrm{Re}\chi_k(\tau)\mathrm{Re}R_k(\tau) + (\mathrm{Re}R_k(\tau))^2\right) \\
=& \frac{\pi|\tau|}{4a^2}(J_\nu(-k\tau))^2 + a^{-2}\left(2\mathrm{Re}\chi_k(\tau)\mathrm{Re}R_k(\tau) + (\mathrm{Re}R_k(\tau))^2\right) \\
=& \frac{\pi e^{-3Ht_0} z^{2\nu} e^{-(2\nu+3)H\tilde{t}}}{H 2^{2+2\nu}}\left(\frac{1}{\Gamma(\nu+1)^2} + O(z^2)\right) + O(z^2) O\left(e^{-Ht_0(2\alpha-2)}\right)
\end{aligned}$$

[11]Note that we use here again the notation (4.10).

$$\begin{aligned}(\mathrm{Im}T_k)^2 =&a^{-2}\left(\mathrm{Im}\chi_k(\tau)\right)^2 + a^{-2}\left(2\mathrm{Im}\chi_k(\tau)\mathrm{Im}R_k(\tau) + (\mathrm{Im}R_k(\tau))^2\right)\\ =&\frac{\pi|\tau|}{4a^2}\left(Y_\nu(-k\tau)\right)^2 + a^{-2}\left(2\mathrm{Im}\chi_k(\tau)\mathrm{Im}R_k(\tau) + (\mathrm{Im}R_k(\tau))^2\right)\\ =&\frac{\pi e^{-3Ht_0}z^{-2\nu}e^{(2\nu-3)H\tilde{t}}}{H2^{2-2\nu}\sin^2(\pi\nu)}\left(\frac{1}{\Gamma(-\nu+1)^2}+O(z^2)\right)+O(z^{-2\nu})O\left(e^{-Ht_0(2\alpha-5/2+\nu)}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{Re}T_k\mathrm{Im}T_k =&a^{-2}\mathrm{Im}\chi_k(\tau)\mathrm{Re}\chi_k(\tau) + a^{-2}\left(\mathrm{Re}\chi_k(\tau)\mathrm{Im}R_k(\tau)+\mathrm{Im}\chi_k(\tau)\mathrm{Re}R_k(\tau)+\mathrm{Im}R_k(\tau)\mathrm{Im}R_k(\tau)\right)\\ =&\frac{\pi|\tau|}{4a^2}J_\nu(-k\tau)Y_\nu(-k\tau)+a^{-2}(...)\\ =&-\frac{\pi e^{-3Ht_0}e^{-3H\tilde{t}}}{4H\sin(\pi\nu)}\left(\frac{1}{\Gamma(1-\nu)\Gamma(1+\nu)}+O(z^{\min(2,2\nu)})\right)+O(z^0)O\left(e^{-2Ht_0}\right)\end{aligned}$$

We conclude that, for $z\to 0$ and $t_0\to-\infty$,

$$\mathrm{Im}I_1 = -e^{-3Ht_0}O\left(z^0\right)\left(1+O\left(e^{-2Ht_0}\right)\right)$$

and

$$|\mathrm{Re}I_1| = e^{-3Ht_0}O(z^{-2\nu})\left(1+O\left(e^{-Ht_0(2\alpha-5/2+\nu)}\right)\right),$$

and that $\mathrm{Re}I_1$ is negative for z small enough. Obviously, the term

$$I_2 \doteq \int \mathrm{d}t f_{t_0}(t)k^2a^{-2}T_k(t)^2$$

contributes less singular terms in z to c_2. Finally, the last contribution

$$I_3 \doteq \int \mathrm{d}t f_{t_0}(t)(\dot{T}_k(t))^2$$

can only add a negative contribution to $\mathrm{Re}c_2$ having at most the same divergence for $z\to0$ and $t_0\to-\infty$ like $\mathrm{Re}I_1$, whereas $\mathrm{Im}I_3$ behaves like $\mathrm{Im}I_1$ in this respect. This can be deduced by estimating the first τ derivative of the series in (4.26), which furnishes corresponding estimates for the real and imaginary parts of $R'_k(\tau)$. Following the argumentation in subsection 4.1 we conclude that

$$\cos(\arg c_2) = -1+O(z^{4\nu}),$$

like in the pure de Sitter case. We also have $u(z,t_0)=O(z^{-2\nu})$ for sufficiently small t_0, which can be proven in the same manner like for the other values of ν, using lemma 4.2.3. Therefore, the same line of argumentation like in subsection 4.1 can be invoked to conclude that

$$z\mapsto z^2\left(\mu^2\mathrm{Re}\left(\overline{T}_{k(z)}(t)T_{k(z)}(t')\right)+\mu\mathrm{Re}\left(\lambda T_{k(z)}(t)T_{k(z)}(t')\right)\right)$$

is integrable for $z\to0$. The proof of regularity of μ^2 and $\mu|\lambda|$ for finite z works in the same manner like for the other regimes of ν. □

The last lemma that we need establishes a sufficient decay behaviour of the Bogolubov coefficients λ and μ in z:

Lemma 4.2.7. *Consider the SLE ω_{t_0} for the minimally coupled Klein Gordon field of mass m, induced by the smearing function f_{t_0} on the asymptotic de Sitter spacetime specified by either (4.20) or (4.21). Assume in addition that the assumption on $V(\tau)$ in lemma 4.2.4 is satisfied. Let $\mu(k,t_0)$ and $\lambda(k,t_0)$ be the Bogolubov coefficients parametrising ω_{t_0}, where the modes $\rho_k(\tau)$, given by (4.26), are used as reference modes. Then there exists T such that for $t_0 < T$ μ and $\mu|\lambda|$ decay at least like z^{-2}, where $z \doteq kH^{-1}e^{-Ht_0}$.*

Proof. For the calculation of c_1 and c_2 we need properties of the modes $\rho_k(\tau)$ for $t(\tau) \in [-L + t_0, L + t_0]$. In order to apply lemma 4.2.4, $|\tau|$ must be sufficiently big, which can be achieved by choosing t_0 small enough, $t_0 < T$ (recall that t_0 will be sent to $-\infty$ later). Inserting the results of lemma 4.2.4, one can now show that $|c_2|$ can be estimated from above by a product of e^{-3Ht_0} times a function that decays at least like z^{-1} (and is independent from t_0). This result can be established by inserting the formulae for $\rho_k(\tau)$ and $\rho_k'(\tau)$ from lemma 4.2.4 into the definition of c_2 and by performing as many partial integrations (w.r.t. the integration variable τ) as needed (at most two). As an example we consider the following (potentially worst behaving) contribution for c_2:

$$\left|\int f_{t_0} T_k(t)^2 k^2 a^{-2} \mathrm{d}t\right| = \left|\int f_{t_0} \rho_k(\tau(t))^2 k^2 a^{-4} \mathrm{d}t\right|$$
$$\leq \frac{k}{2}\Bigg(\left|\int f_{t_0} a^{-3} e^{-2ik\tau} \mathrm{d}\tau\right| + \frac{1}{k}\left|\int f_{t_0} a^{-3} e^{-2ik\tau}\left(\frac{m^2 - 2H^2}{H^2\tau} - \int_{-\infty}^{\tau} V(\tau')\mathrm{d}\tau'\right)\mathrm{d}\tau\right|$$
$$+ \frac{1}{k^2}\left|\int f_{t_0} a^{-3} C \tau^{-2} \mathrm{d}\tau\right|\Bigg)$$

Performing two times partial integration, we get

$$\begin{aligned}\frac{k}{2}\left|\int f_{t_0} a^{-3} e^{-2ik\tau} \mathrm{d}\tau\right| &\leq \frac{1}{8k}\int \left|\left(f_{t_0} a^{-3}\right)''\right| \mathrm{d}\tau \\ &= \frac{1}{8k}\int \mathrm{d}t \left|\frac{\ddot{f}_{t_0}}{a^2} - 5\dot{f}_{t_0}\frac{a'}{a^4} - 3f\left(\frac{a''}{a^5} - 4\frac{a'^2}{a^6}\right)\right|\end{aligned} \tag{4.39}$$

Going over to z and using the asymptotic behaviour of $a(t)$ in cosmological time for $t \to -\infty$ we obtain

$$\left|\frac{1}{k}\int \mathrm{d}t \ddot{f}_{t_0} a^{-2}\right| \leq \frac{e^{-3Ht_0}}{z}\int_{-L}^{L} \mathrm{d}\tilde{t}\ddot{f}(\tilde{t})\left(e^{-2H\tilde{t}} + Ce^{Ht_0}e^{-H\tilde{t}}\right),$$

where the integral can be estimated from above by a t_0- independent constant. Taking additionally the asymptotic behaviour of the scale factor into account, the remaining terms of (4.39) can be treated similarly, requiring even less partial integrations. For c_1 we obtain in the same manner

$$c_1 \geq \int \mathrm{d}t f_{t_0} k^2 a^{-4} |\rho_k(\tau(t))|^2 \geq C' e^{-3t_0} z, \quad z \to \infty$$

for some constant C' so that

$$\frac{|c_2|^2}{c_1^2} = O(z^{-4}) \tag{4.40}$$

if t_0 and z are sufficiently big. Using a Taylor expansion of μ and $|\lambda|\mu$ in the variable $|c_2|^2/c_1^2$ yields the claim. □

After this rather lengthy preparations we are now able to generalise theorem 4.1 to asymptotic de Sitter spacetimes.

Theorem 4.2.8. *Let f be an arbitrary real positive test function of compact support on the real line and let ω_{t_0} denote the SLE of the minimally coupled free Klein-Gordon field of mass $m^2 > 0$ (which implies Re$\nu < 3/2$), induced by the shifted function $f_{t_0} = f(t - t_0)$, on the spatially flat asymptotic de Sitter spacetime $(\mathcal{M}, g)$ introduced above. Depending on the value of ν, the perturbation potential V (defined by (4.23)), induced by the scale factor describing $(\mathcal{M}, g)$ should fulfil the following asymptotic requirements:*

$$\begin{aligned} V &= O(\tau^{-3}), \quad \mathrm{Re}\nu = 0,\ \mathrm{Im}\nu \neq 0 \\ V &= O(\tau^{-4}), \quad \mathrm{Re}\nu \in [0, 1/2) \\ V &= O(\tau^{-7}), \quad \mathrm{Re}\nu \in [1/2, 3/2) \\ V' &= O(\tau^{-3}), \quad \forall\nu \end{aligned}$$

Then

$$\lim_{t_0 \to -\infty} \mathcal{W}_2^{t_0} \to \mathcal{W}_2^{\lambda_\mathcal{M}}$$

in the natural topology of $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$.

Proof. In analogy to subsection 4.1 we decompose

$$\mathcal{W}_2^{t_0}(x, y) = \mathcal{W}_2^{\lambda_\mathcal{M}}(x, y) + \mathcal{R}^{t_0}(x, y), \tag{4.41}$$

where the bidistribution $\mathcal{R}^{t_0}$ is given by

$$\begin{aligned} &\mathcal{R}^{t_0}(x, y) \doteq \\ &\frac{2}{(2\pi)^3} \int d^3\mathbf{k} \Big(\mu^2 \mathrm{Re}\left(\overline{T}_k(t) T_k(t')\right) + \mu \mathrm{Re}\left(\lambda T_k(t) T_k(t')\right) \Big) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}, \end{aligned} \tag{4.42}$$

where μ, λ depend on t_0, k and describe the SLE ω_{t_0} w.r.t. the modes $T_k(t) = a^{-1}\rho_k(\tau(t))$, with ρ_k given by (4.26). These modes in turn describe the preferred state $\lambda_\mathcal{M}$ on $(\mathcal{M}, g)$. We have to show that, for any two compactly supported test functions $g, h \in \mathcal{D}(\mathcal{M})$, there holds $\lim_{t_0 \to -\infty} \mathcal{R}^{t_0}(g, h) = 0$. In order to analyse $\mathcal{R}^{t_0}$ we will use the above lemmas for $\rho_k(\tau)$ and the Bogolubov coefficients describing ω_{t_0}, which hold under the condition that τ and t_0 are sufficiently small, respectively. Thus, we will choose t_0 small enough *and* restrict g, h to $\mathcal{D}(\mathcal{M}_T, g)$, $\mathcal{M}_T \doteq \{(t, \mathbf{x}) \in \mathcal{M} : t < T\}$, where T is sufficiently small. We denote the cartesian product of the temporal supports (in cosmological time) of g and h by $D(g, h)$. Furthermore, $|\hat{h}(t, \mathbf{k})|$ can be bounded from above for all $\mathbf{k}$ by a smooth and compactly supported function which we denote by $\hat{h}(t)$; the same holds for g. Thus we have

$$\begin{aligned}
&|\mathcal{R}^{t_0}(g,h)| \\
\leq &C \int\limits_{\mathbb{R}_+} \mathrm{d}k k^2 | \iint\limits_{D(g,h)} \mathrm{d}t\mathrm{d}t' a^3(t)a^3(t') \left(\mu^2 \mathrm{Re}\left(\overline{T}_k(t)T_k(t')\right) + \mu \mathrm{Re}\left(\lambda T_k(t)T_k(t')\right)\right) \hat{g}(t)\hat{h}(t')| \\
= &H^3 e^{3Ht_0} C \int\limits_{\mathbb{R}_+} \mathrm{d}z z^2 | \iint\limits_{D(g,h)} \mathrm{d}t\mathrm{d}t' a^3(t)a^3(t') \Bigg(\mu^2 \mathrm{Re}\left(\overline{T}_{k(z)}(t)T_{k(z)}(t')\right) \\
&\qquad\qquad\qquad + \mu \mathrm{Re}\left(\lambda T_{k(z)}(t)T_{k(z)}(t')\right) \Bigg) \hat{g}(t)\hat{h}(t')|.
\end{aligned}$$

We split the z integration into two parts. Call $\mathcal{R}_1^{t_0}$ the contribution coming from $z \leq z^*$ and $\mathcal{R}_2^{t_0}$ the corresponding second part. We first discuss $\mathcal{R}_1^{t_0}$ for all possible regimes of ν. For ν imaginary, $\mathcal{R}_1^{t_0}$ is a product of e^{3Ht_0} with an integral which can be bounded independently of t_0, due to the lemmas 4.2.1 and 4.2.5; therefore it converges to zero for $t_0 \to -\infty$. If $\nu \in (0, 1/2)$ we have for $(t,t') \in D(g,h)$:

$$|T_{k(z)}(t)T_{k(z)}(t')| \leq C'(z^{2\nu}e^{2\nu Ht_0} + z^{-2\nu}e^{-2\nu Ht_0} + 2).$$

Together with the behaviour of μ^2 and $\mu|\lambda|$ for small z and their regularity for finite z (lemma 4.2.5) this implies convergence of $\mathcal{R}_1^{t_0}$ to zero as well. For $\nu \in (1/2, 3/2)$ we have to invoke additionally lemma 4.2.6 to ensure integrability for $z \to 0$. More precisely,

$$\begin{aligned}
&z^2 \left(\mu^2 \mathrm{Re}\left(\overline{T}_{k(z)}(t)T_{k(z)}(t')\right) + \mu \mathrm{Re}\left(\lambda T_{k(z)}(t)T_{k(z)}(t')\right)\right) \\
= &\mathrm{Re}T_k(t)\mathrm{Re}T_k(t')(\mu^2 + \mu|\lambda| \cos \arg\lambda) + \mathrm{Im}T_k(t)\mathrm{Im}T_k(t')(\mu^2 + \mu|\lambda| \cos \arg c_2) \\
&- \mu|\lambda| \sin \arg\lambda(\mathrm{Re}T_k(t)\mathrm{Im}T_k(t') + \mathrm{Im}T_k(t)\mathrm{Re}T_k(t'))
\end{aligned} \tag{4.43}$$

Since $z \leq z^* = k^* e^{-Ht_0}$ we have $k \leq k^*$ and thus $|\mathrm{Re}T_k(t)| = O(k^\nu)$ and $|\mathrm{Im}T_k(t)| = O(k^{-\nu})$. Furthermore, $u(z) = O(z^{-2\nu})$ and the same holds then for μ^2 and $|\lambda|\mu$. Finally, according to lemma 4.2.6 we have $\cos(\arg c_2) = -1 + O(z^{4\nu})$ and boundedness of $|\mu^2 - \mu|\lambda||$. Thus, we have

$$\begin{aligned}
&e^{3Ht_0} \int_0^{z^*} \mathrm{d}z z^2 | \iint\limits_{D(h,g)} \mathrm{d}t\mathrm{d}t'(\mu^2 + \mu|\lambda| \cos(\arg c_2)\mathrm{Im}T_{k(z)}(t)\mathrm{Im}T_{k(z)}(t')| \\
&\leq Ce^{(3-2\nu)Ht_0} \int_0^{z^*} \mathrm{d}z z^{2-2\nu} |\mu^2 - \mu|\lambda| + O(z^{2\nu})|.
\end{aligned}$$

The z integral exists and the whole term converges to zero for $t_0 \to -\infty$. The same arguments apply for the remaining terms contained in (4.43).

Now we come to the contribution $\mathcal{R}_2^{t_0}$. The following discussion will be valid $\forall \nu$. The contributions of the integrand (4.43) containing μ^2 are clearly integrable, since $|T_k(t)| = O(k^{-1/2})$ for $k > k^*$ and $\mu^2 = O(z^{-4})$. Regarding the second contribution, the measure factor z^2 cancels with the asymptotics of μ. Since we can choose T (which is the upper bound of the temporal support of the test functions g, h) sufficiently small and $k > k^*$, we may apply lemma 4.2.4 to treat the k asymptotics of $T_k(t)$:

$$T_k(t) = \frac{e^{-i\pi/4}}{a\sqrt{k}} e^{-ik\tau}(1 + O(k^{-1})).$$

The remaining contribution to $\mathcal{R}_2^{t_0}$ reads

$$e^{3Ht_0}\int_{z^*}^{\infty}\mathrm{d}z z^2 \Big|\iint\limits_{D(g,h)}\mathrm{d}t\mathrm{d}t' a^3(t)a^3(t')\mu \mathrm{Re}(\lambda T_k(t)T_k(t'))\hat{h}(t)\hat{g}(t')\Big|$$
$$=\int_{k^*}^{\infty}\mathrm{d}k k^2 \Big|\iint\limits_{D(g,h)}\mathrm{d}t\mathrm{d}t' a^3(t)a^3(t')\mu \mathrm{Re}(\lambda T_k(t)T_k(t'))\hat{h}(t)\hat{g}(t')\Big|.$$

Using the asymptotic expression for $\rho_k(\tau)$ in lemma 4.2.4 (recall that $T_k(t) = a^{-1}\rho_k(\tau(t))$ and performing at most one partial integration in the integration variable $\tau(t)$ (note that $\hat{h}$ and $\hat{g}$ have compact support) one easliy establishes the existence of the above $k-$ integral. Now for $k \geq k^*$ and t_0 sufficiently small, $z(k)$ is sufficiently big and we have $\mu|\lambda| \leq Cz^{-2} = k^{-2}e^{2Ht_0}$ by lemma 4.2.7, which proves the convergence of $\mathcal{R}_2^{t_0}$ to zero for $t_0 \to -\infty$. In order to complete the proof we remark that $\mathcal{W}_2^{t_0}$ is fully determined by knowing its restriction to $\mathcal{D}(\mathcal{M}_T \times \mathcal{M}_T)$, since the Klein–Gordon field on a globally hyperbolic spacetime fulfils the time slice axiom. □

We would like to add the following remark: In order to show the above convergence of SLE's to $\lambda_{\mathcal{M}}$, we had to require a more restrictive asymptotic behaviour for V (and thus for the scale factor) than the authors of [DMP09a; DMP09b]. The technical reason for doing so was to control the behaviour of the expression $c_1^2 - |c_2|^2$, which enters in the definition of the Bogolubov coefficients of ω_{t_0}. More precisely, we had to control it *uniformly* for small k and all t_0. This made it necessary to obtain estimates of $\rho_k(\tau)$ for its real and imaginary part separately (lemma 4.2.2 and 4.2.3). Maybe more abstract arguments could be used to prove that $c_1^2 - |c_2|^2$ is larger than some finite positive constant independently of k and t_0, in which case our additional requirements could be dropped. However, for the discussion of the large k-behaviour of the Bogolubov coefficients, the condition on V' seems to be indispensable.

5. The Semiclassical Backreaction Problem for SLE's

One of the most interesting problems in the realm of QFT on curved spacetimes, both physically and mathematically, is certainly the study of the semiclassical Einstein equation

$$G_{ab} = -8\pi G\omega(: T_{ab} :), \tag{5.1}$$

even when restricting to free field theories. We do not want to motivate here how (5.1) could be derived from a potential quantum theory of gravity (the interested reader is referred to [Hac10, sec IV.1] and the literature cited there.) As a matter of fact, (5.1) is the simplest way to describe the interaction of quantum matter with the spacetime metric in a semiclassical manner. Since the expectation value of the quantum stress energy tensor is a probabilistic quantity, it is expected that the semiclassical Einstein equation will be meaningful if the relative fluctuations of $: T_{ab} :$ in the Hadamard state ω are small. The Hadamard property at least guarantees that they are finite [Wal77]. However, the smallness can only be checked after one has found a particular solution to (5.1), since the properties of the state itself depend on the spacetime $(\mathcal{M}, g)$. Thus, one of the big conceptual problems for dealing with the backreaction problem is how to disentangle the choice of a Hadamard state ω and the specific background $(\mathcal{M}, g)$. In other words, we would like to find a possibility of fixing the state ω *without* knowing the spacetime explicitly. There are several ways to deal with this problem. First of all one may restrict the analysis to a particular class $\mathcal{C}$ of spacetimes. For instance, one might require that all $(\mathcal{M}, g)$ belonging to $\mathcal{C}$ possess a unique geometrical structure, such as a cosmological past horizon (see definition 1.3.1), which in turn can be used to single out a unique preferred state $\lambda_{\mathcal{M}}$, as we discussed earlier. Such a state can then be fixed and one can try to find a solution of (5.1) for $\omega = \lambda_{\mathcal{M}}$ and $(\mathcal{M}, g) \in \mathcal{C}$. Another possibility is to restrict to cosmological (i.e. FRW-) spacetimes. In this case (5.1) simplifies considerably to the semiclassical Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\omega(: \rho :). \tag{5.2}$$

It turns out that for the conformally coupled massless scalar field (which is then conformally invariant), the semiclassical Einstein equation can be solved without knowing ω explicitly. This problem was studied in [Wal78],[Sta80] and more recently in [DFP08], where the latter authors carefully analysed the renormalisation freedom and showed that it can be used to obtain stable de Sitter solutions without the need of introducing a cosmological constant. In [DFP08] the case of a massive field was also discussed. By using adiabatic states in the high mass regime it was shown that the exact choice of the state does not perturb the dynamics of the solution significantly.

In the following we want to pursue the idea to take SLE's as reference states for the semiclassical Einstein equation in the class of FRW spacetimes. Since the test function f (depending on cosmological time), which induces the SLE ω_f, is a spacetime–independent object, the same is then true for ω_f. That is, we have an unambiguous and physically well motivated prescription at our disposal to prescribe a Hadamard state in order to study (5.1) in FRW spacetimes. While this point of view is conceptually appealing, there remains however the task of doing analytic computations! It is useful to look first for applications which are both physically interesting and admit for simplifications. Our idea is to start with a fixed background FRW–spacetime $(\mathcal{M}_0, g_0)$, with scale function $a_0(t)$, and to formulate the backreaction problem perturbatively over $(\mathcal{M}_0, g_0)$, the perturbation parameter being a perturbation function $\delta a(t)$ of the background scale factor $a_0(t)$. This perturbative approach can be tackled by analytic means and allows for the investigation of *stability questions* of the background spacetimes. In the case of Minkowski space $(\mathbb{R}^4, \eta)$, we know already that this spacetime is a fixpoint of (5.1) if ω is the Minkowski vacuum ω_{Mink} and provided that the renormalisation constants are chosen appropriately. Furthermore, we know that each SLE ω_f reduces to ω_{Mink} on $(\mathbb{R}^4, \eta)$. An obvious question is then the following: Do there exist other nontrivial fixpoints of (5.1) for the fixed state ω_f in the class of FRW spacetimes? The question of stability of $(\mathbb{R}^4, \eta)$ can then be answered by investigating properties of the (nontrivial) perturbative solutions $\delta a(t)$ of a perturbative version of (5.1). While we do not aim to give a mathematically precise notion of stability of spacetimes here, we can nevertheless impose some conditions that a "stable solution" would have to fulfil. Namely, in case of the perturbation background $(\mathbb{R}^4, \eta)$, δa should be globally defined on $\mathbb{R}$ and stay bounded together with its derivatives. While this merely excludes runaway solutions, one could in addition require that δa and its derivatives converge to zero for $t \to \pm\infty$. In the following we will work out this perturbative ansatz for the case of Minkowski space as reference background. We consider the class of FRW spacetimes with flat spatial sections and a scale factor of the form

$$a = 1 + \delta a, \tag{5.3}$$

i.e. we consider spatially homogenous and isotropic perturbations of Minkowski space, where δa is contained in a suitable function space. In particular, δa is supposed to be smooth and $\delta a \ll 1$. Furthermore we consider again the minimally coupled massive $(m^2 > 0)$ Klein Gordon field, which we prescribe to be in a SLE ω_f induced by the test function f. Our first goal is to derive a functional Taylor expansion of the renormalised energy density

$$a \mapsto \omega_f(:\rho(t):)[a], \; a \in C^\infty(\mathbb{R}, \mathbb{R}_+)$$

in the vicinity of the Minkowski scale factor $a_0 = 1$. The energy density, obtained via a point splitting procedure, is also a smooth function. We assume that $\omega_f(:\rho(t):)[a]$ posesses an expansion

$$\omega_f(:\rho(t):)[a_0 + \delta a] = \omega_f(:\rho(t):)[a_0] + \left\langle \frac{\delta}{\delta a}\omega_f(:\rho(t):)[a]\bigg|_{a_0}, \delta a \right\rangle + ...$$

It will turn out that – in order to obtain a nontrivial semiclassical Einstein equation – this has to be done up to second order in δa. To proceed, we introduce the following notation: For any functional of the scale factor $\mathcal{F}[a] = \mathcal{F}[1 + \delta a]$ we use the notation $\mathcal{F}^{<n>}[\delta a]$ for its Taylor expansion up to order n, where we ocassionally supress the argument δa. For the contribution of order n we will write $\mathcal{F}^{(n)}$. Unfortunately, we denote by $\delta a^{(n)}$ also the $n-$ th derivative of the metric perturbation with respect to conformal time τ. But since this notation will only be used for δa and the test function f, it will not lead to confusions. The strategy of our calculation will be the following: First we calculate the correction to the Minkowski mode functions $\chi_k^{(0)}$, which are the basic objects for the calculation of the energy density. We will do this in conformal time τ, defined by $\mathrm{d}\tau = \mathrm{d}ta^{-1}$. Having obtained the modes as functionals of δa up to second order, we calculate the Bogolubov coefficients λ, μ describing the SLE for the test function f according to Olbermann's prescription. λ and μ will then be functionals of δa. These are the ingredients for the calculation of the unrenormalised energy density. A similar functional expansion in δa has then also to be done for the truncated Hadamard parametrix $\mathcal{G}_1$. Since we already know its form both in the position- and momentum space representation for our considered class of spacetimes (we refer the reader to section 2.4.2), this will be a rather easy task. Finally we determine the renormalisation freedom for the energy density coming from the tt component of the covariantly conserved tensors g_{ab}, G_{ab}, I_{ab} and J_{ab}. We will then be in the position to write down the perturbed semiclassical Einstein equation, being a nonlinear integro–differential equation for δa. This equation will then be analysed with regard to asymptotic properties of its solutions and we will furthermore present a numerical solution.

5.1. Perturbative Semiclassical Einstein Equation over Minkowski Space

The Mode Functions

Our starting point is the full mode equation

$$\chi_k''(\tau) + (k^2 + Q(\tau))\chi_k(\tau) = 0. \tag{5.4}$$

As already discussed in the previous chapters, we work with the auxiliary modes $\chi_k(\tau)$ in conformal time τ, which are linked with the original modes $S_k(t)$ via

$$\chi_k(\tau(t)) = a(t)S_k(t). \tag{5.5}$$

$S_k(t)$ in turn has to fulfil the differential equation (2.13) and the normalisation condition (2.14). The latter translates to the requirement

$$\chi_k'\overline{\chi_k} - \chi_k\overline{\chi_k'} = i. \tag{5.6}$$

The "time dependent mass" $Q(\tau)$, defined by

$$Q(\tau) \doteq a(t(\tau))^2 \left(m^2 - \frac{R(\tau)}{6}\right),$$

reads with $a(t(\tau)) = 1 + \delta a(\tau)$:

$$Q(\tau) = m^2 + \underbrace{2m^2\delta a + m^2(\delta a)^2 - \frac{\delta a''}{1+\delta a}}_{\doteq V_p(\tau)}.$$

We perform the expansion

$$(a^2R)^{<2>} = 6\delta a''(1-\delta a)$$

and obtain

$$Q^{<2>} = m^2 + \underbrace{2\delta a m^2 - \delta a''}_{\doteq Q^{(1)}} + \underbrace{m^2\delta a^2 + \delta a''\delta a}_{\doteq Q^{(2)}} \tag{5.7}$$

For $\delta a = 0$, the special solutions of (5.4) are chosen to be the Minkowski modes

$$\chi_k^{(0)}(\tau) \doteq \frac{1}{\sqrt{2\omega}} e^{i\omega\tau},$$

where throughout this chapter we define $\omega \doteq \sqrt{k^2+m^2}$. For $\delta a \neq 0$ a formal solution of (5.4) may be given by

$$\begin{aligned}\chi_k(\tau) =& \chi_k^{(0)}(\tau) \\ &+ \sum_{n=1}^{\infty}(-1)^n \int_0^\tau \mathrm{d}t_1 ... \int_0^{t_{n-1}} \mathrm{d}t_n S_k(\tau,t_1)...S_k(t_{n-1},t_n)V_p(t_1)...V_p(t_n)\chi_k^{(0)}(t_n)\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}S_k(t_1,t_2) &\doteq i\left(\overline{\chi}_k^{(0)}(t_1)\chi_k^{(0)}(t_2) - \overline{\chi}_k^{(0)}(t_2)\chi_k^{(0)}(t_1)\right) \\ &= \frac{1}{\omega}\sin(\omega(t_1-t_2))\end{aligned}$$

Starting from (5.8) and inserting $V_p = Q^{(1)} + Q^{(2)} + O(\delta a^3)$ we obtain the following expression for the first and second order contributions to χ_k and χ_k':

$$\begin{aligned}
\chi_k^{(1)}(\tau) =& -\frac{1}{\sqrt{2\omega}\omega}\left(\left(I^{(1)} + iII^{(1)}\right)\sin\omega\tau - \left(II^{(1)} + iIII^{(1)}\right)\cos\omega\tau\right) \\
\chi_k'^{(1)}(\tau) =& -\frac{1}{\sqrt{2\omega}}\left(\left(I^{(1)} + iII^{(1)}\right)\cos\omega\tau + \left(II^{(1)} + iIII^{(1)}\right)\sin\omega\tau\right) \\
\chi_k^{(2)}(\tau) =& \frac{1}{\sqrt{2\omega}\omega^2}\left(\left(U - \omega\left(I^{(2)} + iII^{(2)}\right)\right)\sin\omega\tau - \left(V - \omega\left(II^{(2)} + iIII^{(2)}\right)\right)\cos\omega\tau\right) \\
\chi_k'^{(2)}(\tau) =& \frac{1}{\sqrt{2\omega}\omega}\left(\left(U - \omega\left(I^{(2)} + iII^{(2)}\right)\right)\cos\omega\tau + \left(V - \omega\left(II^{(2)} + iIII^{(2)}\right)\sin\omega\tau\right)\right),
\end{aligned}$$

where we defined

$$\begin{aligned}
I^{(i)}(\tau) &= \int_0^\tau \mathrm{d}\tau' \cos^2\omega\tau' Q^{(i)}(\tau') \\
II^{(i)}(\tau) &= \int_0^\tau \mathrm{d}\tau' \sin\omega\tau' \cos\omega\tau' Q^{(i)}(\tau') \\
III^{(i)}(\tau) &= \int_0^\tau \mathrm{d}\tau' \sin^2\omega\tau' Q^{(i)}(\tau') \\
U(\tau) &= \int_0^\eta \mathrm{d}\tau' Q^{(1)}\left(\sin\omega\tau'\cos\omega\tau'\left(I^{(1)} + iII^{(1)}\right) - \cos^2\omega\tau'\left(II^{(1)} + iIII^{(1)}\right)\right) \\
V(\tau) &= \int_0^\tau \mathrm{d}\tau' Q^{(1)}\left(\sin^2\omega\tau'\left(I^{(1)} + iII^{(1)}\right) - \sin\omega\tau'\cos\omega\tau'\left(II^{(1)} + iIII^{(1)}\right)\right)
\end{aligned}$$

It is a straightforward calculation to show that condition (5.6) is satisfied[1] up to second order by the above constructed $\chi_k^{(1)}$ and $\chi_k^{(2)}$.

The Bogolubov Coefficients for an SLE and the Unrenormalised Energy Density

Now we turn to the calculation of the state ω_f, for whose construction we use the reference modes $\chi_k^{<2>}$. The SLE ω_f, described by the modes $T_k(t)$, is parametrised by Bogolubov coefficients λ_k, μ_k via the relation

$$T_k(t) = \frac{1}{a(t)} \left(\lambda_k \chi_k(\tau(t)) + \mu_k \overline{\chi}_k(\tau(t))\right),$$

with

$$|\lambda_k|^2 - |\mu_k|^2 = 1,$$

and its (unrenormalised) energy density per mode reads

$$\begin{aligned}\rho_k =& \frac{1}{2}\left(|\dot{T}_k|^2 + (k^2 a^{-2} + m^2)|T_k|^2\right) \\ =& \left(\frac{1}{2} + \mu_k^2\right)\left(|\dot{S}_k|^2 + (k^2 a^{-2} + m^2)|S_k|^2\right) + \mathrm{Re}\left(\lambda_k \mu_k \left(\dot{S}_k^2 + (k^2 a^{-2} + m^2) S_k^2\right)\right),\end{aligned} \tag{5.9}$$

with $S_k(t)$ defined in (5.5). For the calculation of λ_k and μ_k describing ω_f we need first to expand the expressions

$$|\dot{S}_k|^2 + (k^2 a^{-2} + m^2)|S_k|^2 \qquad \text{and} \qquad \dot{S}_k^2 + (k^2 a^{-2} + m^2) S_k^2$$

up to order two in δa by inserting $\chi_k^{<2>}$. We obtain

$$\begin{aligned}\left(|\dot{S}_k|^2 + (k^2 a^{-2} + m^2)|S_k|^2\right)^{(0)} &= \omega, \\ \left(|\dot{S}_k|^2 + (k^2 a^{-2} + m^2)|S_k|^2\right)^{(1)} &= -\left(3\omega + \frac{k^2}{\omega}\right)\delta a + 2\mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k'^{(1)} + \omega^2 \overline{\chi_k^{(0)}}\chi_k^{(1)}\right)\end{aligned}$$

and

$$\begin{aligned}&\left(|\dot{S}_k|^2 + (k^2 a^{-2} + m^2)|S_k|^2\right)^{(2)} \\ =& \left(\frac{13}{2}\omega + \frac{7k^2}{2\omega}\right)(\delta a)^2 + \frac{(\delta a')^2}{2\omega} - 8\delta a \mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k'^{(1)}\right) - 4\delta a(2k^2 + m^2)\mathrm{Re}\left(\chi_k^{(0)}\overline{\chi_k}^{(1)}\right) \\ & - 2\delta a' \mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k^{(1)} + \overline{\chi_k^{(0)}}\chi_k'^{(1)}\right) + |\chi_k'^{(1)}|^2 + \omega^2 |\chi_k^{(1)}|^2 + 2\mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k'^{(2)} + \omega^2 \overline{\chi_k^{(0)}}\chi_k^{(2)}\right)\end{aligned}$$

[1] This actually follows already from the ansatz (5.8).

We calculate the following terms:

$$\begin{aligned}
\mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k'^{(1)}\right) &= -\frac{1}{2}\left(\sin\omega\tau\cos\omega\tau\left(III^{(1)}-I^{(1)}\right)+II^{(1)}(\cos^2\omega\tau-\sin^2\omega\tau)\right)\\
\mathrm{Re}\left(\omega^2\overline{\chi_k^{(0)}}\chi_k^{(1)}\right) &= \frac{1}{2}\left(\sin\omega\tau\cos\omega\tau\left(III^{(1)}-I^{(1)}\right)+II^{(1)}(\cos^2\omega\tau-\sin^2\omega\tau)\right)\\
\mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k^{(1)}\right) &= \frac{1}{2\omega}\left(I^{(1)}\sin^2\omega\tau+III^{(1)}\cos^2\omega\tau-2II^{(1)}\sin\omega\tau\cos\omega\tau\right)\\
\mathrm{Re}\left(\overline{\chi_k^{(0)}}\chi_k'^{(1)}\right) &= -\frac{1}{2\omega}\left(III^{(1)}\sin^2\omega\tau+I^{(1)}\cos^2\omega\tau+2II^{(1)}\sin\omega\tau\cos\omega\tau\right)\\
|\chi_k'^{(1)}|^2+\omega^2|\chi_k^{(1)}|^2 &= \frac{1}{2\omega}\left(2\left(II^{(1)}\right)^2+\left(I^{(1)}\right)^2+\left(III^{(1)}\right)^2\right)\\
\mathrm{Re}\left(\overline{\chi_k'^{(0)}}\chi_k'^{(2)}+\omega^2\overline{\chi_k^{(0)}}\chi_k^{(2)}\right) &= \frac{1}{2\omega}\left(\left(II^{(1)}\right)^2-I^{(1)}III^{(1)}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(|\dot S_k|^2+(k^2a^{-2}+m^2)|S_k|^2\right)^{(0)} &= \omega\\
\left(\dot S_k^2+(k^2a^{-2}+m^2)S_k^2\right)^{(0)} &= 0\\
\left(|\dot S_k|^2+(k^2a^{-2}+m^2)|S_k|^2\right)^{(1)} &= -\left(3\omega+\frac{k^2}{\omega}\right)\delta a\\
\left(\dot S^2+(k^2a^{-2}+m^2)S^2\right)^{(1)} &= \frac{\delta a m^2}{\omega}e^{2i\omega\tau}+2II^{(1)}+i\left(III^{(1)}-I^{(1)}\right)-i\delta a' e^{2i\omega\tau}
\end{aligned}$$

and

$$\begin{aligned}
&\left(|\dot S_k|^2+(k^2a^{-2}+m^2)|S_k|^2\right)^{(2)}\\
&=\left(\frac{13}{2}\omega+\frac{7k^2}{2\omega}\right)(\delta a)^2+\frac{(\delta a')^2}{2\omega}+\frac{1}{2\omega}\left(4\left(II^{(1)}\right)^2+\left(I^{(1)}-III^{(1)}\right)^2\right)\\
&\quad+2\delta a\frac{m^2}{\omega^2}\left(\left(III^{(1)}-I^{(1)}\right)\sin\omega\tau\cos\omega\tau+II^{(1)}(\cos^2\omega\tau-\sin^2\omega\tau)\right)\\
&\quad-\frac{\delta a'}{\omega}\left(\left(III^{(1)}-I^{(1)}\right)(\cos^2\omega\tau-\sin^2\omega\tau)-4II^{(1)}\sin\omega\tau\cos\omega\tau\right)
\end{aligned}$$

To proceed, we write $c_i = c_i^{(0)} + c_i^{(1)}$ with[2]

$$\begin{aligned}
c_1^{(i)} &\doteq \int \mathrm{d}t f(t)\left(|\dot S_k|^2+(k^2a^{-2}+m^2)|S_k|^2\right)^{(i)}\\
c_2^{(i)} &\doteq \int \mathrm{d}t f(t)\left(\dot S_k^2+(k^2a^{-2}+m^2)S_k^2\right)^{(i)},
\end{aligned}$$

[2]We dropped here the prefactor 1/2 of Olbermanns definition.

and remark that $c_1^0 = \omega$ and $c_2^0 = 0$. For calculating $\rho_k^{<2>}$ we need the following functions of λ and μ, which we expand[3] in $c_i^{(1)}$:

$$\begin{aligned}(\mu^2)^{<2>} &= \frac{\left(\mathrm{Re}c_2^{(1)}\right)^2 + \left(\mathrm{Im}c_2^{(1)}\right)^2}{4\omega^2}\\ (\lambda\mu)^{<1>} &= \frac{i\mathrm{Im}c_2^{(1)} - \mathrm{Re}c_2^{(1)}}{2\omega}\end{aligned}$$

The explicit expressions for $c_i^{(1)}$ read

$$\begin{aligned}\mathrm{Re}c_2^{(1)} &= \int \mathrm{d}\tau f(t(\tau)) \left(2II^{(1)} + \frac{m^2}{\omega}\cos(2\omega\tau)\delta a + \sin(2\omega\tau)\delta a'\right)\\ \mathrm{Im}c_2^{(1)} &= \int \mathrm{d}\tau f(t(\tau)) \left(III^{(1)} - I^{(1)} + \frac{m^2}{\omega}\sin(2\omega\tau)\delta a - \cos(2\omega\tau)\delta a'\right).\end{aligned}$$

Note that in the last two equations we used $\mathrm{d}t = \mathrm{d}\tau(1+\delta a)$. Thus, in the required perturbation order of $\mathrm{Re}c_2$ and $\mathrm{Im}c_2$ we are allowed to use $\mathrm{d}t = \mathrm{d}\tau$ since

$$\left(\dot{S}_k^2 + (k^2 a^{-2} + m^2)S_k^2\right)^{(0)} = 0$$

Inserting the results into (5.9) gives for every order

$$\begin{aligned}\rho_k^{(0)} =& \frac{\omega}{2}\\ \rho_k^{(1)} =& -\frac{\delta a}{2}\left(3\omega + \frac{k^2}{\omega}\right)\\ \rho_k^{(2)} =& \left(\frac{13}{4}\omega + \frac{7k^2}{4\omega}\right)(\delta a)^2 + \frac{(\delta a')^2}{4\omega}\\ &+ \frac{\delta a m^2}{2\omega^2}\left(\sin 2\omega\tau\left(III^{(1)} - I^{(1)} - \mathrm{Im}c_2^{(1)}\right) + \cos 2\omega\tau\left(2II^{(1)} - \mathrm{Re}c_2^{(1)}\right)\right)\\ &- \frac{\delta a'}{2\omega}\left(\cos 2\omega\tau\left(III^{(1)} - I^{(1)} - \mathrm{Im}c_2^{(1)}\right) - \sin 2\omega\tau\left(2II^{(1)} - \mathrm{Re}c_2^{(1)}\right)\right)\\ &+ \frac{1}{4\omega}\left(\left(2II^{(1)} - \mathrm{Re}c_2^{(1)}\right)^2 + \left(III^{(1)} - I^{(1)} - \mathrm{Im}c_2^{(1)}\right)^2\right)\end{aligned}$$

The Hadamard Parametrix and the Renormalisation Freedom

The next task is to renormalise the point split energy density by means of the Hadamard subtraction scheme. We start by collecting the divergent terms[4] of $\rho_k^{(i)}$. We introduce the decomposition

$$\rho^{(i)}(k) = \Omega^{(i)}(k) + R^{(i)}(k), \tag{5.10}$$

[3] In order to obtain ρ_k up to second order we need only the first order terms of the c_i.

[4] Divergent w.r.t. the integration $\int_0^\infty \mathrm{d}kk^2...$

where $\Omega^{(i)}(k)$ is supposed to contain the divergencies[5] in k. For zeroth and second order we make the obvious definitions

$$\begin{aligned}\Omega^{(0)}(k) &\doteq \frac{\omega}{2}\\ \Omega^{(1)}(k) &\doteq -\frac{\delta a}{2}\left(3\omega + \frac{k^2}{\omega}\right).\end{aligned}$$

In order to extract $\Omega^{(2)}(k)$ from $\rho_k^{(2)}$, we calculate

$$\begin{aligned}2II^{(1)} - \mathrm{Re}c_2^{(1)} &= -\frac{Q^{(1)}(\tau)\cos(2\omega\tau)}{2\omega} + \frac{1}{2\omega}X_1(\tau) - \int A(\tau)f(t(\tau))\mathrm{d}\tau\\ III^{(1)} - I^{(1)} - \mathrm{Im}c_2^{(1)} &= -\frac{Q^{(1)}(\tau)\sin(2\omega\tau)}{2\omega} + \frac{1}{2\omega}Y_1(\tau) - \int B(\tau)f(t(\tau))\mathrm{d}\tau\end{aligned} \tag{5.11}$$

where we defined

$$\begin{aligned}A(\tau) &\doteq \frac{\delta a''\cos(2\omega\tau)}{2\omega} + \sin(2\omega\tau)\delta a'\\ B(\tau) &\doteq \frac{\delta a''\sin(2\omega\tau)}{2\omega} - \cos(2\omega\tau)\delta a'\\ X_i(\tau) &\doteq \int_0^\tau \mathrm{d}\tau'\cos(2\omega\tau')\left(\partial_\tau^i Q^{(1)}\right)(\tau') - \int \mathrm{d}\tau' f\int_0^{\tau'}\mathrm{d}\tau''\cos(2\omega\tau'')\left(\partial_\tau^i Q^{(1)}\right)(\tau'')\\ Y_i(\tau) &\doteq \int_0^\tau \mathrm{d}\tau'\sin(2\omega\tau')\left(\partial_\tau^i Q^{(1)}\right)(\tau') - \int \mathrm{d}\tau' f\int_0^{\tau'}\mathrm{d}\tau''\sin(2\omega\tau'')\left(\partial_\tau^i Q^{(1)}\right)(\tau'')\end{aligned}$$

Inserting identities (5.11) into the expression obtained for $\rho_k^{(2)}$, its singular part may be defined as

$$\begin{aligned}\Omega^{(2)}(k) \doteq &\left(\frac{13}{4}\omega + \frac{7k^2}{4\omega}\right)(\delta a)^2 + \frac{(\delta a')^2}{4\omega} - \frac{m^2\delta a Q^{(1)}(\tau)}{4\omega^3} + \frac{\delta a' Q'^{(1)}(\tau)}{8\omega^3}\\ &+ \frac{\left(Q^{(1)}(\tau)\right)^2}{16\omega^3}.\end{aligned}$$

Consequently,

$$\begin{aligned}R^{(2)}(k) = &-\frac{\delta a'}{8\omega^3}\left(\cos(2\omega\tau)(X_2(\tau) + F_1) + \sin(2\omega\tau)(Y_2(\tau) + F_2)\right)\\ &-\frac{B(\tau)}{2\omega}\int \mathrm{d}\tau' fB - \frac{A(\tau)}{2\omega}\int \mathrm{d}\tau' fA + \frac{\delta a''}{8\omega^3}\left(\cos(2\omega\tau)X_1 + \sin(2\omega\tau)Y_1\right)\\ &+\frac{1}{4\omega}\left(\left(\frac{X_1(\tau)}{2\omega} - \int \mathrm{d}\tau' fA\right)^2 + \left(\frac{Y_1(\tau)}{2\omega} - \int \mathrm{d}\tau' fB\right)^2\right),\end{aligned} \tag{5.12}$$

with

$$\begin{aligned}F_1 &\doteq \int \mathrm{d}\tau f(t(\tau))Q'^{(1)}(\tau)\cos(2\omega\tau)\\ F_2 &\doteq \int \mathrm{d}\tau f(t(\tau))Q'^{(1)}(\tau)\sin(2\omega\tau).\end{aligned}$$

[5]Note that such a decomposition is not unique; thus (5.10) is meaningless until $\Omega^{(i)}(k)$ is defined seperately.

We can immediately read off this formula that $R^{(2)}(k)$ depends on the perturbation δa via its derivatives, beginning with the first one. To proceed, recall that the temporal coincidence limit of the unrenormalised point-split energy density of order i is the distribution

$$[\rho^{(i)}]_\tau = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k} e^{-k\epsilon} e^{i\mathbf{kr}}\rho_k^{(i)},$$

whose singular part, according to (5.10), is defined by

$$W_\Omega^{(i)} \doteq \lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k} e^{-k\epsilon} e^{i\mathbf{kr}}\Omega^{(i)}(k).$$

In order to link $W_\Omega^{(i)}$ with the to-be-subtracted Hadamard singularity $[\mathcal{R}\mathcal{G}_1]_\tau$ we apply lemma 2.4.5. Accordingly, we first have to determine the asymptotics of $\Omega^{(i)}$ in k:

$$\begin{aligned}
\Omega^{(0)}(k) &= \frac{k}{2} + \frac{m^2}{4k} - \frac{m^4}{16k^3} + O(k^{-5})\\
\Omega^{(1)}(k) &= -2k\delta a - \frac{m^2\delta a}{2k} + O(k^{-5})\\
\Omega^{(2)}(k) &= 5k\delta a^2 + \frac{3m^2\delta a^2 + \delta a'^2}{4k} + \frac{\delta a''^2 + 2m^2\delta a'^2 - 2\delta a'\delta a'''}{16k^3} + O(k^{-5}).
\end{aligned}$$

In the following, we use the notations of lemma 2.4.5. Since we are interested in the coincidence limit $r\to 0$ of $[\rho^{(i)}]_\tau - [\mathcal{R}\mathcal{G}_1]_\tau^{(i)}$, we only need the contribution of $R_{\Omega,L}^{2k'-1}$ (which is the intergral kernel of the distribution $\mathcal{R}_{\Omega,L}^{2k'-1}$) which remains finite for $r\to 0$; i.e. we choose $k'=1$. We then obtain

$$\begin{aligned}
R_1^{(0)} =& \frac{m^4}{64}\left(4\log mL - \log 16 - 3 + 4\gamma\right)\\
R_1^{(1)} =& \frac{m^4\delta a}{16}\\
R_1^{(2)} =& -\frac{m^4}{32}\delta a^2 - \frac{m^2}{16}\left(1+\gamma-\log 4 + 2\log mL\right)\delta a'^2\\
&+\frac{1}{8}\left(\gamma + \log 4 + \log mL - 1\right)\left(\delta a'\delta a''' - \frac{1}{2}\delta a''^2\right).
\end{aligned}$$

We have thus identified the singular distributions $W_\Omega^{(i)}$ with distributions in position space. More explicitly,

$$\begin{aligned}
W_\Omega^{(0)} =& \frac{1}{4\pi^2}\left(-\frac{2}{r_+^4} + \frac{m^2}{2r_+^2} + \frac{m^4}{16}\mathrm{lo}_0\right) + \frac{m^4}{32\pi^2}\left(\log mL - \log 2 - \frac{3}{4} + \gamma\right)\\
W_\Omega^{(1)} =& \frac{1}{4\pi^2}\left(\frac{8\delta a}{r_+^4} - \frac{m^2\delta a}{r_+^2} + \frac{m^4\delta a}{8}\right)\\
W_\Omega^{(2)} =& \frac{1}{4\pi^2}\left(-\frac{20\delta a^2}{r_+^4} + \frac{3m^2\delta a^2 + \delta a'^2}{2r_+^2} - \frac{\delta a''^2 + 2m^2\delta a'^2 - 2\delta a'\delta a'''}{16}\mathrm{lo}_0\right)\\
&+\frac{1}{16\pi^2}\left(\delta a'\delta a''' - \frac{1}{2}\delta a''^2\right) - \frac{m^2\delta a'^2}{32\pi^2}(1+\gamma-\log 4 + 2\log mL) - \frac{m^4\delta a^2}{64\pi^2}
\end{aligned}$$

On the other hand, according to (2.35) and (A.6), we know that

$$
\begin{aligned}
&[\mathcal{R}\mathcal{G}_1]_\tau - \frac{1}{3}[P_x\mathcal{G}_1] = \\
&\frac{1}{4\pi^2}\Bigg(-\frac{2}{C^2 r_+^4} + \left(\frac{C'^2}{8C^4} + \frac{m^2}{2C}\right)\frac{1}{r_+^2} \\
&\quad + \left(\frac{m^4}{16} - \frac{m^2 C'^2}{32C^3} + \frac{9C'^4}{256C^6} - \frac{C'^2 C''}{16C^5} - \frac{C''^2}{64C^4} + \frac{C' C'''}{32C^4}\right)(\mathrm{lo}_0 + \log C) \\
&\quad + \frac{11C''^2}{960C^4} - \frac{C'^4}{24C^6} + \frac{37C'^2 C''}{480C^5} - \frac{5m^2 C'^2}{96C^3} - \frac{11C''' C'}{480C^4}\Bigg),
\end{aligned}
$$

where

$$
C(\tau) \doteq a^2(t(\tau)).
$$

Expansion in δa up to second order gives

$$
\begin{aligned}
\left([\mathcal{R}\mathcal{G}_1]_\tau - \frac{1}{3}[P_x\mathcal{G}_1]\right)^{(0)} &= \frac{1}{4\pi^2}\left(-\frac{2}{r_+^4} + \frac{m^2}{2r_+^2} + \frac{m^4}{16}\mathrm{lo}_0\right) \\
\left([\mathcal{R}\mathcal{G}_1]_\tau - \frac{1}{3}[P_x\mathcal{G}_1]\right)^{(1)} &= \frac{1}{4\pi^2}\left(\frac{8\delta a}{r_+^4} - \frac{m^2\delta a}{r_+^2} + \frac{m^4}{8}\delta a\right) \\
\left([\mathcal{R}\mathcal{G}_1]_\tau - \frac{1}{3}[P_x\mathcal{G}_1]\right)^{(2)} &= \frac{1}{4\pi^2}\Bigg(-\frac{20\delta a^2}{r_+^4} + \frac{\delta a'^2 + 3m^2\delta a^2}{2r_+^2} \\
&\quad + \left(\frac{\delta a'\delta a'''}{8} - \frac{\delta a''^2}{16} - \frac{m^2\delta a'^2}{8}\right)\mathrm{lo}_0 \\
&\quad - \frac{m^4\delta a^2}{16} + \frac{11\delta a''^2}{240} - \frac{5m^2\delta a'^2}{24} - \frac{11\delta a'\delta a'''}{120}\Bigg),
\end{aligned}
$$

and we see that the distributions $W_\Omega^{(i)}$ have the correct singularities in every order. Furthermore, the expansion of the renormalisation freedom (B.1) up to second order in δa reads

$$
C_{tt}^{<2>} = \mathrm{A}m^4 - 3\mathrm{B}m^2\delta a'^2 + 6(3\Gamma + \Delta)\left(\delta a''^2 - 2\delta a'''\delta a'\right) \tag{5.13}
$$

The renormalised quantum expectation value in order i is given by

$$
\omega_f(:\rho:)^{(i)}(\tau) = \lim_{r\to 0}\left([\rho]_\tau - [\mathcal{R}\mathcal{G}_1]_\tau + \frac{1}{3}[P_x\mathcal{G}_1]\right)^{(i)} + C_{tt}^{(i)}(\tau).
$$

Thus,

$$
\begin{aligned}
\omega_f(:\rho:)^{(0)} =& \frac{m^4}{32\pi^2}\left(\log ML - \log 2 - \frac{3}{4} + \gamma\right) + Am^4 \\
\omega_f(:\rho:)^{(1)} =& 0 \\
\omega_f(:\rho:)^{(2)} =& \delta a'^2 \frac{m^2}{4\pi^2}\left(\frac{1}{12} - \frac{\gamma - \log 4 + 2\log mL}{8}\right) \\
&+ \frac{1}{4\pi^2}\left(\delta a'\delta a''' - \frac{1}{2}\delta a''^2\right)\left(\frac{2\gamma + 2\log 4 + 2\log mL}{4} + \frac{11}{120}\right) \\
&- 3\mathrm{B}m^2\delta a'^2 - 12(3\Gamma + \Delta)\left(\delta a'''\delta a' - \frac{1}{2}\delta a''^2\right) \\
&+ R^{(2)}[f, \delta a'],
\end{aligned}
\tag{5.14}
$$

where

$$
R^{(2)}[f, \delta a'] \doteq \frac{1}{2\pi^2}\int \mathrm{d}k k^2 R^{(2)}(k), \tag{5.15}
$$

with $R^{(2)}(k)$ defined in (5.12). We have thus collected all the necessary ingredients for our first goal, the formulation of the semiclassical Einstein equation in our perturbational setting.

The Renormalised Semiclassical Einstein Equation

By putting together the pieces of the previous subsections and remarking that the expansion of the left hand side of (5.2) reads

$$
(H^2)^{<2>} = \delta a'^2,
$$

we get the following result:

Proposition 5.1.1. *Let $(\mathcal{M}, g)$ be a FRW spacetime which has flat spatial sections and a scale factor of the form $a = 1+\delta a$, where δa is a smooth perturbation function. Then the semiclassical Einstein equation for the minimally coupled Klein Gordon field in the state of low energy ω_f with $m^2 > 0$ reads*

$$
0 = \frac{m^4}{32\pi^2}\left(\log mL - \log 2 - \frac{3}{4} + \gamma\right) + Am^4 \tag{5.16}
$$

in zeroth order and

$$
0 = m^2\delta a'^2 R_1(B, m) + \left(\delta a'\delta a''' - \frac{1}{2}\delta a''^2\right) R_2(3\Gamma + \Delta, m) + R^{(2)}[f, \delta a'], \tag{5.17}
$$

in second order order perturbation theory in δa, while the first order is trivial. Here we defined the modified renormalisation constants

$$
\begin{aligned}
R_1(B, m) &\doteq \frac{1}{48\pi^2} - \frac{\gamma - \log 4 + 2\log mL}{32\pi^2} - 3\mathrm{B} - \frac{3}{8\pi m^2 G} \\
R_2(3\Gamma + \Delta, m) &\doteq \frac{2\gamma + 2\log 4 + 2\log mL}{16\pi^2} + \frac{11}{480\pi^2} - 12(3\Gamma + \Delta),
\end{aligned}
$$

where A, B, Γ and Δ are the original renormalisation constants appearing in (B.1) and $R^{(2)}[f, \delta a']$ is given by (5.15).

5.1.1. Coherent States

We have derived the semiclassical Friedmann equation for the perturbation δa of the scale factor over Minkowski space in second order perturbation theory, using SLE's as reference states. It was necessary to go to order two since the first order was trivial. We considered a quantum mechanical pure vacuum state, namely the SLE ω_f. However, starting from such a state, we can construct a more general class of states, namely the *coherent* states[6]. These states can be interpreted as quantum mechanical description of classical field configurations, in the sense that expectation values of the quantised field in a coherent state fulfils the classical equation of motion. We want now to discuss how such a coherent state might change our perturbative backreaction problem. Coherent states can be defined as follows: Consider the automorphism map α_ψ on the field algebra $\mathcal{A}(\mathcal{M}, g)$, given by

$$\alpha_\psi(\Phi) = \Phi + \psi \mathbb{1}, \tag{5.18}$$

where ψ is a real smooth solution of the Klein–Gordon–equation. Given a quantum state ω, we can define the coherent state ω_ψ by

$$\omega_\psi = \omega \circ \alpha_\psi. \tag{5.19}$$

If we want this state to be homogenous and isotropic, then ψ must have this property as well. That is, it must satisfy the differential equation for the zero mode,

$$\ddot{\psi} + 3H\dot{\psi} + m^2\psi = 0. \tag{5.20}$$

Since ψ is uniquely determined by two initial values, we obtain a two–parameter family of coherent homogenous isotropic states. Now we require our reference quantum state ω to be quasifree. For the two point function of ω_ψ it follows

$$\begin{aligned}
\mathcal{W}_2^{\omega_\psi}(x,y) &= \omega_\psi(\Phi(x)\Phi(y)) \\
&= \omega\left(\alpha_\psi\Phi(x)\alpha_\psi\Phi(y)\right) \\
&= \omega((\Phi(x) + \psi(x)\mathbb{1})(\Phi(y) + \psi(y)\mathbb{1})) \\
&= \mathcal{W}_2^{\omega}(x,y) + \psi(x)\psi(y),
\end{aligned}$$

where we used the fact that ω is linear and normalised, its one point function vanishes and $\psi(x)$ is a $\mathbb{C}-$ number. The point splitting definition of the energy density in the state ω_ψ thus yields

$$\omega_\psi(:\rho:) = \omega(:\rho:) + \frac{1}{2}\left(\dot{\psi}^2 + m^2\psi^2\right).$$

In order to look at the backreaction problem via perturbation theory over Minkowski space, we have to determine the general solution of (5.20) as series in δa (and its derivatives). This time there is no constraint on the Wronskian of ψ and ψ has to be real. Defining again the auxiliary function λ by

$$\lambda(\tau) \doteq a(t(\tau))\psi(t(\tau)),$$

[6] We restrict our considerations here to *pure* coherents states. Note that it is straightforward to construct mixed coherent states by summing over pure coherent states with a normalised weight. This allows in particular for the definition of isotropic and homogenous mixed coherent states parametrised by infinitely many parameters.

we have

$$\lambda'' + a^2(m^2 - R/6)\lambda = 0.$$

We have to find again a perturbative expression of λ in δa for the fixed initial conditions

$$\lambda(0) = \beta \qquad\qquad \lambda'(0) = \alpha m,$$

with $(\alpha, \beta) \in \mathbb{R}^2$. Following the steps from the above calculation of the modes we obtain

$$\begin{aligned}
\lambda^{(0)}(\tau) =& \alpha \sin(m\tau) + \beta \cos(m\tau) \\
\lambda'^{(0)}(\tau) =& m\left(\alpha \cos(m\tau) - \beta \sin(m\tau)\right) \\
\lambda^{(1)}(\tau) =& \frac{1}{m}\left(\cos(m\tau)\left(\alpha III^{(1)} + \beta II^{(1)}\right) - \sin(m\tau)\left(\alpha II^{(1)} + \beta I^{(1)}\right)\right) \\
\lambda'^{(1)}(\tau) =& -\left(\cos(m\tau)\left(\alpha II^{(1)} + \beta I^{(1)}\right) + \sin(m\tau)\left(\alpha III^{(1)} + \beta II^{(1)}\right)\right).
\end{aligned}$$

The coherent part of the energy density reads

$$\rho_\psi \doteq \frac{1}{2}\left(\dot{\psi}^2 + m^2\psi^2\right) = \frac{1}{2a^2}\left(\frac{\lambda'^2}{a^2} - 2\frac{\lambda'\lambda a'}{a^3} + \left(\frac{a'^2}{a^4} + m^2\right)\lambda^2\right).$$

Thus we obtain

$$\begin{aligned}
\rho_\psi^{(0)} =& \frac{m^2(\alpha^2 + \beta^2)}{2} \\
\rho_\psi^{(1)} =& -m^2\delta a\left(\alpha^2 + \beta^2 + (\alpha\cos(m\tau) - \beta\sin(m\tau))^2\right) \\
& - m\delta a'\left(\alpha\beta\cos(2m\tau) + \frac{1}{2}(\alpha^2 - \beta^2)\sin(2m\tau)\right) \\
& + m\Bigg(\alpha\beta\left(III^{(1)} - I^{(1)}\right) + II^{(1)}(\beta^2 - \alpha^2)\Bigg).
\end{aligned}$$

Now consider the semiclassical Friedmann equation up to first order δa in the coherent state over ω_f belonging to ψ. It reads

$$0 = \frac{m^4}{32\pi^2}\left(\log mL - \log 2 - \frac{3}{4} + \gamma\right) + Am^4 + \rho_\psi^{(0)} + \rho_\psi^{(1)}.$$

If we insist that the renormalisation parameter A should be fixed by the first equation[7] of (5.16), we obtain

$$0 = \rho_\psi^{(0)} + \rho_\psi^{(1)} \tag{5.21}$$

Differentiating w.r.t. τ gives

$$0 = -3m\delta a'(\alpha\cos(m\tau) - \beta\sin(m\tau))^2,$$

whose only solution is $\delta a = \text{const}$. Inserting this into (5.21) it follows

$$\delta a = \frac{\alpha^2 + \beta^2}{2(2\alpha^2 + \beta^2)}.$$

[7] This is reasonable since A should not depend on the choice of the coherent state.

We have $1/4 \leq \delta a \leq 1/2$. This fits quite nice in the results we have obtained so far, since the solution space of (5.17) is invariant under a shift of the scale factor by a constant. That is, taking into account the contribution of a coherent state does not restrict the possible solutions of (5.17).

5.1.2. An Application of the Perturbative Formula for $\omega_f(:\rho:)$

Before we proceed with the analysis of the semiclassical Friedmann equation, we would like to pursue a small excursion. Namely, as a byproduct of our analysis, we can use our perturbative formula (5.14) for the energy density in the SLE ω_f for a cross–check of the results we obtained for the energy density of SLE's on de Sitter space in chapter 3. Let us write the de Sitter scale factor as a perturbation of Minkowski space in the vicinity of $t = 0$:

$$a(t) = 1 + e^{Ht} - 1$$

Consequently, our perturbation δa is given by

$$\delta a = e^{Ht} - 1.$$

We consider furthermore the same test function as in chapter 3, namely a Gaussian centered at $t = 0$ with smearing width ϵ:

$$f(t) = \frac{1}{\sqrt{\pi}\epsilon} e^{-\frac{t^2}{\epsilon^2}}.$$

We assume again that

$$H\epsilon \ll 1.$$

Since we want to calculate the energy density in the state ω_f only in the vicinity of $t = 0$, we assume also $Ht \ll 1$. Thus we have in zeroth order Ht:

$$\begin{aligned} \delta a' &\approx H & \delta a'' &\approx 2H^2 \\ \delta a''' &\approx 6H^3 & \delta a'''' &\approx 24H^4 \end{aligned}$$

and

$$\mathrm{d}\tau \approx \mathrm{d}t.$$

Inserting these approximations in (5.12) we obtain after some lengthy calculation

$$\begin{aligned} R^{(2)}(k) =& e^{-2\epsilon^2\omega^2} \frac{1}{4\omega} \left(\frac{H^2}{\omega^2} + \left(H + \frac{m^2 H - 3H^3}{2\omega^2} \right)^2 \right) \\ & - e^{-\epsilon^2\omega^2} \cos(2\omega t) \left(\frac{H^2}{2\omega} + \frac{H^4}{2\omega^3} + \frac{m^2H^2 - 3H^4}{4\omega^3} \left(1 + \frac{m^2 - 3H^2}{2\omega^2} \right) \right) \\ & - e^{-\epsilon^2\omega^2} \sin(2\omega t) \frac{m^2H^3 - 6H^5}{2\omega^4} + \frac{H^2(m^2 - 3H^2)^2}{16\omega^5} \end{aligned}$$

We define our energy density as

$$\omega(:\rho:)^{(2)} = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k k^2 \tilde{R}^{(2)}(k),$$

where

$$\tilde{R}^{(2)}(k) \doteq R^{(2)}(k) - \frac{H^2(m^2 - 3H^2)^2}{16\omega^5},$$

i.e. we subtracted a constant term. Figure 5.1 shows the result for the same choice of m and $\alpha = \epsilon H$ that was used for the corresponding plot 3.4. A comparison of the two plots shows

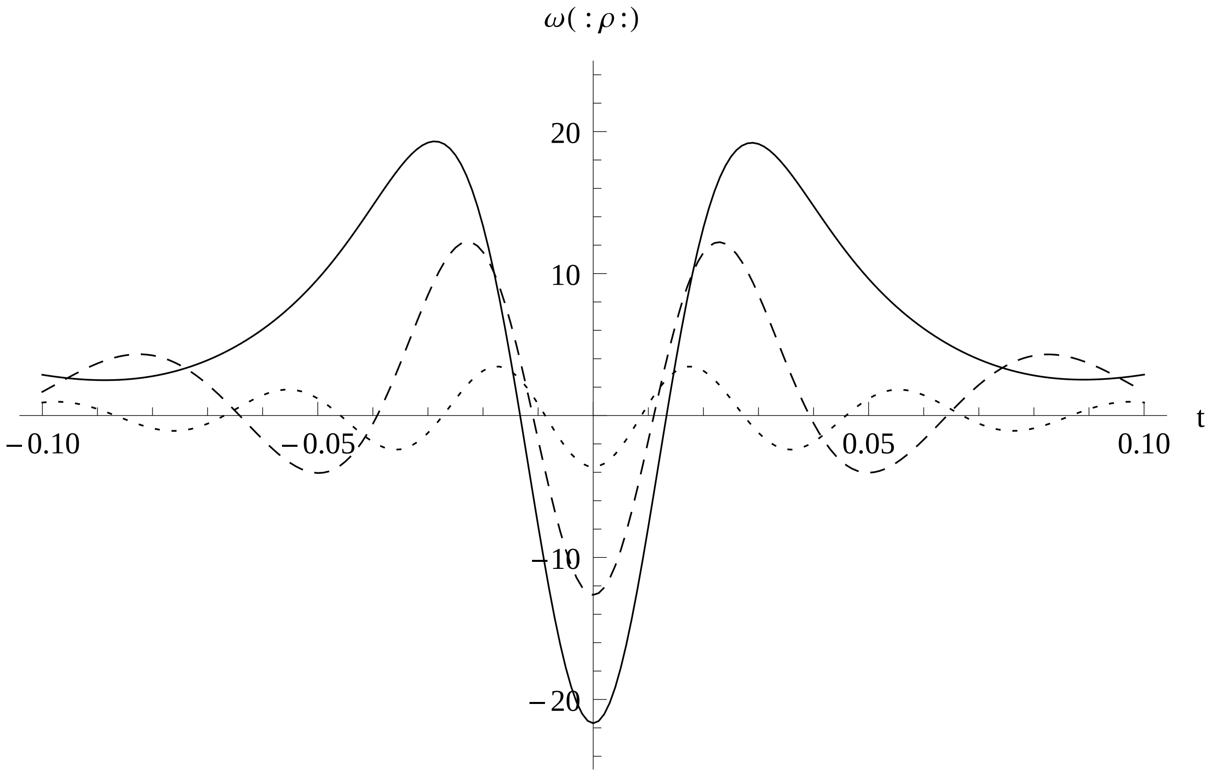

Figure 5.1.: Renormalised energy density for SLE's in units H^4 against cosmological time in units H^{-1}. These plots were calculated in the perturbative approach. The testfunction f is centered at $t = 0$ and the smearing width is $\alpha = \epsilon H = 0.02$. The solid, dashed and dotted curves represent the mass values $n = 500$, $n = 2000$ and $n = 5000$, respectively, where $n = \frac{m^2}{H^2} - 2$.

indeed that our cross–check was succesful. The deviations of the results (the curves in figure 5.1 are symmetric in t in contrast to 3.4) stem from the fact that we took for simplicity only the zeroth order of the Taylor expansions of δa and its derivatives into account.

5.2. Asymptotic Analysis of the Semiclassical Einstein Equation

We have now the task of finding solutions to the system of equations (5.16) and (5.17). The first of them (zeroth order) obviously fixes the renormalisation constant A as a function of m, ensuring that Minkowski space is a solution of the semiclassical Einstein equation for the Minkowski vacuum. The first order perturbation theory gives no information at all, so the question of stability can only be answered by investigating the second order equation. Since $R^{(2)}[f, \delta a']$ contains integrals of $\delta a'$ with f, (5.17) is a *ordinary homogenous nonlinear integro–differential equation* for $\delta a'$ and therefore explicit solutions can in general only be obtained numerically. As already remarked (and indicated in the notation), $R^{(2)}[f, \delta a']$ does not depend on δa itself but rather on its higher derivatives, beginning with the first one. However, a closer inspection of (5.17) reveals that it can be converted into a *linear* equation: Differentiating w.r.t. τ on both sides yields after a rather long calculation

$$0 = \delta a' \delta a'' 2m^2 R_1 + \delta a' \delta a'''' R_2 + \delta a' \left((\mathcal{F}[f, \delta a'](\tau) + \mathcal{G}[f, \delta a'](\tau) \right), \tag{5.22}$$

where the functionals $\mathcal{F}$ and $\mathcal{G}$ (both linear in f and $\delta a'$) are given by

$$\begin{aligned} \mathcal{F}[f, \delta a'](\eta) \doteq& \frac{1}{2\pi^2} \int_0^\infty \frac{\mathrm{d}k k^2}{16\omega^4} \left(\cos(2\omega\eta)(Y_4 - 2m^2 Y_2) - \sin(2\omega\eta)(X_4 - 2m^2 X_2) \right) \\ \mathcal{G}[f, \delta a'](\eta) \doteq& -\frac{1}{2\pi^2} \int_0^\infty \frac{\mathrm{d}k k^2}{16\omega^4} \int_{-d}^{d} \mathrm{d}\tau' \sin(2\omega(\tau' - \tau)) \times \\ & \Big(5f\delta a^{(5)} + 10 f' \delta a^{(4)} + \delta a'''(10 f'' - 12m^2 f) \\ & + \delta a''(5f''' - 12m^2 f') + \delta a'(f^{(4)} - 4m^2 f'' + 4m^4 f) \Big) \end{aligned}$$

Recall that $[-d, d]$ is the support of f. Excluding the trivial solution $\delta a' = 0$ we are left with a linear integro differential equation for $\delta a'$, which is much simpler to handle (both numerically and analytically) than the original one. In the derivation of the above result we used as many partial integrations (of the terms which are integrals in time such as X_i and Y_i) as necessary in order to obtain an absolutely convergent k–integral. The price to pay in the occurrence of higher derivatives of δa in $\mathcal{F}$ and $\mathcal{G}$ up to order six. We can now interchange the k– and τ–integration in the functionals $\mathcal{F}$ and $\mathcal{G}$: Introducing the integral kernel

$$Z(m, x) \doteq \int_0^\infty \frac{\mathrm{d}k k^2}{\omega^4} \sin(2\omega x) \tag{5.23}$$

and rewriting

$$X_i(\tau) = \int_d^\tau \mathrm{d}\tau' \cos(2\omega\tau') \left(\partial_\tau^i Q^{(1)} \right)(\tau') + \int_{-d}^{d} \mathrm{d}\tau' F(\tau') \cos(2\omega\tau') \left(\partial_\tau^i Q^{(1)} \right)(\tau')$$

(and analogously for Y_i), where $F(\tau) \doteq \int_{-d}^{\tau} \mathrm{d}\tau' f(t(\tau'))$, we obtain finally

$$\begin{aligned} \mathcal{F} + \mathcal{G} =& \frac{1}{32\pi^2} \int_d^\tau \mathrm{d}\tau' Z(m, \tau' - \tau)(4m^2 \delta a^{(4)} - \delta a^{(6)} - 4m^4 \delta a'') \\ & - \frac{1}{32\pi^2} \int_{-d}^{d} \mathrm{d}\tau' Z(m, \tau' - \tau) \Big(\delta a^{(6)} F + 5 f \delta a^{(5)} + \delta a^{(4)}(10 f' - 4m^2 F) \\ & + \delta a'''(10 f'' - 12 m^2 f) + \delta a''(5 f''' - 12 m^2 f' + 4m^4 F) \\ & + \delta a'(f^{(4)} - 4m^2 f'' + 4m^4 f) \Big) . \end{aligned} \tag{5.24}$$

The so obtained linear equation of motion for δa will be used for a later numerical treatment. For our subsequent asymptotic analysis, we have to use nevertheless the original form (5.17), since (5.22) has a larger space of solutions. In particular, it allows for solutions approaching zero for large times, which is not compatible with (5.17) as we shall show now. In order to do so, we will assume asymptotic properties of potential solutions δa and investigate if these properties are compatible with (5.17) by looking at the asymptotic limit $\tau \to \pm\infty$. We start with the following asymptotic requirement:

Case 1: $\delta a'$ and its Derivatives Converge to Zero for $\tau \to \pm\infty$

Let us assume that $\delta a'$ and its higher derivatives have the following asymptotic behaviour for $\tau \to= \pm\infty$:

$$\delta a' = O\left((\log|\tau|)^{-\alpha}\right), \qquad \delta a^{(n)} = O\left(\frac{1}{|\tau|(\log|\tau|)^{1+\alpha}}\right), \qquad n \geq 2, \qquad \alpha > 0 \tag{5.25}$$

This defines a class of scale factor perturbations which we would call (asymptotically) stable. Our aim is now to check if such a decay behaviour is compatible with (5.17).

Lemma 5.2.1. *Let δa belong to the class of stable perturbations in the sense of (5.25) and let ψ be a smooth function vanishing at a and whose derivatives have compact support in $[a, b]$. Then for $n \geq 1$ the function*

$$G : \tau \mapsto \int_0^\infty \frac{\mathrm{d}k k^2}{\omega^3} \sin(2\omega\tau + \phi) \int_a^\tau \mathrm{d}\tau' \psi(\tau')\delta a^{(n)}(\tau') \sin(2\omega\tau' + \gamma)$$

converges to zero for $\tau \to \infty$.

Proof. Partial integration yields

$$\begin{aligned}
&\int_a^\tau \mathrm{d}\tau' \psi(\tau')\delta a^{(n)}(\tau') \sin(2\omega\tau' + \gamma) \\
&= -\frac{\cos(2\omega\tau + \gamma)}{2\omega}\psi(\tau)\delta a^{(n)}(\tau) + \frac{1}{2\omega}\int_a^\tau \mathrm{d}\tau' \left(\psi(\tau')\delta a^{(n)}\right)'(\tau') \cos(2\omega\tau' + \gamma).
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\int_a^\tau \mathrm{d}\tau' \left(\psi\delta a^{(n)}\right)'(\tau') \cos(2\omega\tau' + \gamma) \\
&= \underbrace{\int_a^\infty \mathrm{d}\tau' \left(\psi\delta a^{(n)}\right)'(\tau') \cos(2\omega\tau' + \gamma)}_{\doteq I(\omega)} - \underbrace{\int_\tau^\infty \mathrm{d}\tau' \left(\psi\delta a^{(n)}\right)'(\tau') \cos(2\omega\tau' + \gamma)}_{\doteq J(\tau,\omega)}.
\end{aligned}$$

Since $\left(\psi\delta a^{(n)}\right)' \in L_1(a, \infty)$, we can estimate $|I(\omega)|$ by a constant and

$$|J(\omega, \tau)| \leq \int_\tau^\infty \mathrm{d}\tau' \left|\left(\psi\delta a^{(n)}\right)'(\tau')\right|.$$

It follows

$$\begin{aligned}
|G(\tau)| \leq &\left(|\psi(\tau)\delta a^{(n)}(\tau)| + \int_\tau^\infty \mathrm{d}\tau' \left|\left(\psi\delta a^{(n)}\right)'(\tau')\right|\right) \int_0^\infty \frac{\mathrm{d}k k^2}{2\omega^4} \\
&+ \left|\int_0^\infty \frac{\mathrm{d}k k^2}{2\omega^4} \sin(2\omega\tau + \phi) I(\omega)\right|.
\end{aligned}$$

It is clear that the first term vanishes in the limit $\tau \to \infty$ while the second one does so by the Riemann-Lebesgue lemma. □

Since we may rewrite

$$X_1(\omega,\tau) = \int_{-d}^{\tau} \mathrm{d}\tau' F(\tau')Q'(\tau')\cos(2\omega\tau'),\ \tau > d$$
$$X_1(\omega,\tau) = \int_{\tau}^{d} \mathrm{d}\tau'(F(\tau')-1)Q'(\tau')\cos(2\omega\tau'),\ \tau < -d$$

and analogously for Y_1, Y_2, X_2, where

$$F(\tau) \doteq \int_{-\infty}^{\tau} f(\tau')\mathrm{d}\tau',$$

we arrive at the following

Proposition 5.2.2. *Let δa have the asymptotic form (5.25). Then the semiclassical Friedmann equation (5.17) reads in the limit $\tau \to \pm\infty$*

$$0 = \lim_{\tau\to\pm\infty}\int_0^{\infty}\frac{\mathrm{d}kk^2}{8\pi^2\omega}\left(\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' fA\right)^2 + \left(\frac{Y_1}{2\omega} - \int \mathrm{d}\tau' fB\right)^2\right).$$

Proof. The assertion follows directly from the application of the above lemma to the contributions of $R^{(2)}(k)$. □

Let us now analyse the above asymptotic form. Since for $\tau \to \infty$,

$$|X_1(\omega,\tau)| \leq \frac{1}{2\omega}\int_{-d}^{\tau}\mathrm{d}\tau'|(F\partial_\tau Q^{(1)})'(\tau')| \leq \frac{C}{\omega}$$

for all τ and $\int \mathrm{d}\tau fA$ falls off faster than any inverse power of k (similar arguments work again for $Y_1, \int \mathrm{d}\tau fB$ and $\tau \to -\infty$), we may interchange limit and integration by dominated convergence and get the conditions

$$\lim_{\tau\to\pm\infty}\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' fA\right) = 0$$

and

$$\lim_{\tau\to\pm\infty}\left(\frac{Y_1}{2\omega} - \int \mathrm{d}\tau' fB\right) = 0$$

for almost every k if δa is supposed to be a solution of (5.17) with decay properties (5.25). After some partial integrations it follows

$$\lim_{\tau\to\infty}\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' fA\right) = \frac{1}{2\omega}\int_{-\infty}^{\infty}\cos(2\omega\tau')\left(2m^2\delta a' F - (F\delta a')''\right)\mathrm{d}\tau'$$
$$= -\frac{1}{4\omega^2}\int_{-\infty}^{\infty}\sin(2\omega\tau')\left(2m^2\delta a' F - (F\delta a')''\right)'\mathrm{d}\tau',$$

and similarly

$$\lim_{\tau\to\infty}\left(\frac{Y_1}{2\omega}-\int \mathrm{d}\tau' fB\right)=\frac{1}{4\omega^2}\int_{-\infty}^{\infty}\cos(2\omega\tau')\left(2m^2\delta a'F-(F\delta a')''\right)'\mathrm{d}\tau'.$$

This means that the Fourier transform of the smooth L_1 function

$$\psi'(\tau)\doteq\left(2m^2\delta a'F-(F\delta a')''\right)'$$

can only have support on $(-2m,2m)$:

$$\psi'(\tau)=\int_{-2m}^{2m}\mathrm{d}k(\cos k(\tau+d)h(k)+\sin k(\tau+d)g(k))$$

Since ψ' and all its derivatives vanish at $\tau=-d$, we conclude that $h(k)$ and $g(k)$ are antisymmetric and symmetric, respectively and thus $\psi'=0$ for all τ. This in turn means $\psi=0$ ($\psi=\text{const}\neq 0.$ would contradict the assumed decay property of δa) and thus $\delta a'F=ce^{-\sqrt{2}m\tau}$. Since this function has to vanish at $\tau=-d$, we conclude $\delta a'=0$. We summarise our finding in the following theorem:

Theorem 5.2.3. *Let δa be a nontrivial solution of the perturbative semiclassical Einstein equation (5.17), i.e. $\delta a\neq\text{const}$. Then it cannot be stable in the sense of (5.25).*

This result seems to be surprising, since it implies that Minkowski space is not stable when coupled to a scalar field which is in an isotropic homogenous state of low energy. However, our situation does not really correspond to the setting usually considered in physics when analysing stability: The term "pertubation" usually refers to a *local* displacement of a physical system, which is in our case the spacetime metric. But for the sake of simplicity, we restricted our considerations only to FRW spacetimes and quantum states sharing its symmetries with the benefit of being left with a single dynamical quantity δa. But the perturbation of the spacetime described by a perturbation of the scale factor over Minkowski background is a highly nonlocal one, since it affects simultanously all space. Thus, the perturbation has no "possibility to spread out" and vanish for large times. Taking this picture into account, it is not astonishing that the perturbation cannot entirely relax back to Minkowski geometry.

Case 2: δa with Oscillatory Asymptotics

As shown in the previous subsection, we cannot have stability of Minkowski space in the sense that every solution of (5.17) converges to zero in the past and future. This asymptotic behaviour is not compatible with the structure of (5.17), due to a "memory effect" which is a rather general feature of integro–differential equations. As we will argue in this section, one way out is to allow for an oscillatory asymptotics of solutions for δa, which could then provide a weaker notion of stability of Minkowski space in the sense that δa and its derivatives stay at least bounded. By assuming δa to have a purely oscillatory asymptote[8], we may write down the following ansatz for δa without loss of generality:

$$\delta a=\delta a_{sd}+\alpha_-C_-(\tau)\sin(\lambda_-\tau+\phi_-)+\alpha_+C_+(\tau)\sin(\lambda_+\tau+\phi_+).\qquad(5.26)$$

[8] By "purely" we mean a constant frequency.

δa_{sd} is a smooth function of sufficient fast decrease in $|\tau|$ according to the criterion (5.25), whereas $C_\pm$ are smooth cutoff functions whose derivatives are compactly supported in $[d,e]$ and $[-e,-d]$, respectively (recall that $[-d,d]$ is the support of f). Note that we have the freedom to fix the form of $C_\pm$. Choosing $C_\pm = 1$ for $\tau \gtrless \pm e$, we may fix $C_\pm$ by requiring its first derivative to be a normalised bump function. The ansatz (5.26) leads to the following

Proposition 5.2.4. *Let δa be a solution of (5.17) which is of the form (5.26). Then its frequency $\lambda_\pm \doteq x_\pm m > 0$ and amplitude $\alpha_\pm$ must fulfil the consistency equations*

$$0 = R_1 x_\pm^2 - R_2 \frac{x_\pm^4}{2} - \frac{x_\pm^2(2+x_\pm^2)^2}{32\pi^2} I_1(x_\pm) \tag{5.27}$$

and

$$0 = \alpha_\pm^2 \left(\frac{x_\pm^4(2_\pm + x_\pm^2)}{32\pi^2}((2+x_\pm^2)I_2(x_\pm) - 2I_1(x_\pm)) - \frac{R_2 x_\pm^4}{2} \right) + U^\pm, \tag{5.28}$$

where the functions

$$\begin{aligned} I_1(x) &\doteq \int_0^\infty \frac{z^2}{\sqrt{z^2+1}^3(x^2-4(z^2+1)^2)} \mathrm{d}z \\ I_2(x) &\doteq \int_0^\infty \frac{\mathrm{d}z^2}{\sqrt{z^2+1}^3(x^2-4(z^2+1)^2)^2} \mathrm{d}z \end{aligned}$$

are defined for $x \in [0,2)$ and

$$U^\pm \doteq \frac{1}{8\pi^2 m^4} \int_0^\infty \frac{\mathrm{d}k k^2}{\omega} \left(Y_1^{\pm 2} + Y_2^{\pm 2} + 2\alpha_\pm (Y_1^\pm Z_1^\pm + Y_2^\pm Z_2^\pm) + \alpha_\pm^2 (Z_1^{\pm 2} + Z_2^{\pm 2}) \right).$$

Furthermore, for fixed $C_\pm$ the additional condition

$$\frac{1}{8\pi^2} \int_0^\infty \frac{\mathrm{d}k k^2}{\omega} \left(Y_1^\pm Z_1^\pm + Y_2^\pm Z_2^\pm \right) = 0 \tag{5.29}$$

must hold, which determines the phases $\phi_\pm$ uniquely. The expressions $Y_i^\pm$ and $Z_i^\pm$ are defined *by*

$$
\begin{aligned}
Y_1^+ \doteq& -\frac{1}{4\omega^2}\int_{-d}^{\infty} \mathrm{d}\tau \sin(2\omega\tau)(2m^2 F\delta a'_{sd} - (F\delta a'_{sd})'')' \\
Y_1^- \doteq& -\frac{1}{4\omega^2}\int_{-\infty}^{d} \mathrm{d}\tau \sin(2\omega\tau)(2m^2 (F-1)\delta a'_{sd} - ((F-1)\delta a'_{sd})'')' \\
Y_2^+ \doteq& \frac{1}{4\omega^2}\int_{-d}^{\infty} \mathrm{d}\tau \cos(2\omega\tau)(2m^2 F\delta a'_{sd} - (F\delta a'_{sd})'')' \\
Y_2^- \doteq& \frac{1}{4\omega^2}\int_{-\infty}^{d} \mathrm{d}\tau \cos(2\omega\tau)(2m^2 (F-1)\delta a'_{sd} - ((F-1)\delta a'_{sd})'')' \\
Z_1^\pm \doteq& \pm(m^2+2\omega^2)\Bigg\{ \cos\phi_\pm \left(\frac{\mathrm{Im}\hat{C}'_\pm(\lambda_\pm+2\omega)}{\lambda_\pm+2\omega} - \frac{\mathrm{Im}\hat{C}'_\pm(\lambda_\pm-2\omega)}{\lambda_\pm-2\omega}\right) \\
&+ \sin\phi_\pm \left(\frac{\mathrm{Re}\hat{C}'_\pm(\lambda_\pm+2\omega)}{\lambda_\pm+2\omega} - \frac{\mathrm{Re}\hat{C}'_\pm(\lambda_\pm-2\omega)}{\lambda_\pm-2\omega}\right)\Bigg\} \\
Z_2^\pm \doteq& (m^2+2\omega^2)\Bigg\{ -\cos\phi_\pm \left(\frac{\mathrm{Re}\hat{C}'_\pm(\lambda_\pm+2\omega)}{\lambda_\pm+2\omega} + \frac{\mathrm{Re}\hat{C}'_\pm(\lambda_\pm-2\omega)}{\lambda_\pm-2\omega}\right) \\
&+ \sin\phi_\pm \left(\frac{\mathrm{Im}\hat{C}'_\pm(\lambda_\pm+2\omega)}{\lambda_\pm+2\omega} + \frac{\mathrm{Im}\hat{C}'_\pm(\lambda_\pm-2\omega)}{\lambda_\pm-2\omega}\right)\Bigg\}
\end{aligned}
$$

Proof. We have to insert ansatz (5.26) in (5.17) and to evaluate every term for $\tau \to \pm\infty$, which is a lenghty but elementary task. We first calculate the following expressions, where the subscripts $\pm$ refer to $\tau \gtrless 0$:

$$
\begin{aligned}
\delta a' =& \delta a'_{sd} + C'_\pm \alpha_\pm \sin(\lambda_\pm\tau+\phi_\pm) + C_\pm\alpha_\pm\lambda_\pm\cos(\lambda_\pm\tau+\phi_\pm) \\
\delta a'' =& \delta a''_{sd} + 2C'_\pm\alpha_\pm\lambda_\pm\cos(\lambda_\pm\tau+\phi_\pm) + C''_\pm\alpha_\pm\sin(\lambda_\pm\tau+\phi_\pm) \\
&- C_\pm\alpha_\pm\lambda_\pm^2\sin(\lambda_\pm\tau+\phi_\pm) \\
\delta a''' =& \delta a'''_{sd} - 3C'_\pm\alpha_\pm\lambda_\pm^2\sin(\lambda_\pm\tau+\phi_\pm) + 3C''_\pm\alpha_\pm\lambda_\pm\cos(\lambda_\pm\tau+\phi_\pm) \\
&+ C'''_\pm\alpha_\pm\sin(\lambda_\pm\tau+\phi_\pm) - C_\pm\alpha_\pm\lambda_\pm^3\cos(\lambda_\pm\tau+\phi_\pm) \\
\delta a'''' =& \delta a''''_{sd} + C''''_\pm\alpha_\pm\sin(\lambda_\pm\tau+\phi_\pm) - 6C''_\pm\alpha_\pm\lambda_\pm^2\sin(\lambda_\pm\tau+\phi_\pm) \\
&+ 4C'''_\pm\alpha_\pm\lambda_\pm\cos(\lambda_\pm\tau+\phi_\pm) - 3C'_\pm\alpha_\pm\lambda_\pm^3\cos(\lambda_\pm\tau+\phi_\pm) \\
&+ C_\pm\alpha_\pm\lambda_\pm^4\sin(\lambda_\pm\tau+\phi_\pm)
\end{aligned}
$$

and thus

$$
\begin{aligned}
Q^{(1)} =& Q^{(1)}_{sd} + C_\pm\alpha_\pm(2m^2+\lambda_\pm^2)\sin(\lambda_\pm\tau+\phi_\pm) \\
Q'^{(1)} =& Q'^{(1)}_{sd} + C_\pm\alpha_\pm\lambda_\pm(2m^2+\lambda_\pm^2)\cos(\lambda_\pm\tau+\phi_\pm) \\
Q''^{(1)} =& Q''^{(1)}_{sd} - C_\pm\alpha_\pm\lambda_\pm^2(2m^2+\lambda_\pm^2)\sin(\lambda_\pm\tau+\phi_\pm)
\end{aligned}
$$

with

$$\begin{aligned}
Q_{sd}^{(1)} \doteq & 2m^2\delta a_{sd} - \delta a_{sd}'' - 2C_\pm' \alpha_\pm \lambda_\pm \cos(\lambda_\pm \tau + \phi_\pm) \\
& - C_\pm'' \alpha_\pm \sin(\lambda_\pm \tau + \phi_\pm) \\
Q_{sd}'^{(1)} \doteq & 2m^2\delta a_{sd}' - \delta a_{sd}''' + C_\pm' \alpha_\pm (2m^2 + 3\lambda_\pm^2) \sin(\lambda_\pm \tau + \phi_\pm) \\
& - 3C_\pm'' \alpha_\pm \lambda_\pm \cos(\lambda_\pm \tau + \phi_\pm) - C_\pm''' \alpha_\pm \sin(\lambda_\pm \tau + \phi_\pm) \\
Q_{sd}''^{(1)} \doteq & 2m^2\delta a_{sd}'' - \delta a_{sd}'''' + C_\pm' \alpha_\pm \lambda_\pm (4m^2 + 3\lambda_\pm^2) \cos(\lambda_\pm \tau + \phi_\pm) \\
& + C_\pm'' \alpha_\pm \sin(\lambda_\pm \tau + \phi_\pm)(2m^2 + 6\lambda_\pm^2) - 4C_\pm''' \alpha_\pm \lambda_\pm \cos(\lambda_\pm \tau + \phi_\pm) \\
& - C_\pm'''' \alpha_\pm \sin(\lambda_\pm \tau + \phi_\pm)
\end{aligned}$$

Now we can analyse the contributions of the energy density $\omega_f(:\rho:)^{(2)}$ by plugging in the ansatz for δa. We start with the expressions X_1 and Y_1. For $\tau > e$ we have

$$\begin{aligned}
X_1(\tau) = & \int_{-d}^{\tau} \mathrm{d}\tau' F(\tau') \cos(2\omega\tau') \left(2m^2 \delta a_{sd}' - \delta a_{sd}'''\right)(\tau') \\
& + \frac{\alpha_+ \lambda_+ (2m^2 + \lambda_+^2)}{\lambda_+^2 - 4\omega^2} \left(\lambda_+ \sin(\lambda_+ \tau + \phi_+) \cos(2\omega\tau) - 2\omega \cos(\lambda_+ \tau + \phi_+) \sin(2\omega\tau)\right) \\
& + 2\alpha_+ \omega (m^2 + 2\omega^2) \Bigg\{ \cos\phi_+ \left(\frac{\mathrm{Im}\hat{C}_+'(\lambda_+ + 2\omega)}{\lambda_+ + 2\omega} - \frac{\mathrm{Im}\hat{C}_+'(\lambda_+ - 2\omega)}{\lambda_+ - 2\omega} \right) \\
& + \sin\phi_+ \left(\frac{\mathrm{Re}\hat{C}_+'(\lambda_+ + 2\omega)}{\lambda_+ + 2\omega} - \frac{\mathrm{Re}\hat{C}_+'(\lambda_+ - 2\omega)}{\lambda_+ - 2\omega} \right) \Bigg\} \\
Y_1(\tau) = & \int_{-d}^{\tau} \mathrm{d}\tau' F(\tau') \sin(2\omega\tau') \left(2m^2 \delta a_{sd}' - \delta a_{sd}'''\right)(\tau') \\
& + \frac{\alpha_+ \lambda_+ (2m^2 + \lambda_+^2)}{\lambda_+^2 - 4\omega^2} \left(\lambda_+ \sin(\lambda_+ \tau + \phi_+) \sin(2\omega\tau) + 2\omega \cos(\lambda_+ \tau + \phi_+) \cos(2\omega\tau)\right) \\
& + 2\alpha_+ \omega (m^2 + 2\omega^2) \Bigg\{ -\cos\phi_+ \left(\frac{\mathrm{Re}\hat{C}_+'(\lambda_+ + 2\omega)}{\lambda_+ + 2\omega} + \frac{\mathrm{Re}\hat{C}_+'(\lambda_+ - 2\omega)}{\lambda_+ - 2\omega} \right) \\
& + \sin\phi_+ \left(\frac{\mathrm{Im}\hat{C}_+'(\lambda_+ + 2\omega)}{\lambda_+ + 2\omega} + \frac{\mathrm{Im}\hat{C}_+'(\lambda_+ - 2\omega)}{\lambda_+ - 2\omega} \right) \Bigg\},
\end{aligned}$$

and for $\tau < -e$:

$$\begin{aligned}
X_1(\tau) = &\int_\tau^d \mathrm{d}\tau'(F(\tau')-1)\cos(2\omega\tau')\left(2m^2\delta a'_{sd} - \delta a'''_{sd}\right)(\tau') \\
&+ \frac{\alpha_-\lambda_-(2m^2+\lambda_-^2)}{\lambda_-^2-4\omega^2}\left(\lambda_-\sin(\lambda_-\tau+\phi_-)\cos(2\omega\tau) - 2\omega\cos(\lambda_-\tau+\phi_-)\sin(2\omega\tau)\right) \\
&- 2\alpha_-\omega(m^2+2\omega^2)\Bigg\{\cos\phi_-\left(\frac{\mathrm{Im}\hat{C'_-}(\lambda_-+2\omega)}{\lambda_-+2\omega} - \frac{\mathrm{Im}\hat{C'_-}(\lambda_--2\omega)}{\lambda_--2\omega}\right) \\
&+ \sin\phi_-\left(\frac{\mathrm{Re}\hat{C'_-}(\lambda_-+2\omega)}{\lambda_-+2\omega} - \frac{\mathrm{Re}\hat{C'_-}(\lambda_--2\omega)}{\lambda_--2\omega}\right)\Bigg\} \\
Y_1(\tau) = &\int_\tau^d \mathrm{d}\tau'(F(\tau')-1)\sin(2\omega\tau')\left(2m^2\delta a'_{sd} - \delta a'''_{sd}\right)(\tau') \\
&+ \frac{\alpha_-\lambda_-(2m^2+\lambda_-^2)}{\lambda_-^2-4\omega^2}\left(\lambda_-\sin(\lambda_-\tau+\phi_-)\sin(2\omega\tau) + 2\omega\cos(\lambda_-\tau+\phi_-)\cos(2\omega\tau)\right) \\
&- 2\alpha_-\omega(m^2+2\omega^2)\Bigg\{-\cos\phi_-\left(\frac{\mathrm{Re}\hat{C'_-}(\lambda_-+2\omega)}{\lambda_-+2\omega} + \frac{\mathrm{Re}\hat{C'_-}(\lambda_--2\omega)}{\lambda_--2\omega}\right) \\
&+ \sin\phi_-\left(\frac{\mathrm{Im}\hat{C'_-}(\lambda_-+2\omega)}{\lambda_-+2\omega} + \frac{\mathrm{Im}\hat{C'_-}(\lambda_--2\omega)}{\lambda_--2\omega}\right)\Bigg\}
\end{aligned}$$

Similar identities can be established for X_2 and Y_2. Application of lemma 5.2.1 yields then for $\tau \to \pm\infty$:

$$\delta a'' \int_0^\infty \frac{\mathrm{d}k k^2}{16\pi^2\omega^3}\left(\cos(2\omega\tau)X_1 + \sin(2\omega\tau)Y_1\right) \to -\frac{\alpha_\pm^2\lambda_\pm^4(2m^2+\lambda_\pm^2)\sin^2(\lambda_\pm\tau+\phi_\pm)}{16\pi^2} I_1(m^{-1}\lambda_\pm)$$

and

$$-\delta a' \int_0^\infty \frac{\mathrm{d}k k^2}{16\pi^2\omega^3}\left(\cos(2\omega\tau)X_2 + \sin(2\omega\tau)Y_2\right) \to -\frac{\alpha_\pm^2\lambda_\pm^4(2m^2+\lambda_\pm^2)\cos^2(\lambda_\pm\tau+\phi_\pm)}{16\pi^2} I_1(m^{-1}\lambda_\pm)$$

Now we turn to the last remaining contribution[9] for $R^{(2)}[f, \delta a']$,

$$\int_0^\infty \frac{\mathrm{d}k k^2}{8\pi^2\omega}\left(\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' f A\right)^2 + \dots\right).$$

As already shown in the last subsection, we may interchange the limit in τ and the integration

[9]We left out the explicit discussion of the contributions of $R^{(2)}[f, \delta a']$ which vanish for $\tau \to \pm\infty$. However, this is a rather easy task.

in k. We obtain for $\tau \gtrless \pm e$:

$$\begin{aligned}
\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' f A\right) =& Y_1^{\pm}(\tau) + \alpha_{\pm} Z_1^{\pm} \\
&+ \frac{\alpha_{\pm}\lambda_{\pm}(2m^2 + \lambda_{\pm}^2)}{2\omega(\lambda_{\pm}^2 - 4\omega^2)} \big(\lambda_{\pm} \sin(\lambda_{\pm}\tau + \phi_{\pm}) \cos(2\omega\tau) \\
&- 2\omega \cos(\lambda_{\pm}\tau + \phi_{\pm}) \sin(2\omega\tau)\big) \\
\left(\frac{Y_1}{2\omega} - \int \mathrm{d}\tau' f B\right) =& Y_2^{\pm}(\tau) + \alpha_{\pm} Z_2^{\pm} \\
&+ \frac{\alpha_{\pm}\lambda_{\pm}(2m^2 + \lambda_{\pm}^2)}{2\omega(\lambda_{\pm}^2 - 4\omega^2)} \big(\lambda_{\pm} \sin(\lambda_{\pm}\tau + \phi_{\pm}) \sin(2\omega\tau) \\
&+ 2\omega \cos(\lambda_{\pm}\tau + \phi_{\pm}) \cos(2\omega\tau)\big)
\end{aligned}$$

The functions $Y_i^{\pm}(\tau)$ are obtained from the definitions of the $Y_i^{\pm}$ in proposition 5.2.4 by replacing the respective infinite integration limit by τ. It follows

$$\begin{aligned}
&\int_0^{\infty} \frac{\mathrm{d}k k^2}{8\pi^2\omega} \left(\left(\frac{X_1}{2\omega} - \int \mathrm{d}\tau' f A\right)^2 + \left(\frac{Y_1}{2\omega} - \int \mathrm{d}\tau' f B\right)^2 \right) \\
\to& U^{\pm} + \frac{\alpha_{\pm}^2 \lambda_{\pm}^4 (2m^2 + \lambda^2)^2}{32\pi^2} I_2(m^{-1}\lambda_{\pm}) - \frac{\alpha_{\pm}^2 \lambda_{\pm}^2 (2m^2 + \lambda^2)^2 \cos^2(\lambda_{\pm}\tau + \phi_{\pm})}{32\pi^2} I_1(m^{-1}\lambda_{\pm}).
\end{aligned}$$

Finally, the local terms appearing on the right hand side of (5.17) have the following asymptotes for $\tau \to \pm\infty$:

$$\begin{aligned}
(\delta a')^2 &\to \alpha_{\pm}^2 \lambda_{\pm}^2 \cos^2(\lambda_{\pm}\tau + \phi_{\pm}) \\
\delta a' \delta a''' - \frac{1}{2}(\delta a'')^2 &\to -\frac{\alpha_{\pm}^2 \lambda_{\pm}^4}{2} (\cos^2(\lambda_{\pm}\tau + \phi_{\pm}) + 1).
\end{aligned}$$

Now we have analysed all contributions of (5.17) in their asymptotic limit. Comparison of the coefficients for the oscillating part ($\propto \cos^2(\lambda_{\pm}\tau + \phi_{\pm})$) leads to (5.27); from the remaining constant term we get (5.28). (5.27) determines the possible values for $\lambda_{\pm}$ as a function of R_1 and R_2, whereas (5.28) is a condition on the amplitude $\alpha_{\pm}$. I_1 and I_2 can be explicitly calculated:

$$\begin{aligned}
I_1(x_{\pm}) &= \frac{1}{x_{\pm}^2} \left(\frac{\sqrt{4 - x^2}}{|x_{\pm}|} \arccos \frac{\sqrt{4 - x_{\pm}^2}}{2} - 1 \right) \\
I_2(x_{\pm}) &= -\frac{3}{2x_{\pm}^4} + \frac{(6 - x_{\pm}^2)(\pi - 2 \arcsin \frac{\sqrt{4 - x_{\pm}^2}}{2})}{2|x_{\pm}|^5 \sqrt{4 - x_{\pm}^2}}
\end{aligned}$$

Since solutions of (5.17) possess the symmetry $\delta a \to -\delta a$, we have to require that (5.28) must have exactly two symmetric solutions for $\alpha_{\pm}$. Since it is a quadratic equation in α of the general form

$$p\alpha^2 + q\alpha + r = 0,$$

this implies $q = 0$, which can only be true if (5.29) holds, which in turn leads to the consistency equation (5.29) for the phases $\phi_{\pm}$. □

By inspecting (5.28) and taking the definitions of $I_1(x)$ and $I_2(x)$ into account, we arrive at the following

Proposition 5.2.5. *A necessary condition on the renormalisation constants for the existence of solutions of (5.17) which are asymptotically oscillatory with frequency λ is*

$$0 > \left(\frac{\lambda^4(2m^2+\lambda^2)}{32\pi^2 m^4} \left((2m^2+\lambda^2) I_2 m^2(\lambda/m) - 2m^2 I_1(\lambda/m)\right) - \frac{R_2\lambda^4}{2} \right), \tag{5.30}$$

where $\lambda \in (0, 2m)$ is determined by (R_1, R_2) via (5.27). This implies in particular $R_2 > 0$.

Proof. (5.30) is obviously necessary in order for (5.28) to have real solutions for $\alpha_\pm$ under the condition (5.29). However, the expression

$$\left(2+x^2\right) I_2(x) - 2I_1(x)$$

can be checked to be positive for $x \in (0, 2)$ and thus R_2 must be positive. □

The existence of oscillatory asymptotics can in principle also depend on the test function f inducing the SLE as well as on the initial conditions of δa, since the functions $Z_i^\pm$ depend on the solution δa via $\phi_\pm$. In other words, if a certain choice of (R_1, R_2) enables an oscillatory stable solution for given f and initial conditions for δa, there might exist $\tilde{f}$ and/or other initial conditions for δa which furnish only unstable solutions.

We have thus shown that Minkowski space is "weakly stable" under homogenous and isotropic perturbations induced by a state of low energy, by proving the compatibility of asymptotically oscillatory solutions with the semiclassical Friedmann equation in perturbation theory. This notion of stability is however quite weak, since the excitation δa of Minkowski space described by a nontrivial solution δa will persist forever. An interesting question, which is however beyond the scope of our perturbative ansatz over Minkowski background, would be the following: Do "unstable solutions" solutions δa move to some "attracting" scale factor? In order to answer this question, one could perform the perturbation ansatz for other backgrounds like de Sitter space.

5.3. Numerical Solution

As we already observed above, we may derive from (5.17) the following linear ordinary homogenous integro differential equation for $\delta a'$:

$$0 = 2m^2 R_1 \delta a'' + R_2 \delta a'''' + \mathcal{H}[f, \delta a', ..., \delta a^{(6)}], \tag{5.31}$$

where we defined, with (5.24),

$$\begin{aligned}\mathcal{H} \doteq&\mathcal{F}+\mathcal{G}\\ =&\frac{1}{32\pi^2}\int_d^\tau \mathrm{d}\tau' Z(m,\tau'-\tau)(4m^2\delta a^{(4)}-\delta a^{(6)}-4m^4\delta a'')\\ -&\frac{1}{32\pi^2}\int_{-d}^d \mathrm{d}\tau' Z(m,\tau'-\tau)\Big(\delta a^{(6)}F+5f\delta a^{(5)}+\delta a^{(4)}(10f'-4m^2F)\\ &\qquad+\delta a'''(10f''-12m^2f)+\delta a''(5f'''-12m^2f'+4m^4F)\\ &\qquad+\delta a'(f^{(4)}-4m^2f''+4m^4f)\Big)\end{aligned}$$

and the kernel $Z(m,x)$ is given by (5.23). Equation (5.31) is of third order in $\delta a'$, so it requires three initial conditions. For the numerical approach we choose the following method: We discretise the symmetric time domain $D \doteq (-T,T)$ into n intervals. Call the corresponding supporting points $\tau(i)$, with $i=0,...,n$. Next we approximate $\delta a'$ by a piecewise polynomial of degree five (also known as *spline*), i.e. for $\tau\in[\tau(i),\tau(i+1)]$ and $i=0,...,n-1$ we set

$$\delta a_s'(\tau) \doteq \sum_{\kappa=0}^{5} a_{i,\kappa}(\tau-\tau(i))^\kappa.$$

In order for $\delta a_s'$ to be four times continously differentiable, the following constraints must hold for the coefficients $a_{i,\kappa}$ for $i=0,...,n-2$:

$$\begin{aligned}\sum_{\kappa=0}^{5} a_{i,\kappa}(\tau(i+1)-\tau(i))^\kappa &= a_{i+1,0}\\ \sum_{\kappa=1}^{5} \kappa a_{i,\kappa}\left(\tau(i+1)-\tau(i)\right)^{(\kappa-1)} &= a_{i+1,1}\\ \sum_{\kappa=2}^{5} \kappa(\kappa-1) a_{i,\kappa}\left(\tau(i+1)-\tau(i)\right)^{(\kappa-2)} &= 2a_{i+1,2}\\ \sum_{\kappa=3}^{5} \kappa(\kappa-1)(\kappa-2) a_{i,\kappa}\left(\tau(i+1)-\tau(i)\right)^{(\kappa-3)} &= 6a_{i+1,3}\\ \sum_{\kappa=4}^{5} \kappa(\kappa-1)(\kappa-2)(\kappa-3) a_{i,\kappa}\left(\tau(i+1)-\tau(i)\right)^{(\kappa-4)} &= 24a_{i+1,4}\end{aligned} \tag{5.32}$$

Inserting this approximation for $\delta a'$ into (5.31), the right hand side becomes a function of the coefficients $a_{i,\kappa}$, with $i=0,...,n-1$ and $\kappa=0,...,5$, subject to the constraints (5.32). Now $\delta a'$ is an exact solution of (5.31) on $(-T,T)$ if and only if[10]

$$0=\int_{-T}^{T}\mathrm{d}\tau\left(2m^2R_1\delta a''+R_2\delta a^{(4)}+\mathcal{H}[f,\delta a']\right)^2.$$

[10] We could choose any strictly positive function of the right hand side x of (5.31); the most simple choice for our subsequent numerical treatment is however x^2.

We may thus obtain a approximate solution for $\delta a'$ by searching for the global minimum of the function

$$M(\{a_{i,\kappa}\}) \doteq \int_{-T}^{T} \mathrm{d}\tau \left(2m^2 R_1 \delta a_s'' + R_2 \delta a_s^{(4)} + \mathcal{H}[f, \delta a_s']\right)^2 \tag{5.33}$$

under the constraints (5.32) and three additional initial and/or boundary conditions for $\delta a_s'$.

We implemented this approach using Mathematica. The normalised test function f was chosen to be a piecewise polynomial of degree five, being four times continously differentiable and vanishing together with its first four derivatives at $t = \pm 2\delta$:

$$f(t) = \begin{cases} \frac{(t+2\delta)^5}{16\delta^6} & -2\delta \le t \le -\delta \\ \frac{1}{16\delta} + \frac{5(t+\delta)}{16\delta^2} + \frac{5(t+\delta)^2}{8\delta^3} + \frac{5(t+\delta)^3}{8\delta^4} + \frac{5(t+\delta)^4}{16\delta^5} - \frac{15(t+\delta)^5}{16\delta^6} & -\delta \le t \le 0 \\ \frac{1}{16\delta} + \frac{5(\delta-t)}{16\delta^2} + \frac{5(\delta-t)^2}{8\delta^3} + \frac{5(\delta-t)^3}{8\delta^4} + \frac{5(\delta-t)^4}{16\delta^5} - \frac{15(\delta-t)^5}{16\delta^6} & 0 \le t \le \delta \\ \frac{(2\delta-t)^5}{16\delta^6} & \delta \le t \le 2\delta \end{cases}$$

The supporting points were chosen as follows:

$$\tau(i) = \begin{cases} -2\delta - h\left(\frac{n}{2} - 2\right) + ih & 0 \le i \le \frac{n}{2} - 2 \\ -2\delta + \delta\left(i - \frac{n}{2} + 2\right) & \frac{n}{2} - 2 < i < \frac{n}{2} + 2 \\ 2\delta - h\left(\frac{n}{2} + 2\right) + ih & \frac{n}{2} + 2 \le i \le n \end{cases} \tag{5.34}$$

That is, the step width is δ within the support of f and h elsewhere. The functional M is calculated by Mathematica's numerical integration routine for a fixed choice of m, supporting points $\tau(i)$ and the above test function f, whereas the renormalisation constants R_1 and R_2 remain free parameters of M. Being a positive polynomial of second degree in the variables $a_{i,\kappa}$, M always posseses a global minimum under the linear constraints (5.32) plus the three initial conditions for $\delta a_s'$. After choosing numerical values for R_1 and R_2, the minimisation of M can be performed using Mathematica's built in function "Minimize".

In the following we present some plots for $\delta a'$ obtained by the above numerical method. The time t and the mass m refer to a fixed choice of a lengthscale. The first plot 5.2 suggests that the numerical results become asymptotically reliable if the ratio of the step width h and the expected oscillation period[11] $T = \frac{2\pi}{mx}$ is smaller than $\approx 1/6$. For the choice of m, R_1 and R_2 made for plot 5.2, equation (5.27) has the solutions $x_1 = 0.91$ and $x_2 = 1.78$. Only the first one satisfies the constraint (5.30), and the corresponding oscillation period is $T = \frac{2\pi}{mx_1} = 3.45$. The solid curve shows exactly this period and we conjecture that this plot shows the oscillatory regime. Note that by choosing an appropriate ratio of the initial conditions we may adjust the solution $\delta a'$ in such a way that it asymptotically oscillates about zero. This is necessary if $\delta a'$ shall also be a solution to the original equation (5.17). In figure 5.3 we compared the solutions for a fixed choice of m, R_1 and R_2, when varying the smearing width δ. It seems that the ratio of mass and smearing width has an important influence on the solution. Namely, if the smearing width becomes larger for fixed mass, the contributions of the functional $\mathcal{H}$ in (5.31) will become smaller and the character of the dynamics of (5.31) will be dominated by the choice of R_1 and R_2. The nonlocal contribution $\mathcal{H}$ may thus be seen as the quantum contribution to the Friedmann equation, since only there the form of the test function f enters. A similar interpretation applies

[11] Recall that x is predicted by formula (5.27).

to figure 5.4. There we held the mass and the test function fixed, while the magnitude of R_1 and R_2 was varied in such a way that the asymptotic frequency of the solution stays the same. Here we see that increasing R_1 and R_2 will dominate the influence of the test function f at some stage. The solution will then look like a pure sinusoidal curve. This can already be inferred from equation (5.31). The term $2m^2 R_1 \delta a'' + R_2 \delta a''''$ will then dominate the functional $\mathcal{H}$ and (5.31) gets the character of the equation of motion of the harmonic oscillator with frequency $\lambda = \sqrt{\frac{2m^2 R_1}{R_2}}$.

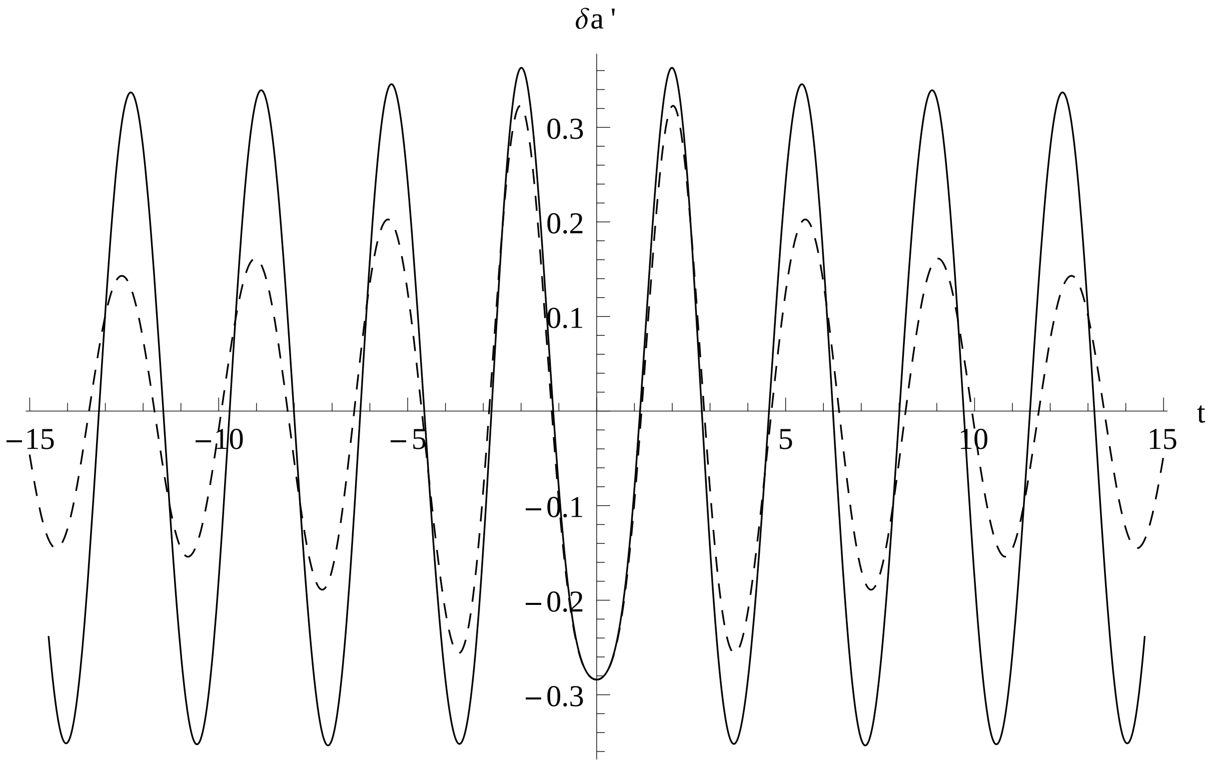

Figure 5.2.: Numerical solution for $\delta a'$. We chose the parameters $\delta = 1,\ m = 2,\ R_1 = 0.001,\ R_2 = 0.008$ and initial conditions $\delta a'(0) = -0.284, \delta a''(0) = 0$ and $\delta a'''(0) = 0.1$. The curves were obtained for step widths $h = 1$ (dashed) and $h = 0.5$ (solid) and the minimisation values M_{min} are 5.28×10^{-5} and 4.19×10^{-5}, respectively.

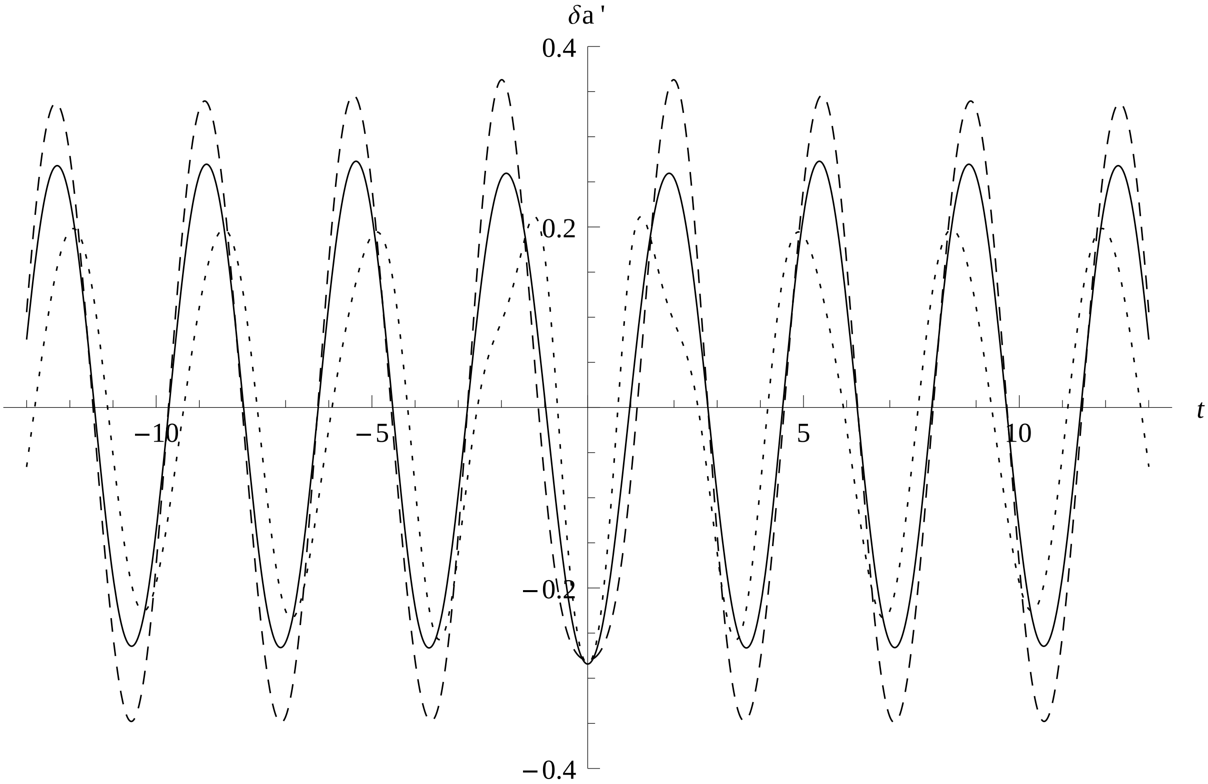

Figure 5.3.: Numerical solutions for $\delta a'$. We chose the parameters $h = 0.5,\ m = 2,\ R_1 = 0.001,\ R_2 = 0.008$. Solid: $\delta = 2, \delta a'(0) = -0.285,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.379$. Dashed: $\delta = 1, \delta a'(0) = -0.285,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.1$. Dotted: $\delta = 0.5, \delta a'(0) = -0.285,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.580$.

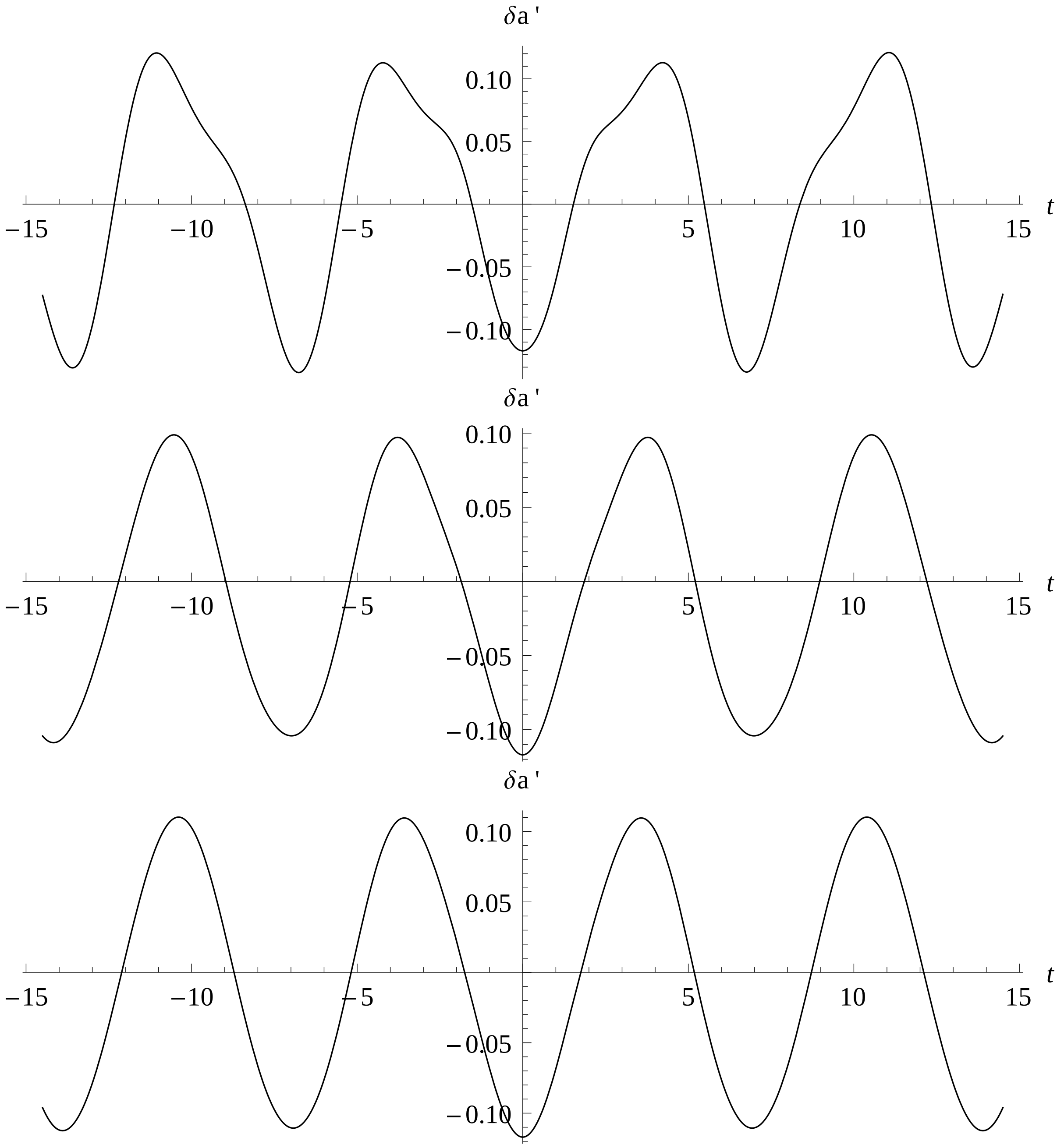

Figure 5.4.: Numerical solutions for $\delta a'$. For all three plots we chose the parameters $h = 0.5,\ m = 1$ and $\delta = 1$. The first plot was obtained for $R_1 = 0.0015,\ R_2 = 0.0092$ and the initial conditions $\delta a'(0) = -0.117,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.057$. The second plot shows the solution for $R_1 = 0.007,\ R_2 = 0.023$ and $\delta a'(0) = -0.117,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.056$. For the third plot we chose $R_1 = 0.015,\ R_2 = 0.042$ and initial conditions $\delta a'(0) = -0.117,\ \delta a''(0) = 0,\ \delta a'''(0) = 0.056$.

A. Calculation of $[\mathcal{R}\mathcal{G}_1^s]_\tau$ and $[P_x\mathcal{G}_1]$ on Spatially Flat FRW Spacetime

In this appendix we will work out the steps already discussed in section 2.4.2. This will lead to an explicit expression of the Hadamard parametrix and its derivatives in terms of the scale factor and its derivatives. We use the notation introduced in section 2.4.2. Lets start with the calculation of $[P_x\mathcal{G}_1]$. In conformal time τ we have the following relations:

$$P_x \doteq \Box_x + m^2 = \frac{1}{C}\partial_{\tau\tau} + \frac{C'}{C^2}\partial_\eta - \frac{1}{C}\triangle + m^2$$
$$R = \frac{3C''}{C^2} - \frac{3C'^2}{2C^3}.$$

We introduce the notation

$$G_1^s \doteq 4\pi^2\sqrt{C(\tau)C(\tau')}\mathcal{G}_1^s.$$

To first order in the signed squared geodesic distance σ, G_1^s is given by

$$\tilde{G}_1^s(\tau,\tau',r) = \left(\frac{\tilde{q}}{\tilde{\sigma}_+}\right)^s + \tilde{R}_\Delta + \frac{1}{L^2}\left(\overset{o}{v}_0 + \overset{o}{v}_1\frac{\tilde{\rho}}{L^2} + \tilde{\rho}^2\tilde{R}_v\right)\left(\log\left(\frac{\tilde{\sigma}_+}{L^2}\right)\right)^s. \tag{A.1}$$

For the functions $\tilde{R}_\Delta$ and $\tilde{q}$ we have the following expansions [Sch10]:

$$\begin{aligned}\tilde{q}(\tau,\tau',r) =& C(\tau') + \frac{C'(\tau')}{2}(\tau-\tau') + \left(\frac{C''(\tau')}{6} - \frac{C'(\tau')^2}{48C(\tau')}\right)(\tau-\tau')^2 \\ &+ \frac{C'(\tau')^2}{48C(\tau')}r^2 + \ldots\end{aligned} \tag{A.2}$$

$$\begin{aligned}&\tilde{R}_\Delta(\tau,\tau',r) = \\ &\frac{C(\tau')R(\tau')}{72} + \frac{(CR)'(\tau')}{144}(\tau-\tau') \\ &+ \left((CR)''(\tau') - \frac{3}{4}\left(\frac{C''(\tau')}{C(\tau')}\right)^2 + \frac{C''(\tau')C'(\tau')^2}{C^3(\tau')} - \frac{3}{8}\left(\frac{C'(\tau')}{C(\tau')}\right)^4\right)\frac{(\tau-\tau')^2}{480} \\ &+ \left(\frac{15}{8}\left(\frac{C'(\tau')}{C(\tau')}\right)^4 - \frac{11}{3}\frac{C''(\tau')C'(\tau')^2}{C^3(\tau')} + \frac{3}{4}\left(\frac{C''(\tau')}{C(\tau')}\right)^2 + \frac{C'(\tau')C'''(\tau')}{C^2(\tau')}\right)\frac{r^2}{480} \\ &+ \ldots\end{aligned} \tag{A.3}$$

They allow for the calculation of coincidence limits of derivatives of $\tilde{q}$ and $\tilde{R}_\Delta$ w.r.t. τ and r. Furthermore, from (2.33) and (2.34) one can deduce the following coincidence limits:

$$[\overset{o}{v}_0]_\tau = \frac{L^2}{4}Q_{m,\xi} \qquad [\partial_\tau \overset{o}{v}_0]_\tau = \frac{L^2}{8}Q'_{m,\xi} \tag{A.4}$$
$$[\partial_{\tau\tau} \overset{o}{v}_0]_\tau = \frac{L^2}{12}Q''_{m,\xi} \qquad [\overset{o}{v}_1]_\tau = \frac{L^4}{32}(Q''_{m,\xi}/3 + Q^2_{m,\xi})$$

For the subsequent calculation we make use of the identity

$$P_x \frac{f(\tau,\mathbf{x},\tau',\mathbf{x}')}{\sqrt{C(\tau)C(\tau')}} = \frac{1}{\sqrt{C(\tau)C(\tau')}C(\tau)}\left(\overset{o}{\Box}_x + Q(\tau)\right) f(\tau,\mathbf{x},\tau',\mathbf{x}'),$$

with $\overset{o}{\Box}_x \doteq \partial_\tau^2 - \triangle_{\mathbf{x}}$. Furthermore, there holds

$$[(\overset{o}{\Box}_x + Q(\tau))(\overset{o}{v}_0 + \overset{o}{v}_1\rho)] = 0. \tag{A.5}$$

By means of a rather lengthy calculation, using the information about temporal derivatives and restrictions of the involved distributions according to lemmas 2.4.3 and 2.4.4 and the distributional identities

$$\overset{o}{\Box}_x \left(\frac{\tilde{q}}{\tilde{\sigma}_+}\right)^s = 0, \qquad \left[\overset{o}{\Box}_x \left(\log\frac{\sigma_+}{q}\right)^s\right]_\tau = -\frac{4}{r_+^2},$$

one can explicitly convince oneself that all singular terms that appear in $P_x\mathcal{G}_1^s$ cancel each other (this is a priori clear by construction of $\mathcal{G}_k$). The finite contribution which survives the coincidence limit $x \to x'$ is

$$\begin{aligned}
[P_x\mathcal{G}_1^s] &= \frac{1}{4\pi^2C^2}\left[\left(\overset{o}{\Box}_x + Q(\tau)\right)\left(R_\Delta + (\overset{o}{v}_0 + \overset{o}{v}_1\rho + O(\rho^2))\log q\right) - 12\overset{o}{v}_1\right] \\
&= \frac{1}{4\pi^2C^2}\Bigg(\left[\overset{o}{v}_0\right]\left(\left[\frac{q''q - q'^2}{q^2}\right] - \left[\frac{q\partial_r^2 q - (\partial_r q)^2}{q^2} + \frac{2\partial_r q}{rq}\right]\right) \\
&\qquad + 2\left[\partial_\tau\overset{o}{v}_0\right]\left[\frac{q'}{q}\right] + \left[\left(\partial_\tau^2 - \partial_r^2 - \frac{2}{r}\partial_r + Q\right)R_\Delta\right] - 12\left[\overset{o}{v}_1\right]\Bigg).
\end{aligned}$$

It can be written as function of the scale factor and the curvature scalar:

$$\begin{aligned}
[P_x\mathcal{G}_1^s] &= \frac{1}{4\pi^2}\left(\frac{\Box R}{40} - \frac{3}{8}m^4 + \frac{1}{8}m^2R - \frac{19}{640}\frac{C'^4}{C^6} + \frac{C'^2C''}{10C^5} - \frac{3}{32}\frac{C''^2}{C^4}\right) \\
&= \frac{1}{4\pi^2}\left(\frac{9}{40}\dot{H}^2 + \frac{3}{20}\dddot{H} + \frac{7}{20}H^2\dot{H} + \frac{21}{20}H\ddot{H} - \frac{29}{20}H^4 - \frac{3}{8}m^4 + \frac{m^2}{4}(6H^2 + 3\dot{H})\right)
\end{aligned} \tag{A.6}$$

This expression agrees with the one given in Appendix A of [DFP08] for $\xi = 0$, taking into account their sign convention $(-,+,+,+)$ and the relation $[v_1] = -\frac{1}{3}[P_x\mathcal{G}_1^s]$, where $[v_1]$ is defined in this reference. For the special case $m^2 = 2H^2$ and $C = (H\tau)^{-2}$ we obtain

$$[P_x\mathcal{G}_1^s] = \frac{1}{4\pi^2}\frac{1}{20}H^4. \tag{A.7}$$

Now we turn to the calculation of $[\mathcal{R}\mathcal{G}_1^s]_\tau$, which is the "counterterm" for the unrenormalised point split energy density, $[\mathcal{R}\mathcal{W}_2^s]_\tau$, restricted to $\tau = \tau'$. We will derive an expression for general masses m but minimal coupling $\xi = 0$. Expressing the differential operator $\mathcal{R}$ in conformal coordinates we obtain

$$\begin{aligned}
[\mathcal{R}\mathcal{G}_1^s]_\tau =& \frac{1}{2C}\left[\left(\partial_{\tau\tau'} - \partial_r^2 - \frac{2\partial_r}{r} + Cm^2\right)\tilde{\mathcal{G}}_1^s\right]_\tau \\
=& \left(\frac{C'^2}{8C^4} + \frac{m^2}{2C} - \frac{1}{2C^2}\left(\partial_{rr} + \frac{2\partial_r}{r}\right)\right)[\tilde{G}_1^s]_\tau - \frac{C'}{2C^3}[\partial_\tau \tilde{G}_1^s]_\tau \\
&+ \frac{1}{2C^2}[\partial_{\tau\tau'}\tilde{G}_1^s]_\tau.
\end{aligned} \tag{A.8}$$

The ingredients for the calculation are again the lemmas 2.4.3 and 2.4.4, the knowledge of R_Δ and q (given in (A.3) and (A.2)) and the coincidence limits (A.4). We also need to take derivatives w.r.t. τ', which we obtain in the following way: Differentiating (2.33) with respect to τ' yields

$$(\tau - \tau')\partial_{\tau'\tau}\overset{o}{v}_0 - \partial_\tau \overset{o}{v}_0 + \partial_{\tau'}\overset{o}{v}_0 = 0,$$

which gives the result $[\partial_\tau \overset{o}{v}_0] = [\partial_{\tau'}\overset{o}{v}_0]$. Similarly, one deduces the relation $[\partial_{\tau\tau'}\overset{o}{v}_0] = \frac{1}{2}[\partial_\tau^2 \overset{o}{v}_0]$. We are now in the position to compute the contributions appearing in (A.8):

$$\begin{aligned}
[G_1]_\tau &= \frac{1}{r_+^2} + [R_\Delta]_\tau + [\overset{o}{v}_0]_\tau(\log[q]_\tau + \mathrm{lo}_0) \\
[\triangle G_1]_\tau &= \frac{2}{r_+^4} + [\triangle R_\Delta]_\tau + \frac{2[\overset{o}{v}_0]}{r_+^2} + [\overset{o}{v}_0][\triangle \log q]_\tau + [\overset{o}{v}_1](6(\log[q]_\tau + \mathrm{lo}_0) + 10) \\
[\partial_\tau G_1]_\tau &= [\partial_\tau R_\Delta]_\tau + [\partial_\tau \overset{o}{v}_0]_\tau(\log[q]_\tau + \mathrm{lo}_0) + [\overset{o}{v}_0]\left[\frac{\partial_\tau q}{q}\right]_\tau \\
[\partial_{\tau\tau'}G_1]_\tau &= -\frac{2}{r_+^4} + [\partial_{\tau\tau'}R_\Delta]_\tau + \frac{1}{2}[\partial_\tau^2 \overset{o}{v}_0](\log[q]_\tau + \mathrm{lo}_0) + 2[\partial_\tau \overset{o}{v}_0]\left[\frac{\partial_\tau q}{q}\right]_\tau \\
&\quad + [\overset{o}{v}_0]\left([\partial_{\tau\tau'}\log q]_\tau + \frac{2}{r_+^2}\right) + 2[\overset{o}{v}_1](\log[q] + \mathrm{lo}_0 + 1)
\end{aligned}$$

The Laplacian $\triangle$ refers to $\mathbf{r} = |\mathbf{x} - \mathbf{x}'|$ and acts on spherically symmetric functions as $\partial_{rr} + 2\partial_r/r$. The result, expressed in terms of the curvature scalar and the scale factor, reads

$$\begin{aligned}
[\mathcal{R}\mathcal{G}_1^s]_\tau = \frac{1}{4\pi^2}\Bigg\{ &- \frac{2}{C^2 r_+^4} + \left(\frac{C'^2}{8C^4} + \frac{m^2}{2C}\right)\frac{1}{r_+^2} \\
&+ \left(\frac{m^4}{16} - \frac{m^2 C'^2}{32C^3} + \frac{9C'^4}{256C^6} - \frac{C'^2 C''}{16C^5} - \frac{C''^2}{64C^4} + \frac{C'C'''}{32C^4}\right)(\mathrm{lo}_0 + \log C) \\
&+ \frac{\Box R}{120} + \frac{m^2 C''}{8C^2} - \frac{19C''^2}{960C^4} - \frac{11m^2 C'^2}{96C^3} - \frac{33C'^4}{640C^6} - \frac{11C'''C'}{480C^4} \\
&+ \frac{53C'^2C''}{480C^5} - \frac{m^4}{8}\Bigg\}.
\end{aligned}$$

We reexpress the result in terms of the Hubble function $H \doteq \frac{\dot{a}}{a}$:

$$\begin{aligned}[\mathcal{R}\mathcal{G}_1^s]_\tau = \frac{1}{4\pi^2}\Bigg\{ & -\frac{2}{a^4 r_+^4} + \frac{H^2+m^2}{2a^2}\frac{1}{r_+^2} \\ & + \left(\frac{m^4}{16} - \frac{m^2 H^2}{8} + \frac{\ddot{H}H}{8} + \frac{3H^2\dot{H}}{8} - \frac{\dot{H}^2}{16}\right)(\mathrm{lo}_0 + \log a^2) \\ & + \frac{\Box R}{120} + m^2\left(\frac{7}{24}H^2 + \frac{1}{4}\dot{H}\right) - \frac{m^4}{8} + \frac{H^4}{80} - \frac{11H\ddot{H}}{120} \\ & - \frac{61H^2\dot{H}}{120} - \frac{19\dot{H}^2}{240}\Bigg\}\end{aligned}$$

For the scale factor $a(t) = \exp(Ht)$ and arbitrary mass m this reduces to

$$\begin{aligned}[\mathcal{R}\mathcal{G}_1^s]_\tau = \frac{1}{4\pi^2}\Bigg\{ & -\frac{2}{a^4 r_+^4} + \frac{H^2+m^2}{2a^2}\frac{1}{r_+^2} + \left(\frac{m^4}{16} - \frac{m^2H^2}{8}\right)(\mathrm{lo}_0 + \log a^2) \\ & + \frac{7m^2H^2}{24} - \frac{m^4}{8} + \frac{H^4}{80}\Bigg\}.\end{aligned}$$

Note that for the cases $m^2 = 0$ and $m^2 = 2H^2$ the logarithmic singularity is absent.

B. Renormalisation Freedom of $\omega(:\rho:)$ on Spatially Flat FRW Spacetime

The $tt-$ components of the covariantly conserved tensors I_{ab} and J_{ab} are readily calculated. Using

$$\begin{aligned}
R &= 6(2H^2+\dot{H}) \\
\ddot{R} &= \Box R - 3H\dot{R} \\
R_{tt} &= 3\left(H^2+\dot{H}\right) \\
R_{\mu\nu}R^{\mu\nu} &= 12\left(3H^4+3H^2\dot{H}+\dot{H}^2\right) \\
R^{\mu\nu}R_{\mu t\nu t} &= 9H^4+3\dot{H}^2+12H^2\dot{H} \\
(\partial_t)^a(\partial_t)^b\Box R_{ab} &= \frac{1}{2}\Box R - 6\dot{H}^2 - 6\ddot{H}H - 30\dot{H}H^2
\end{aligned}$$

it follows

$$\begin{aligned}
I_{tt} &= 18\dot{H}^2 - 36\ddot{H}H - 108\dot{H}H^2 \\
J_{tt} &= 6\dot{H}^2 - 12\ddot{H}H - 36\dot{H}H^2
\end{aligned}$$

We see that I_{tt} and J_{tt} are not independent (this could have been inferred by more abstract arguments) and thus our renormalisation freedom is given by

$$C_{tt} = Am^4 - 3Bm^2H^2 + (3\Gamma+\Delta)(6\dot{H}^2 - 12\ddot{H}H - 36\dot{H}H^2) \tag{B.1}$$

In the Sitter space ($H = \text{const}$) there obviously holds $I_{tt} = J_{tt} = 0$. Thus, the remaining renormalisation freedom for the energy density consists in the $tt-$ components of g_{ab} and G_{ab}.

Conclusions

In this thesis we showed that is is possible to reach at various results of interest for states of low energy for the minimally coupled Klein Gordon field on cosmological spacetimes in a rigorous manner. To begin with, we introduced the necessary concepts of quantum field theory on curved spacetimes with focus on the algebraic approach. We then discussed how expectation values of observables involving products of fields at the same point can be defined. We argued that this is always possible if the state ω is of Hadamard type. For concrete calculations, one has therefore to compute the Hadamard singularity $\mathcal{G}$, which is needed to define the expectation values of the above observables via a point splitting procedure. Following this approach, we calculated the expecation value of the energy density in SLE's on spatially flat de Sitter space. To this end, we used a calculational method developed in [Sch10], which permits to determine the Hadamard singularity $\mathcal{G}$ on spatially flat FRW spacetimes and to convert it into an integral over momentum space. Similar calculations in this realm are usually done using adiabatic states together with a subtraction recipe called adiabatic renormalisation. In contrast, our results were obtained in a conceptually clean way. Moreover, SLE's are an improved version of adiabatic states and can be controlled by the choice of an appropriate test function, which permits in principle to model many physical situations.

Inspired by the results obtained for the energy density, we could prove that SLE's converge to the distinguished Bunch Davies state on de Sitter space if the support of the smearing function is shifted to the infinite past. We then generalised this result to the class of asymptotic de Sitter spacetimes, where an analogon of the Bunch Davies state exists due to the geometric structure of a past cosmological horizon. However, in order to do so, we had to impose conditions on the asymptotic behaviour of the scale factor and its derivatives which are stricter than those which are necessary for the existence of the preferred state $\lambda_{\mathcal{M}}$. Our convergence result allows for an interpretation of $\lambda_{\mathcal{M}}$ as state of low energy in the infinite past, independently of the form of the smearing function. This is reminiscent to the situation in Minkowski space, where the distinguished state (the Minkowski vacuum) is also a state of low energy for all smearing functions f.

Finally we discussed the problem of semiclassical backreaction for SLE's on FRW spacetimes. Generally, one obstacle for solving the semiclassical Einstein equation is the necessity of fixing a quantum state independently of the spacetime. However, SLE's have the big advantage that they can be specified without having explicit knowledge of the scale factor, which allows for a direct formulation of the semiclassical Friedmann equation by using an SLE as fixed reference state. Since the resulting functional equation for the scale factor is very complicated, we applied this idea to the case of small perturbations δa of the scale factor of a fixed FRW background, for which we chose Minkowski space. In order to obtain an equation of motion for δa, we had to calculate the point split regularised energy density and the renormalisation freedom perturbatively up to second order. The corresponding perturbative semiclassical Friedmann equation was then analysed with respect to the asymptotic behaviour of solutions for large times. It turned out that stable solutions for δa are not possible in the strict sense that the perturbations and its derivatives converge to zero for times which are far away from the support of the test function. However, oscillatory asymptotics are possible if certain constraints on the renormalisation parameters are fulfilled. In this case, the oscillation frequencies are constrained to lie in the interval $(0, 2m)$ and depend on the renormalisation constants. We also demonstrated a numerical method of solving for δa. With these results in mind it seems that, in order to answer the question of semiclassical stability of Minkowski space, it is presumably too restrictive to consider homogenous and isotropic perturbations, since such perturbations cannot "spread out" by construction. Nevertheless, we could show that the problem of backreaction can be treated in a rigorous manner by using SLE's. It would be interesting to pursue this problem by giving up at least the assumption of homogeneity. The corresponding quantum states for the perturbational backreaction problem could be e.g. an isotropic one particle state in the folium of an SLE. Another possibility would be to construct a more general class of SLE's which minimise the time–smeared energy density only in a compact radially symmetric region of space. Moreover, one could try to do the same perturbative investigation for other background spacetimes, such as de Sitter space.

Acknowledgements

First of all I would like to thank Prof. Klaus Fredenhagen for agreeing to supervise my dissertation. He provided many inspiring ideas for my project and I had the great opportunity to learn a lot about quantum field theory from him.

I am also very grateful to the other actual and former members of the AQFT group in Hamburg, which are Kai Keller, Thomas Hack, Falk Lindner, Christian Pfeifer, Katarzyna Rejzner, Kolja Them, Jan Schlemmer, Claudio Dappiaggi, Nicola Pinamonti, Paniz Imani, Mathias Makedonski, Sebastian Schubert, Jan Weise, Reza Safari and Marcos Brum. It was always a very enjoyable atmosphere with interesting discussions.

I would like to express my gratitude to Falk Lindner. Being the voluntary administrator of our group, he spent lots of his time for keeping the computers running!

I am particularly indebted to Thomas Hack, who was always very patient in answering both stupid and intelligent questions. Moreover, he has the matchless talent to see serious things from their funny side. Thank you for the nice jokes ;-).

Many thanks go also to Jan Schlemmer. During his rather short stay in Hamburg he helped not only me with his mathematical knowledge.

I would also like to mention the members of the four o'clock coffee meetings, which are Falk Lindner, Cecelie Hector, Patrick Rieth and Christian Pfeifer. Ich sag' nur: "Oh, es gibt Kuchen!"

Special thanks go to my family and to my friends. Thank you for your invaluable support. Without it, this thesis could not have been written!

Finally, I gratefully acknowledge the financial support of the Studienstiftung des Deutschen Volkes for the full time of my PhD project.

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